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Viscosity Approximation Methods with Meir-Keeler Contractions for Nonexpansive Semigroups

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Abstract : We establish strong convergence theorems of the viscosity approximation method associated with Meir-Keeler contractions for nonexpansive semigroups in a real Hilbert space. The main results improve and extend the corresponding results existed in the literature.

Keywords : Hilbert space; Meir-Keeler contraction; Left regular; Viscosity approximation method; Nonexpansive semigroup.
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1 Introduction

Let H be a real Hilbert space and K a nonempty, closed and convex subset of H. Let $T : K \to K$ be a mapping. Then T is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$. The fixed points set of T is denoted by $F(T)$.

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In 1967, Halpern [1] introduced the following classical iteration for a nonexpansive mapping $T : K \to K$ in a real Hilbert space: $x_0 \in K$ and

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \quad n \geq 0,$$

where $\{\alpha_n\} \subset (0,1)$ and $u \in K$ is fixed.

Let $f : K \to K$ be a contraction (i.e. $\|f(x)-f(y)\| \leq \alpha\|x-y\|$ for all $x, y \in K$ and $\alpha \in [0,1)$). In 2000, Moudafi [2] introduced the viscosity approximation method for a nonexpansive mapping T as follows: $x_0 \in K$ and

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n, \quad n \geq 0,$$

where $\{\alpha_n\} \subset (0,1)$. There have been researches concerning strong convergence of viscosity approximation methods for nonexpansive mappings or nonexpansive semigroups (see, for examples, [3–10]).

A viscosity approximation method with Meir-Keeler contractions was first studied by Suzuki [11]. Very recently, Petruşel and Yao [12] studied the following viscosity approximation method with a generalized contraction: $x_0 \in K$ and

$$x_{n+1} = \lambda_{n+1} f(x_n) + (1-\lambda_{n+1})T_{n+1}x_n, \quad n \geq 0,$$

where $\{\lambda_n\} \subset (0,1)$ and $\{T_n\}_{n=1}^{\infty}$ is a family of nonexpansive mappings on K.

In this paper, motivated by Moudafi [2], Saeidi [7], Suzuki [11] and Petruşel and Yao [12], we consider the following iterative scheme for a nonexpansive semigroup $\mathcal{S} = \{T(t) : t \in S\}$ defined by $x_1 \in K$ and

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T(\mu_n)x_n, \quad n \geq 1,$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $(0,1)$ with $\alpha_n + \beta_n + \gamma_n = 1$ and $f : K \to K$ is a Meir-Keeler contraction.

2 Preliminaries and Lemmas

In this section, we give some preliminaries, definitions, lemmas and propositions which will be used in our main results.

Let S be a semigroup. We denote by $\ell^\infty(S)$ the Banach space of all bounded real-valued functionals on S with supremum norm. For each $s \in S$, we define the left and right translation operators $l(s)$ and $r(s)$ on $\ell^\infty(S)$ by

$$(l(s)f)(t) = f(st) \quad and \quad (r(s)f)(t) = f(ts)$$

for each $t \in S$ and $f \in \ell^\infty(S)$, respectively. Let X be a subspace of $\ell^\infty(S)$ containing 1. An element μ in the dual space X^* of X is said to be a *mean* on X if $\|\mu\| = \mu(1) = 1$. It is well-known that μ is a mean on X if and only if

$$\inf_{s\in S} f(s) \leq \mu(f) \leq \sup_{s\in S} f(s)$$

for each $f \in X$. We often write $\mu_t(f(t))$ instead of $\mu(f)$ for $\mu \in X^*$ and $f \in X$.

