Thai Journal of Mathematics
Volume 3 (2005) Number 2 : 259–274

Strong Convergence of the Modified Mann Iterations in a Banach Space

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Abstract : The Mann iterations for nonexpansive mappings have only weak convergence even in a Hilbert space. We propose two modifications of the Mann iterations in a uniformly smooth Banach space, one for nonexpansive mappings and the other for the resolvent of accretive operators. The two modified Mann iterations are proved to have strong convergence.

Keywords : Modified Mann iterations with errors, nonexpansive mapping, strong convergence, uniformly smooth Banach space, accretive operators.
2000 Mathematics Subject Classification : 47H10, 47H09, 46B20.

1 Introduction

Let X be a real Banach space, C a nonempty closed convex subset of X, and $T : C \to C$ a mapping. Recall that T is *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all x, $y \in C$. A point $p \in C$ is a *fixed point* of T provided $Tp = p$. Denote by $F(T)$ the set of fixed points of T; that is, $F(T) = \{p \in C : Tp = p\}$. It is assumed throughout that T is a nonexpansive mapping such that $F(T) \neq \emptyset$.

Construction of fixed points of nonexpansive mappings is an important subject in the theory of nonexpansive mappings and its applications in a number of applied areas, in particular, in image recovery and signal processing (see, [2, 4, 12, 16, 20, 21]). However, the sequence $\{T^n x\}_{n=0}^{\infty}$ of iterates of the mapping T at a point $x \in C$ may, in general, not behave well. This means that it may not converge (even in the weak topology).

One way to overcome this difficulty is to use Mann's iteration method that produces a sequence $\{x_n\}$ via the recursive manner:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad n \geq 0 \tag{1.1}$$

where the initial guess $x_0 \in C$ is chosen arbitrarily. For example, Reich [14] proved that if X is a uniformly convex Banach space with a Fréchet differentiable norm and if $\{\alpha_n\}$ is chosen such that $\sum_{n=1}^{\infty} \alpha_n (1 - \alpha_n) = \infty$, then the sequence $\{x_n\}$ defined by (1.1) converges weakly to a fixed point of T. However, this scheme has only weak convergence even in a Hilbert space [7].

Some attempts to modify the Mann iteration method (1.1) so that strong convergence is guaranteed have recently been made. Nakajo and Takahashi [11] proposed the following modification of the Mann iteration method (1.1) in a Hilbert space H:

$$\begin{cases} x_0 = x \in C, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n} \cap Q_n(x_o), \end{cases} \tag{1.2}$$

where P_K denotes the metric projection from H onto a closed convex subset K of H. They proved that if the sequence $\{\alpha_n\}$ is bounded above from one, then $\{x_n\}$ defined by (1.2) converges strongly to $P_{F(T)}(x_0)$. Their argument does not work outside the Hilbert space setting. Also, at each iteration step, an additional projection is needed to calculate. Some related work can also be found in Shioji and Takahashi [17] and Kamimura and Takahashi [9].

Without knowing the rate of convergence of (1.2), we propose here a simpler modification of Mann's iteration scheme is a convex combination of a fixed point in C and the Mann's iteration method (1.1) and works in a uniformly smooth Banach space. there is no additional projection involved in our new scheme (see (1.3) below). Let C be a closed convex subset of a Banach space and $T : C \to C$ a nonexpansive mapping such that $F(T) \neq \emptyset$. Define $\{x_n\}$ in the following way:

$$\begin{cases} z_n = a_n x_n + (1-a_n)Tx_n, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tz_n, \\ x_{n+1} = \beta_n u + (1-\beta_n)y_n, \ \ n \geq 0, \end{cases} \tag{1.3}$$

where $u \in C$ is an arbitrary (but fixed) element in C, and $\{a_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ are three sequences in (0,1). The iterative shemes (1.3) are called the modified Mann iteration. We prove, under certain appropriate assumptions on the sequences $\{a_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ which will e made precise in Section 3, that $\{x_n\}$ defined by (1.3) converges to a fixed point of T.

Our second modification of Mann's iteration method (1.1) is an adaption to (1.3) for finding a zero of an $m-accretive$ operator A, for which we assume that the zero set $A^{-1}(0) \neq \emptyset$. Our iteration process $\{x_n\}$ is given by

$$\begin{cases} z_n = J_{r_n}x_n, \\ y_n = a_n x_n + (1-a_n)J_{r_n}x_n, \\ x_n + 1 = \alpha_n u + (1-\alpha_n)y_n, \ \ n \geq 0, \end{cases} \tag{1.4}$$

where for each $r > 0, J_r = (I + rA^{-1})$ is the resolvent of A. Again we prove, in a uniformly smooth Banach space and under certain appropriate assumptions on the sequences $\{a_n\}$, $\{\alpha_n\}$ and $\{\gamma_n\}$ which will be made precise in section 4, that $\{x_n\}$ defined by (1.4) converges strongly to a zero of A.

2 Preliminaries

Let X be a real Banach space. Recall the (normalized) *dualitymap* J from X into X^*, the dual space of X, is given by

$$J(X) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\},\ x \in X.$$

We are going to work in uniformly smooth Banach spaces that can be characterized by duality mappings as follows (see [5] for more details):

Lemma 2.1 *A Banach space X is uniformly smooth if and only if the duality map J is single valued and norm-to-norm uniformly continuous on bounded sets of X.*

In our convergence results in the next sections, we need to estimate the square-norm $\|x_{n+1} - p\|^2$ in terms of the square-norm $\|x_n - p\|^2$, where x_i is the ith iterate for $i \geq 1$, and p is a fixed point of our mapping T. To do this, we need the following well-known (subdifferential) inequality:

Lemma 2.2 *In a Banach space X, there holds the inequality*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle,\ \ x,\ y \in X,$$

where $j(x + y) \in J(x + y)$.

Recall that if C and D are nonempty subsets of a Banach space X such that C is nonempty closed convex and $D \subset C$, then a map $Q : C \to D$ is called a retraction from C onto D provided $Q(x) = x$ for all $x \in D$. A retraction $Q : C \to D$ is sunny [3, 13] provided $Q(x + t(x - Q(x))) = Q(x)$ for all $x \in C$ and $t \geq 0$ whenever $x+t(x-Q(x)) \in C$. A sunny nonexpansive retraction is a sunny retraction, which is also nonexpansive. Sunny nonexpansive retractions play an important role in our argument. They are characterized as follows [3, 8, 13]: If X is a smooth Banach space, then $Q : C \to D$ is a sunny nonexpansive retraction if and only if there holds the inequality

$$\langle x - Qx, J(y - Qx)\rangle \leq 0 \ \textit{for all}\ x \in C \ \ \textit{and} \ \ y \in D.$$

Reich [15] showed that if X is uniformly smooth and if D is the fixed point set of a nonexpansive mapping from C into itself, then there is a sunny nonexpansive retraction from C onto D and it can be constructed as follows.

Lemma 2.3 (Reich [15]) *Let X be a uniformly smooth Banach space and let $T : C \to C$ be a nonexpansive mapping with a fixed foint. For each fixed $u \in C$ and every $t \in (0, 1)$, the unique fixed point $x_t \in C$ of the contraction $C \ni x \mapsto tu + (1-t)Tx$ converges strongly as $t \to 0$ to a fixed point of T. Define $Q : C \to F(T)$ by $Qu = s - \lim_{t\to 0} x_t$. Then Q is the unique sunny nonexpansive retract from C onto $F(T)$; that is, Q satisfies the property:*

$$\langle u - Qu, J(z - Qu)\rangle \leq 0,\ \ u \in C,\ \ z \in F(T).$$

We also need the following lemma that can be found in the existing literature (see, e.g., [18, 19]):

Lemma 2.4 *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of nonnegative real numbers satisfying the property*

$$a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n\delta_n, \quad n \geq 0,$$

where $\{\gamma_n\}_{n=0}^{\infty} \subset (0,1)$ and $\{\delta_n\}_{n=0}^{\infty}$ are such that

(i) $\lim_{n\to\infty} \gamma_n = 0$ *and* $\sum_{n=0}^{\infty} \gamma_n = \infty$,

(ii) either $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=0}^{\infty} |\gamma_n\delta_n| < \infty$.

Then $\{a_n\}_{n=0}^{\infty}$ converges to zero.