Let X be a translation invariant subspace of $\ell^\infty(S)$ (i.e. $l(s)X \subset X$ and $r(s)X \subset X$ for each $s \in S$) containing 1. Then a mean μ on X is said to be *left invariant* (resp. *right invariant*) if $\mu(l(s)f) = \mu(f)$ (resp. $\mu(r(s)f) = \mu(f)$) for each $s \in S$ and $f \in X$. A mean μ on X is said to be *invariant* if μ is both left and right invariant [13–15]. S is said to be *left* (resp. *right*) *amenable* if X has a left (resp. right) invariant mean. S is a amenable if S is left and right amenable. In this case, $\ell^\infty(S)$ also has an invariant mean. It is known that $\ell^\infty(S)$ is amenable when S is commutative semigroup or solvable group. However, the free group or semigroup of two generators is not left or right amenable (see [16, 17]). A net $\{\mu_\alpha\}$ of means on X is said to be *left regular* [16] if

$$\lim_\alpha \|l_s^*\mu_\alpha - \mu_\alpha\| = 0$$

for each $s \in S$, where l_s^* is the adjoint operator of l_s.

Let K be a nonempty, closed and convex subset of H. A family $\mathcal{S} = \{T(s) : s \in S\}$ is called a nonexpansive semigroup on K if for each $s \in S$, the mapping $T(s) : K \to K$ is nonexpansive and $T(st) = T(s)T(t)$ for each $s, t \in S$. We denote by $F(\mathcal{S})$ the set of common fixed points of $\mathcal{S}$, i.e.

$$F(\mathcal{S}) = \bigcap_{s\in S} F(T(s)) = \bigcap_{s\in S} \{x \in K : \ T(s)x = x\}.$$

Throughout this paper, we denote the open ball of radius r centered at 0 by B_r and also denote the closed convex hull of $A \subset H$ by $\overline{co}A$. For $\varepsilon > 0$ and a mapping $T : D \to H$, the set of ε-approximate fixed points of T will be denoted by $F_\varepsilon(T, D)$, i.e. $F_\varepsilon(T, D) = \{x \in D : \ \|x - Tx\| \le \varepsilon\}$.

The following lemmas are important in order to prove our main theorem.

Lemma 2.1 ([17–19])**.** *Let f be a function of a semigroup S into a Banach space E such that the weak closure of $\{f(t) : \ t \in S\}$ is weakly compact and let X be a subspace of $\ell^\infty(S)$ containing all the functions $t \mapsto \langle f(t), x^*\rangle$ with $x^* \in E^*$. Then, for any $\mu \in X^*$, there exists a unique element f_μ in E such that*

$$\langle f_\mu, x^*\rangle = \mu_t\langle f(t), x^*\rangle$$

for all $x^* \in E^*$. Moreover, if μ is a mean on X then

$$\int f(t)\ d\mu(t) \in \overline{co}\{f(t) : \ t \in S\}.$$

We can write f_μ by $\int f(t)\ d\mu(t)$.

Lemma 2.2 ([17–19])**.** *Let K be a closed convex subset of a Hilbert space H, $\mathcal{S} = \{T(s) : \ s \in S\}$ be a nonexpansive semigroup from K into K such that $F(\mathcal{S}) \neq \emptyset$ and X be a subspace of $\ell^\infty(S)$ containing 1 and the mapping $t \mapsto \langle T(t)x, y\rangle$ be an element of X for each $x \in K$ and $y \in H$, and μ be a mean on X. If we write $T(\mu)x$ instead of $\int T_t x\ d\mu(t)$, then the following hold:*

(i) $T(\mu)$ *is a nonexpansive mapping from* K *into* K*;*

(ii) $T(\mu)x = x$ *for each* $x \in F(\mathcal{S})$*;*

(iii) $T(\mu)x \in \overline{co}\{T_t x : \ t \in S\}$ *for each* $x \in K$*;*

(iv) if μ *is left invariant, then* $T(\mu)$ *is a nonexpansive retraction from* K *onto* $F(\mathcal{S})$.

Let K be a nonempty, closed and convex subset of a real Hilbert space H. Then, for any $x \in H$, there exists a unique nearest point in K, denoted by $P_K(x)$, such that

$$\|x - P_K(x)\| \leq \|x - y\|$$

for all $y \in K$. Such a P_K is called the *metric projection* of H onto K. We also know that for $x \in H$ and $z \in K$, $z = P_K x$ if and only if

$$\langle x - z, y - z\rangle \leq 0, \quad \forall y \in K.$$

Lemma 2.3 ([20])**.** *Let* K *be a nonempty, closed and convex of a Hilbert space* H *and let* $T : K \to K$ *be a nonexpansive mapping such that* $F(T) \neq \emptyset$*. If* $\{x_n\}$ *is a sequence in* K *weakly converging to* x *and if* $\{(I - T)x_n\}$ *converges strongly to* y*, then* $(I - T)x = y$*.*

A mapping $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be an *L-function* if $\psi(0) = 0$, $\psi(t) > 0$, for each $t > 0$ and for every $s > 0$ there exists $u > s$ such that $\psi(t) \leq s$, for all $t \in [s, u]$. As a consequence, every L-function ψ satisfies $\psi(t) < t$ for each $t > 0$.

Definition 2.4. Let (X, d) be a metric space. A mapping $f : X \to X$ is said to be

(i) (ψ, L)*-contraction* if $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is an L-function and $d(f(x), f(y)) < \psi(d(x, y))$ for all $x, y \in X$ with $x \neq y$;

(ii) *Meir-Keeler type mapping* if for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for each $x, y \in X$ with $d(x, y) < \varepsilon + \delta$ we have $d(f(x), f(y)) < \varepsilon$.

Remark 2.5. *If* $\psi(t) = \alpha t$, $\alpha \in (0, 1)$*, then we get the usual contraction mapping with coefficient* α*. Other examples of* L*-functions are* $\psi(t) = \frac{t}{1+t}$ *and* $\psi(t) = ln(1 + t)$, $t \in \mathbb{R}_+$*.*

Theorem 2.6 ([21])**.** *Let* (X, d) *be a complete metric space and* $f : X \to X$ *a Meir-Keeler type mapping. Then* f *has a unique fixed point.*

Lim [22] proved the following useful characterization of Meir-Keeler and (ψ, L)-functions:

Theorem 2.7 ([22])**.** *Let* (X, d) *be a metric space and* $f : X \to X$ *a mapping. Then the following assertions are equivalent:*

(i) f *is a Meir-Keeler type mapping;*

(ii) there exists an L-function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* f *is a* (ψ, L)*-contraction.*

The following results was proved in [11]:

Proposition 2.8 ([11]). *Let* K *be a convex subset of a Banach space* E*. Let* $f : K \to K$ *be a Meir-Keeler type mapping. Then for each* $\varepsilon > 0$ *there exists* $r \in (0,1)$ *such that for each* $x, y \in K$ *with* $\|x - y\| \geq \varepsilon$*, we have*

$$\|f(x) - f(y)\| \leq r\|x - y\|.$$

Proposition 2.9 ([11]). *Let* K *be a convex subset of a Banach space* E*, let* $T : K \to K$ *be a nonexpansive mapping, and let* $f : K \to K$ *be a Meir-Keeler type mapping. Then the following hold:*

(i) $T \circ f$ *is a Meir-Keeler type mapping on* K*.*

(ii) For each $\alpha \in (0,1)$*, the mapping* $x \mapsto \alpha f(x) + (1-\alpha)T(x)$ *is a Meir-Keeler type mapping on* K*.*

In the sequel, we need the following crucial lemmas.