3 Modified Mann's Iteration for Nonexpansive Mappings

In this section, we modify the Mann's iteration method (1.1) and prove strong convergence of the iterative sequence to a fixed point of a nonexpansive mapping in a uniformly smooth Banach space under some sufficient conditions. Our new iteration process works in a Banach space setting as opposed to [11], iteration process that works only in the framework of Hilbert spaces.

Theorem 3.1 *Let C be a closed convex subset of a uniformly smooth Banach space X and let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Given points u, $x_0 \in C$ and given sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $(0,1)$ and $\{a_n\}$ in* $[0,1]$. *If following conditions are satisfied:*

(i) $\liminf_{n\to\infty} a_n > o$;

(ii) $\alpha_n \to 0$ *and* $\beta_n \to 0$;

(iii) $\sum_{n=0}^{\infty} \beta_n = \infty$;

(iv) $\sum_{n=0}^{\infty} |a_{n+1}-a_n| < \infty$, $\sum_{n=0}^{\infty} |\alpha_{n+1}-\alpha_n| < \infty$ *and* $\sum_{n=0}^{\infty} |\beta_{n+1}-\beta_n| < \infty$,

then the sequences $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ and $\{z_n\}_{n=0}^{\infty}$ given by

$$\begin{cases} z_n = a_n x_n + (1-a_n)Tx_n, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tz_n, \\ x_{n+1} = \beta_n u + (1-\beta_n)y_n, \ \ n \geq 0, \end{cases} \tag{3.1}$$

converge strongly to a fixed point of T.

Proof. First we observe that $\{x_n\}_{n=0}^{\infty}$ is bounded. Indeed, if we take a fixed point p of T, noting that

$$\begin{aligned}
\|z_n - p\| &= \|a_n x_n + (1-a_n)Tx_n - p\| \\
&= \|a_n(x_n - p) + (1-a_n)(Tx_n - p)\| \\
&\leq a_n\|x_n - p\| + (1-a_n)\|Tx_n - p\| \\
&\leq a_n\|x_n - p\| + (1-a_n)\|x_n - p\| \\
&= \|x_n - p\|,
\end{aligned}$$

and

$$\begin{aligned}
\|y_n - p\| &= \|\alpha_n x_n + (1-\alpha_n)Tz_n - p\| \\
&= \|\alpha_n(x_n - p) + (1-\alpha_n)(Tz_n - p)\| \\
&\leq \alpha_n\|x_n - p\| + (1-\alpha_n)\|Tz_n - p\| \\
&\leq \alpha_n\|x_n - p\| + (1-\alpha_n)\|x_n - p\| \\
&= \|x_n - p\|. \qquad (3.2)
\end{aligned}$$

It follows from (3.2) that

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\beta_n u + (1-\beta_n)y_n - p\| \\
&= \|\beta_n(u - p) + (1-\beta_n)(y_n - p)\| \\
&\leq \beta_n\|u - p\| + (1-\beta_n)\|y_n - p\| \\
&\leq \beta_n\|u - p\| + (1-\beta_n)\|x_n - p\| \\
&\leq \max\{\|u - p\|, \|x_n - p\|\}.
\end{aligned}$$

Now, an induction yields

$$\|x_n - p\| \leq \max\{\|u - p\|, \|x_0 - p\|\}, \ n \geq 0. \qquad (3.3)$$

Hence, $\{x_n\}$ is bounded, so are $\{y_n\}$ and $\{z_n\}$. As a result, we obtain by condition (i),

$$\begin{aligned}
\|x_{n+1} - y_n\| &= \|\beta_n u + (1-\beta_n)y_n - y_n\| \\
&= \|\beta_n u - \beta_n y_n\| \\
&= \beta_n\|u - y_n\| \ \to 0. \qquad (3.4)
\end{aligned}$$

We next show that
$$\|x_n - Tx_n\| \to 0. \qquad (3.5)$$

It suffices to show that
$$\|x_{n+1} - x_n\| \to 0. \qquad (3.6)$$

Indeed, if (3.6) holds, then noting (3.4), we obtain

$$\begin{aligned}
\|x_n - Tx_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \|y_n - Tx_n\| \\
&\leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \|\alpha_n(x_n - Tz_n) + (Tz_n - Tx_n)\| \\
&\leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \alpha_n\|x_n - Tz_n\| + \|z_n - x_n\| \\
&= \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \alpha_n\|x_n - Tz_n\| + (1-a_n)\|Tx_n - x_n\|,
\end{aligned}$$

by condition (i) there exist positive integers n_0 and η such that

$$\eta\|x_n - Tx_n\| \leq a_n\|x_n - Tx_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \alpha_n\|x_n - Tz_n\|0,$$

for all $n \geq n_0$. Hence $\|x_n - Tx_n\| \to 0$, i,e.,(3.5) holds.

In order to prove (3.6), we calculate $x_{n+1} - x_n$. After some manipulations we get

$$\begin{aligned} x_{n+1} - x_n &= (\beta_n - \beta_{n-1})(u - Tz_{n-1}) + (1-\beta_n)\alpha_n(x_n - x_{n-1}) \\ &\quad + \big((\alpha_n - \alpha_{n-1})(1-\beta_n) - (\beta_n - \beta_{n-1})\alpha_{n-1}\big)(x_{n-1} - Tz_{n-1}) \\ &\quad + (1-\alpha_n)(1-\beta_n)(Tz_n - Tz_{n-1}). \end{aligned}$$

It follows that

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq |\beta_n - \beta_{n-1}|\|u - Tz_{n-1}\| + (1-\beta_n)\alpha_n\|x_n - x_{n-1}\| \\ &\quad + |(\alpha_n - \alpha_{n-1})(1-\beta_n) - (\beta_n - \beta_{n-1})\alpha_{n-1}|\|x_{n-1} - Tz_{n-1}\| \\ &\quad + (1-\alpha_n)(1-\beta_n)\|Tz_n - Tz_{n-1}\| \\ &\leq (1-\beta_n)\alpha_n\|x_n - x_{n-1}\| + (1-\alpha_n)(1-\beta_n)\|z_n - z_{n-1}\| \\ &\quad + |\beta_n - \beta_{n-1}|\|u - Tz_{n-1}\| \\ &\quad + |(\alpha_n - \alpha_{n-1})(1-\beta_n) - (\beta_n - \beta_{n-1})\alpha_{n-1}|\|x_{n-1} - Tz_{n-1}\| \\ &\leq (1-\beta_n)\alpha_n\|x_n - x_{n-1}\| \\ &\quad + (1-\alpha_n)(1-\beta_n)\big(\|x_n - x_{n-1}\| + |a_n - a_{n-1}|\|x_{n-1} - Tx_{n-1}\|\big) \\ &\quad + |\beta_n - \beta_{n-1}|\|u - Tz_{n-1}\| \\ &\quad + |(\alpha_n - \alpha_{n-1})(1-\beta_n) - (\beta_n - \beta_{n-1})\alpha_{n-1}|\|x_{n-1} - Tz_{n-1}\| \\ &= (1-\beta_n)\alpha_n\|x_n - x_{n-1}\| \\ &\quad + (1-\alpha_n)(1-\beta_n)\|x_n - x_{n-1}\| \\ &\quad + (1-\alpha_n)(1-\beta_n)|a_n - a_{n-1}|\|x_{n-1} - Tx_{n-1}\| \\ &\quad + |\beta_n - \beta_{n-1}|\|u - Tz_{n-1}\| \\ &\quad + |(\alpha_n - \alpha_{n-1})(1-\beta_n) - (\beta_n - \beta_{n-1})\alpha_{n-1}|\|x_{n-1} - Tz_{n-1}\| \\ &\leq (1-\alpha_n+\alpha_n)(1-\beta_n)\|x_n - x_{n-1}\| \\ &\quad + |a_n - a_{n-1}|\|x_{n-1} - Tx_{n-1}\| \\ &\quad + |\beta_n - \beta_{n-1}|\|u - Tz_{n-1}\| \\ &\quad + |(\alpha_n - \alpha_{n-1})(1-\beta_n) - (\beta_n - \beta_{n-1})\alpha_{n-1}|\|x_{n-1} - Tz_{n-1}\| \\ &= (1-\beta_n)\|x_n - x_{n-1}\| + |a_n - a_{n-1}|\|x_{n-1} - Tx_{n-1}\| \\ &\quad + |\beta_n - \beta_{n-1}|\|u - Tz_{n-1}\| \\ &\quad + |(\alpha_n - \alpha_{n-1})(1-\beta_n) - (\beta_n - \beta_{n-1})\alpha_{n-1}|\|x_{n-1} - Tz_{n-1}\|. \end{aligned}$$