Lemma 2.10 ([23]). *Assume* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n\delta_n, \quad n \geq 1,$$

where $\{\gamma_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence in* $\mathbb{R}$ *such that*

(a) $\sum_{n=1}^{\infty} \gamma_n = \infty$*;*

(b) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\gamma_n\delta_n| < \infty$*.*

Then $\lim_{n\to\infty} a_n = 0$*.*

Lemma 2.11 ([24]). *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space* E *such that*

$$x_{n+1} = (1-\beta_n)y_n + \beta_n x_n, \quad \forall n \geq 1,$$

where $\{\beta_n\}$ *is a real sequence in* $[0,1]$ *with* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$*. If* $\limsup_{n\to\infty} \left(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|\right) \leq 0$*, then* $\lim_{n\to\infty} \|y_n - x_n\| = 0$*.*

3 Main result

In this section, we are now ready to prove our main theorem. In what follows, we suppose that the L-function from the characterization theorem (see Theorem 2.7), as well as, the function ψ from the definition of the (ψ, L)-contraction is continuous and strictly increasing, and $\lim_{t\to\infty} \eta(t) = \infty$, where $\eta(t) = t - \psi(t)$, $t \in \mathbb{R}_+$. In consequence, we have that η is a bijection on $\mathbb{R}_+$. It is remarked that the functions ψ given in Remark 2.5 are all satisfy the above assumption.

Theorem 3.1. *Let K be a nonempty, closed and convex subset of a Hilbert space H. Let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on K such that $F(\mathcal{S}) \neq \emptyset$. Let X be a left invariant subspace of $\ell^\infty(S)$ such that $1 \in X$, and the function $t \mapsto \langle T(t)x, y\rangle$ is an element of X for each $x, y \in K$. Let $\{\mu_n\}$ be a left regular sequence of means on X such that $\|\mu_{n+1} - \mu_n\| \to 0$, as $n \to \infty$. Let $f : K \to K$ be a Meir-Keeler contraction. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real sequences in $(0, 1)$ with $\alpha_n + \beta_n + \gamma_n = 1$ which satisfy the following conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C3) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then the sequence $\{x_n\}$ generated by

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T(\mu_n)x_n, \quad n \geq 1,$$

converges strongly to $p \in F(\mathcal{S})$ which also solves the following variational inequality:

$$\langle f(p) - p, q - p\rangle \leq 0, \quad \forall q \in F(\mathcal{S}). \tag{3.1}$$

Proof. First we show that $\{x_n\}$ is bounded. For each $w \in F(\mathcal{S})$, we see that

$$\begin{aligned}
\|x_{n+1} - w\| &\leq \alpha_n\|f(x_n) - w\| + \beta_n\|x_n - w\| + \gamma_n\|T(\mu_n)x_n - w\| \\
&\leq \alpha_n \left(\psi(\|x_n - w\|) + \|f(w) - w\|\right) + (1 - \alpha_n)\|x_n - w\| \\
&= \big(\|x_n - w\| - \alpha_n\big(\|x_n - w\| - \psi(\|x_n - w\|)\big)\big) + \alpha_n\|f(w) - w\| \\
&= \big(\|x_n - w\| - \alpha_n\big(\eta(\|x_n - w\|)\big)\big) + \alpha_n\eta\left(\eta^{-1}(\|f(w) - w\|)\right) \\
&\leq \max\left\{\|x_n - w\|, \eta^{-1}(\|f(w) - w\|)\right\}.
\end{aligned}$$

By a simple induction, we can show that

$$\|x_n - w\| \leq \max\left\{\|x_1 - w\|, \eta^{-1}(\|f(w) - w\|)\right\}, \quad \forall n \geq 1.$$

Hence the sequence $\{x_n\}$ is bounded. So are $\{f(x_n)\}$ and $\{T(\mu_n)x_n\}$.

We next show that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Observe that

$$\lim_{n\to\infty} \|T(\mu_{n+1})x_n - T(\mu_n)x_n\| = 0. \tag{3.2}$$

Indeed,

$$\begin{aligned}
\|T(\mu_{n+1})x_n - T(\mu_n)x_n\| &= \sup_{\|z\|=1} |\langle T(\mu_{n+1})x_n - T(\mu_n)x_n, z\rangle| \\
&= \sup_{\|z\|=1} |(\mu_{n+1})_s\langle T(s)x_n, z\rangle - (\mu_n)_s\langle T(s)x_n, z\rangle| \\
&\leq \|\mu_{n+1} - \mu_n\| \sup_{s\in S} \|T(s)x_n\|.
\end{aligned}$$

Since $\{x_n\}$ is bounded and $\lim_{n\to\infty}\|\mu_{n+1}-\mu_n\|=0$, (3.2) holds.

Put $w_n=\frac{x_{n+1}-\beta_n x_n}{1-\beta_n}$. Then

$$\begin{aligned}w_{n+1}-w_n &= \frac{\alpha_{n+1}f(x_{n+1})+\gamma_{n+1}T(\mu_{n+1})x_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n f(x_n)+\gamma_n T(\mu_n)x_n}{1-\beta_n}\\ &= \frac{\alpha_{n+1}f(x_{n+1})}{1-\beta_{n+1}}-\frac{\alpha_n f(x_n)}{1-\beta_n}+\frac{\gamma_{n+1}}{1-\beta_{n+1}}[T(\mu_{n+1})x_{n+1}-T(\mu_{n+1})x_n]\\ &\quad +T(\mu_{n+1})x_n-T(\mu_n)x_n+\frac{\alpha_n}{1-\beta_n}T(\mu_n)x_n-\frac{\alpha_{n+1}}{1-\beta_{n+1}}T(\mu_{n+1})x_n\end{aligned}$$

which implies

$$\begin{aligned}\|w_{n+1}-w_n\| \le{}& \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left(\|T(\mu_{n+1})x_n\|+\|f(x_{n+1})\|\right)\\ &+\frac{\alpha_n}{1-\beta_n}\left(\|T(\mu_n)x_n\|+\|f(x_n)\|\right)+\|x_{n+1}-x_n\|\\ &+\|T(\mu_{n+1})x_n-T(\mu_n)x_n\|.\end{aligned}$$

From (3.2), (C1) and (C3) we have

$$\limsup_{n\to\infty}\left(\|w_{n+1}-w_n\|-\|x_{n+1}-x_n\|\right)\le 0.$$

So by Lemma 2.11, we have $\lim_{n\to\infty}\|w_n-x_n\|=0$. It also follows that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.3}$$

We next show that

$$\lim_{n\to\infty}\|x_n-T(t)x_n\|=0,\quad \forall t\in S. \tag{3.4}$$

Let $w\in F(\mathcal{S})$ and put

$$M=\max\{\|x_1-w\|,\eta^{-1}(\|f(w)-w\|)\}.$$

Set $D=\{y\in K:\|y-w\|\le M\}$. It is easily seen that D is a nonempty bounded closed convex set and $\{x_n\}\subset D$. Further, D is invariant under $\mathcal{S}$. To complete our proof, we follow the proof line as in [25] (see also [17, 26, 27]). Let $\varepsilon>0$. From [28], there exists $\delta>0$ such that

$$\overline{co}\,F_\delta(T(t);D)+B_\delta\subseteq F_\varepsilon(T(t);D),\quad \forall t\in S. \tag{3.5}$$

From Corollary 1.1 in [28], there exists a natural number N such that

$$\left\|\frac{1}{N+1}\sum_{i=0}^{N}T(t^i s)y-T(t)\left(\frac{1}{N+1}\sum_{i=0}^{N}T(t^i s)y\right)\right\|\le\delta, \tag{3.6}$$

for all $t, s \in S$ and $y \in D$. Let $t \in S$. Since $\{\mu_n\}$ is left regular, there exists $n_0 \in \mathbb{N}$ such that