Hence,

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq (1-\beta_n)\|x_n - x_{n-1}\| \\ &\quad + \gamma\big(|a_n - a_{n-1}| + |\alpha_n - \alpha_{n-1}| + 2|\beta_n - \beta_{n-1}|\big), \end{aligned} \tag{3.7}$$

where $\gamma > 0$ is a constant such that

$$\gamma \geq \max\Big\{\|x_{n-1} - Tx_{n-1}\|,\ \|u - Tz_{n-1}\|,\ \|x_{n-1} - Tz_{n-1}\|\Big\}$$

for all n. By assumption (ii)-(iv), we have that

$$\lim_{n\to\infty} \beta_n = 0,\ \sum_{n=0}^{\infty} \beta_n = \infty,\ \text{and}\ \sum_{n=0}^{\infty}(|a_n - a_{n-1}| + |\alpha_n - \alpha_{n-1}| + 2|\beta_n - \beta_{n-1}|) < \infty.$$

Hence, Lemma 2.4 is applied to (3.7) and we obtain $\|x_{n+1} - x_n\| \to 0$. Next, we claim that

$$\limsup_{n\to\infty} \langle u - q, J(x_n - q)\rangle \leq 0, \tag{3.8}$$

where $q = Q(u) = s - \lim_{t\to 0} z_t$ with z_t being the fixed point of the contraction $z \mapsto tu + (1-t)Tz$ (see Lemma 2.3).

In order to prove (3.8), we need some more information on q, which is obtained form that of z_t (cf. [17]). Indeed, z_t solves the fixed point equation

$$z_t = tu + (1-t)Tz_t.$$

Thus we have
$$\begin{aligned} z_t - x_n &= Tz_t - tTz_t - x_n + tx_n + tu - tx_n \\ &= (1-t)(Tz_t - x_n) + t(u - x_n). \end{aligned}$$

We apply Lemma 2.2 to get

$$\begin{aligned} \|z_t - x_n\|^2 &= \|(1-t)(Tz_t - x_n) + t(u - x_n)\|^2 \\ &\leq (1-t)^2\|Tz_t - x_n\|^2 + 2t\langle u - x_n, J(z_t - x_n)\rangle \\ &= (1-t)^2\|Tz_t - x_n\|^2 + 2t\langle u - z_t, J(z_t - x_n)\rangle + 2t\|z_t - x_n\|^2 \\ &\leq (1-t)^2(\|Tz_t - Tx_n\| + \|Tx_n - x_n\|)^2 \\ &\quad + 2t\langle u - z_t, J(z_t - x_n)\rangle + 2t\|z_t - x_n\|^2 \\ &\leq (1-t)^2\|z_t - x_n\|^2 + 2\|z_t - x_n\|\|Tx_n - x_n\| + \|Tx_n - x_n\|^2 \\ &\quad + 2t\langle u - z_t, J(z_t - x_n)\rangle + 2t\|z_t - x_n\|^2 \\ &= (1-t)^2\|z_t - x_n\|^2 + \big(2\|z_t - x_n\| + \|Tx_n - x_n\|\big)\|Tx_n - x_n\| \\ &\quad + 2t\langle u - z_t, J(z_t - x_n)\rangle + 2t\|z_t - x_n\|^2 \\ &= (1 - 2t + t^2)\|z_t - x_n\|^2 + a_n(t) \\ &\quad + 2t\langle u - z_t, J(z_t - x_n)\rangle + 2t\|z_t - x_n\|^2, \end{aligned}$$

where

$$a_n(t) = \big(2\|z_t - x_n\| + \|Tx_n - x_n\|\big)\|Tx_n - x_n\| \to 0 \quad \text{as } n \to \infty. \tag{3.9}$$

It follows that

$$\langle z_t - u, J(z_t - x_n)\rangle \leq \frac{t}{2}\|z_t - x_n\|^2 + \frac{1}{2t}a_n(t). \tag{3.10}$$

Let $n \to \infty$ in (3.10) and noting (3.9) yields

$$\limsup_{n\to\infty}\langle z_t - u, J(z_t - x_n)\rangle \le \frac{t}{2}M, \tag{3.11}$$

where $M > 0$ is a constant such that $M \ge \|z_t - x_n\|^2$ for all $t \in (0,1)$ and $n \ge 0$. Since the set $\{z_t - x_n\}$ is bounded, the duality map J is norm-to-norm uniformly continuous on bounded sets of X (Lemma 2.1), and z_t strongly converges to q. By letting $t \to 0$ in (3.11), it is not hard to find that the two limits can be interchanged and (3.8) is thus proven.