$$\|\mu_n - l^*_{t^i}\mu_n\| \leq \frac{\delta}{3(M + \|w\|)}$$

for all $n \geq n_0$ and $i = 1, 2, ..., N$. So we have for all $n \geq n_0$

$$\begin{aligned}
\sup_{y\in D}&\left\|T(\mu_n)y - \int \frac{1}{N+1}\sum_{i=0}^{N} T(t^i s)y \; d\mu_n(s)\right\| \\
&= \sup_{y\in D}\sup_{\|z\|=1}\left|(\mu_n)_s\langle T(s)y, z\rangle - (\mu_n)_s\left\langle \frac{1}{N+1}\sum_{i=0}^{N} T(t^i s)y, z\right\rangle\right| \\
&\leq \frac{1}{N+1}\sum_{i=0}^{N}\sup_{y\in D}\sup_{\|z\|=1}|(\mu_n)_s\langle T(s)y, z\rangle - (l^*_{t^i}\mu_n)_s\langle T(s)y, z\rangle| \\
&\leq \max_{i=1,2,\ldots,N}\|\mu_n - l^*_{t^i}\mu_n\|(M + \|w\|) \leq \frac{\delta}{3}. \qquad (3.7)
\end{aligned}$$

We observe by Lemma 2.2 that

$$\int \frac{1}{N+1}\sum_{i=0}^{N} T(t^i s)y \; d\mu_n(s) \in \overline{co}\left\{\frac{1}{N+1}\sum_{i=0}^{N} T(t)^i(T(s)y) : \; s \in S\right\}. \qquad (3.8)$$

Combining (3.6)-(3.8) we have

$$\begin{aligned}
T(\mu_n)y &= \int \frac{1}{N+1}\sum_{i=0}^{N} T(t^i s)y \; d\mu_n(s) + \left(T(\mu_n)y - \int \frac{1}{N+1}\sum_{i=0}^{N} T(t^i s)y \; d\mu_n(s)\right) \\
&\in \overline{co}\;\left\{\frac{1}{N+1}\sum_{i=0}^{N} T(t)^i(T(s)y) : \; s \in S\right\} + B_{\delta/3} \\
&\subseteq \overline{co}\; F_\delta(T(t); D) + B_{\delta/3}, \qquad (3.9)
\end{aligned}$$

for all $y \in D$ and $n \geq n_0$. Let $t \in S$ and $\varepsilon > 0$. Then there exists $\delta > 0$ which satisfies (3.5). From (C3), there exist $a, b \in (0, 1)$ such that $0 < a \leq \beta_n \leq b < 1$. Put $L = \psi(M) + \|f(w) - w\| + M$. From (3.3) and (C1), there exists $k_0 \in \mathbb{N}$ such that $\|x_n - x_{n+1}\| < \frac{(1-b)\delta}{3b}$ and $\alpha_n < \frac{\delta(1-b)}{3L}$ for all $n > k_0$. It follows that

$$\begin{aligned}
\frac{\alpha_n}{1-\beta_n}\|f(x_n) - T(\mu_n)x_n\| &\leq \frac{\alpha_n}{1-b}(\|f(x_n) - f(w)\| + \|f(w) - w\| + \|w - T(\mu_n)x_n\|) \\
&\leq \frac{\alpha_n}{1-b}(\psi(\|x_n - w\|) + \|f(w) - w\| + \|x_n - w\|) \\
&\leq \frac{\alpha_n}{1-b}(\psi(M) + \|f(w) - w\| + M) \\
&\leq \frac{\delta(1-b)}{3(1-b)L}L = \frac{\delta}{3}, \qquad (3.10)
\end{aligned}$$

for all $n > k_0$. Moreover,

$$\frac{\beta_n}{1-\beta_n}\|x_n - x_{n+1}\| \leq \frac{b}{1-b}\|x_n - x_{n+1}\| \leq \frac{\delta}{3}. \tag{3.11}$$

So from (3.5) and (3.9)-(3.11) we have

$$\begin{aligned} x_{n+1} &= T(\mu_n)x_n + \frac{\beta_n}{1-\beta_n}(x_n - x_{n+1}) + \frac{\alpha_n}{1-\beta_n}(f(x_n) - T(\mu_n)x_n) \\ &\in \overline{co}\ F_\delta(T(t); D) + B_{\delta/3} + B_{\delta/3} + B_{\delta/3} \\ &\subseteq \overline{co}\ F_\delta(T(t); D) + B_\delta \subseteq F_\varepsilon(T(t); D), \end{aligned}$$

for all $n > k_0$. Hence $\limsup_{n\to\infty} \|x_n - T(t)x_n\| \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary,

$$\lim_{n\to\infty} \|x_n - T(t)x_n\| = 0.$$

Since the sequence $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup z \in K$. From Lemma 2.3, we conclude that $z \in F(\mathcal{S})$. On the other hand, by Proposition 2.9 (i), we know that $P_{F(\mathcal{S})}f$ is a Meir-Keeler contraction. So, by Theorem 2.6, there exists a unique element p such that $P_{F(\mathcal{S})}f(p) = p$ which is also equivalent to

$$\langle f(p) - p, q - p\rangle \leq 0, \quad \forall q \in F(\mathcal{S}).$$

So we have

$$\begin{aligned} \limsup_{n\to\infty}\langle f(p) - p, x_n - p\rangle &= \lim_{j\to\infty}\langle f(p) - p, x_{n_j} - p\rangle \\ &= \langle f(p) - p, z - p\rangle \leq 0. \end{aligned} \tag{3.12}$$

We finally show that $x_n \to p$ as $n \to \infty$. Suppose $\{x_n\}$ does not converge strongly to $p \in F(\mathcal{S})$. Then there exists $\varepsilon > 0$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\|x_{n_k} - p\| > \varepsilon$, for all $k \in \{0, 1, ...\}$. By Proposition 2.8, for this ε there exists $r \in (0, 1)$ such that $\|f(x_{n_k}) - f(p)\| \leq r\|x_{n_k} - p\|$. So we have

$$\begin{aligned} \|x_{n_k+1} - p\|^2 &= \|\alpha_{n_k}(f(x_{n_k}) - p) + \beta_{n_k}(x_{n_k} - p) + \gamma_{n_k}(T(\mu_{n_k})x_{n_k} - p)\|^2 \\ &\leq \|\beta_{n_k}(x_{n_k} - p) + \gamma_{n_k}(T(\mu_{n_k})x_{n_k} - p)\|^2 \\ &\quad + 2\alpha_{n_k}\langle f(x_{n_k}) - p, x_{n_k+1} - p\rangle \\ &\leq (\beta_{n_k}\|x_{n_k} - p\| + \gamma_{n_k}\|T(\mu_{n_k})x_{n_k} - p\|)^2 \\ &\quad + 2\alpha_{n_k}\langle f(x_{n_k}) - p, x_{n_k+1} - p\rangle \\ &\leq (1 - \alpha_{n_k})^2\|x_{n_k} - p\|^2 + 2\alpha_{n_k}\langle f(x_{n_k}) - f(p), x_{n_k+1} - p\rangle \\ &\quad + \ 2\alpha_{n_k}\langle f(p) - p, x_{n_k+1} - p\rangle \\ &\leq (1 - \alpha_{n_k})^2\|x_{n_k} - p\|^2 + 2\alpha_{n_k}\|f(x_{n_k}) - f(p)\|\|x_{n_k+1} - p\| \\ &\quad 2\alpha_{n_k}\langle f(p) - p, x_{n_k+1} - p\rangle \\ &\leq (1 - \alpha_{n_k})^2\|x_{n_k} - p\|^2 + \alpha_{n_k}r\left(\|x_{n_k} - p\|^2 + \|x_{n_k+1} - p\|^2\right) \\ &\quad 2\alpha_{n_k}\langle f(p) - p, x_{n_k+1} - p\rangle. \end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n_k+1}-p\|^2 &\leq \frac{1-(2-r)\alpha_{n_k}+\alpha_{n_k}^2}{1-\alpha_{n_k}r}\|x_{n_k}-p\|^2+\frac{2\alpha_{n_k}}{1-\alpha_{n_k}r}\langle f(p)-p, x_{n_k+1}-p\rangle \\ &= \frac{1-\alpha_{n_k}r-2(1-r)\alpha_{n_k}}{1-\alpha_{n_k}r}\|x_{n_k}-p\|^2+\frac{\alpha_{n_k}^2}{1-\alpha_{n_k}r}\|x_{n_k}-p\|^2 \\ &\quad +\frac{2\alpha_{n_k}}{1-\alpha_{n_k}r}\langle f(p)-p, x_{n_k+1}-p\rangle \\ &= \left(1-\frac{2(1-r)\alpha_{n_k}}{1-\alpha_{n_k}r}\right)\|x_{n_k}-p\|^2 \\ &\quad +\frac{2(1-r)\alpha_{n_k}}{1-\alpha_{n_k}r}\left(\frac{1}{1-r}\langle f(p)-p, x_{n_k+1}-p\rangle+\frac{\alpha_{n_k}}{2(1-r)}\|x_{n_k}-p\|^2\right).\end{aligned}$$