Finally, we show that $x_n \to q$ strongly and this concludes the proof. Indeed, using Lemma 2.2 again we obtain

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \|\beta_n u + (1-\beta_n)y_n - q\|^2\\
&= \|(1-\beta_n)(y_n - q) + \beta_n(u-q)\|^2\\
&\le (1-\beta_n)^2\|y_n - q\|^2 + 2\langle\beta_n(u-q), J(x_{n+1}-q)\rangle\\
&\le (1-\beta_n)\|x_n - q\|^2 + 2\langle\beta_n(u-q), J(x_{n+1}-q)\rangle.
\end{aligned}$$

Now we apply Lemma 2.4 and use (3.8) to see that $\|x_n - q\| \to 0$. Since

$$\|y_n - q\| \le \|x_n - q\| \text{ and } \|z_n - q\| \le \|x_n - q\|,$$

it follows that $y_n \to q$ and $z_n \to q$ as $n \to \infty$. □

If $a_n = 1$ for all $n \in \mathbb{N}$ in Theorem 3.1, we obtain the following result.

Theorem 3.2 ([10, Theorem 1]) *Let C be a closed convex subset of a uniformly smooth Banach space X and let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Given a point $u \in C$ and given sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $(0,1)$, The following conditions are satisfied:*

(i) $\alpha_n \to 0$ *and* $\beta_n \to 0$;

(ii) $\sum_{n=0}^{\infty}\alpha_n = \infty$ *and* $\sum_{n=0}^{\infty}\beta_n = \infty$;

(iii) $\sum_{n=0}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$ *and* $\sum_{n=0}^{\infty}|\beta_{n+1} - \beta_n| < \infty$.

Define a sequence $\{x_n\}_{n=0}^{\infty}$ in C by

$$\begin{cases} x_0 = x \in C \textit{ arbitrarily},\\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n,\\ x_{n+1} = \beta_n u + (1-\beta_n)y_n, \quad n \ge 0.\end{cases}$$

Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to a fixed point of T.

4 Convergence to a Zero of Accretive Operators

Let X be a real Banach space. Recall that a (possibly multivalued) operator A with domain $D(A)$ and range $R(A)$ in X is *accretive* if, for each $x_i \in D(A)$ and $y_i \in Ax_i$ (i=1, 2), there exists a $j \in J(x_2 - x_1)$ such that

$$\langle y_2 - y_1, j \rangle \geq 0.$$

(Here J is the duality map.) An accretive operator A is $m-accretive$ if $R(I+rA) = X$ for each $r > 0$. Throughout this section we always assume that A is $m-accretive$ and has a zero (i.e., the inclusion $0 \in Az$ is solvable). The set of zeros of A is denoted by F. Hence,

$$F = \Big\{ z \in D(A) : 0 \in A(z) \Big\} = A^{-1}(0).$$

For each $r > 0$, we denote by J_r the resolvent of A, i.e., $J_r = (I + rA)^{-1}$. Note that if A is $m - accretive$, then $J_r : X \to X$ is nonexpansive and $F(J_r) = F$ for all $r > 0$. We need the resolvent identity (see [1], where more details on accretive operators can be found).

Lemma 4.1 (The Resolvent Identity) *For $\lambda > 0$ and $\mu > 0$ and $x \in X$,*

$$J_\lambda x = J_\mu \Big(\frac{\mu}{\lambda} x + (1 - \frac{\mu}{\lambda}) J_\lambda x \Big). \tag{4.1}$$

Theorem 4.2 *Assume that X is a uniformly smooth Banach space and A is an $m - accretive$ operator in X such that $A^{-1}(0) \neq \emptyset$. Given a point $x_0 \in X$ and given sequences $\{a_n\}$ in [0,1], $\{\alpha_n\}$ in (0,1) and $\{r_n\}$ satisfy the conditions :*

(i) $a_n \to 0$, $\alpha_n \to 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\sum_{n=0}^{\infty} |a_{n+1} - a_n| < \infty$, $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(iii) $r_n \geq \epsilon$ *for some* $\epsilon > 0$ *and for all* $n \geq 1$.