Using (3.12), (C1) and (C2), we can conclude, by Lemma 2.10, that $x_{n_k} \to p$ as $k \to \infty$. This is a contradiction and hence the sequence $\{x_n\}$ converges strongly to $p \in F(\mathcal{S})$. We thus complete the proof. □

Remark 3.2. *A Meir-Keeler contraction in Theorem 3.1 can also be replaced by a (ψ, L)-contraction (see Petrușel and Yao [12], Lim [22] and Reich [29]).*

Using the results proved in [20] (see also [26]), we obtain the following corollaries:

Corollary 3.3. *Let K be a nonempty, closed and convex subset of a Hilbert space H. Let S and T be nonexpansive mappings on K with $ST = TS$ such that $F := F(S) \cap F(T) \neq \emptyset$. Let $f : K \to K$ be a Meir-Keeler contraction. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real sequences in $(0,1)$ with $\alpha_n + \beta_n + \gamma_n = 1$ satisfying* (C1)-(C3). *Then the sequence $\{x_n\}$ defined by*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \left(\frac{1}{n^2}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} S^i T^j x_n\right), \quad n \geq 1,$$

converges strongly to $p \in F$ which also solves the variational inequality (3.1).

Corollary 3.4. *Let K be a nonempty, closed and convex subset of a Hilbert space H and $\mathcal{S} = \{T(t) : t \in \mathbb{R}_+\}$ a strongly continuous nonexpansive semigroup on K such that $F(\mathcal{S}) \neq \emptyset$. Let $f : K \to K$ be a Meir-Keeler contraction. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real sequences in $(0,1)$ with $\alpha_n + \beta_n + \gamma_n = 1$ satisfying* (C1)-(C3). *Then the sequence $\{x_n\}$ defined by*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \left(\frac{1}{t_n}\int_0^{t_n} T(s)x_n \; d(s)\right), \quad n \geq 1,$$

where $\{t_n\}$ is an increasing sequence in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} t_n/t_{n+1} = 1$, converges strongly to $p \in F(\mathcal{S})$ which also solves the variational inequality (3.1).

Corollary 3.5. *Let K be a nonempty, closed and convex subset of a Hilbert space H and $\mathcal{S} = \{T(t) : t \in \mathbb{R}_+\}$ a strongly continuous nonexpansive semigroup on K such that $F(\mathcal{S}) \neq \emptyset$. Let $f : K \to K$ be a Meir-Keeler contraction. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real sequences in $(0,1)$ with $\alpha_n + \beta_n + \gamma_n = 1$ satisfying* (C1)-(C3). *Then the sequence $\{x_n\}$ defined by*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \left(a_n \int_0^\infty exp(-a_n s) T(s) x_n \ d(s) \right), \quad n \geq 1,$$

where $\{a_n\}$ is a decreasing sequence in $(0,\infty)$ such that $\lim_{n\to\infty} a_n = 0$, converges strongly to $p \in F(\mathcal{S})$ which also solves the variational inequality (3.1).

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