Also assume

$$\sum_{n=1}^{\infty} \Big| 1 - \frac{r_{n-1}}{r_n} \Big| < \infty.$$

If $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be defined by

$$\begin{cases} z_n = J_{r_n} x_n, \\ y_n = a_n x_n + (1 - a_n) z_n, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) y_n, \quad n \geq 0, \end{cases} \tag{4.2}$$

then $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to a zero of A.

Proof. First of all we show that $\{x_n\}$ is bounded. Take a $p \in F = A^{-1}(0)$. It follows that

$$\begin{aligned} \|z_n - p\| &= \|J_{r_n} x_n - p\| \\ &= \|J_{r_n} x_n - J_{r_n} p\| \\ &\leq \|x_n - p\|. \end{aligned} \tag{4.3}$$

And

$$\begin{aligned} \|y_n - p\| &= \|a_n x_n + (1 - a_n) J_{r_n} x_n - p\| \\ &= \|a_n (x_n - p) + (1 - a_n)(J_{r_n} x_n - p)\| \\ &\leq a_n \|x_n - p\| + (1 - a_n) \|J_{r_n} x_n - p\| \\ &\leq a_n \|x_n - p\| + (1 - a_n) \|x_n - p\| \\ &= \|x_n - p\|. \end{aligned} \tag{4.4}$$

It follows from (4.4) that

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n u + (1 - \alpha_n) y_n - p\| \\ &= \|\alpha_n (u - p) + (1 - \alpha_n)(y_n - p)\| \\ &\leq \alpha_n \|u - p\| + (1 - \alpha_n) \|y_n - p\| \\ &\leq \alpha_n \|u - p\| + (1 - \alpha_n) \|x_n - p\| \\ &\leq max\{\|u - p\|, \|x_n - p\|\}. \end{aligned}$$

Now, an induction yields

$$\|x_n - p\| \leq \max\{\|u - p\|, \|x_0 - p\|\}, \ n \geq 0. \tag{4.5}$$

Hence, $\{x_n\}$ is bounded, so is $\{y_n\}$, $\{z_n\}$. As a result, we obtain by condition (i),

$$\begin{aligned} \|x_{n+1} - y_n\| &= \|\alpha_n u + (1 - \alpha_n) y_n - y_n\| \\ &= \|\alpha_n u - \alpha_n y_n\| \\ &= \alpha_n \|u - y_n\| \ \to 0. \end{aligned} \tag{4.6}$$

And

$$\begin{aligned} \|y_n - z_n\| &= \|a_n x_n + (1 - a_n) J_{r_n} x_n - J_{r_n} x_n\| \\ &= \|a_n (x_n - J_{r_n} x_n)\| \\ &= a_n \|x_n - z_n\| \ \to 0. \end{aligned} \tag{4.7}$$

Simple calculations show that

$$x_{n+1} - x_n = (\alpha_n - \alpha_{n-1})(u - y_{n-1}) + (1 - \alpha_n)(y_n - y_{n-1}) \tag{4.8}$$

and

$$\begin{aligned} y_n - y_{n-1} &= a_n (x_n - x_{n-1}) + (a_n - a_{n-1})(x_{n-1} - z_{n-1}) \\ &\quad + (1 - a_n)(z_n - z_{n-1}) \end{aligned} \tag{4.9}$$

The resolvent identity (4.1) implies that

$$z_n = J_{r_n}x_n = J_{r_{n-1}}\big(\frac{r_{n-1}}{r_n}x_n + (1-\frac{r_{n-1}}{r_n})J_{r_n}x_n\big),$$

which in turn implies that

$$\begin{aligned}
\|z_n - z_{n-1}\| &= \|J_{r_{n-1}}\big(\frac{r_{n-1}}{r_n}x_n + (1-\frac{r_{n-1}}{r_n})J_{r_n}x_n\big) - J_{r_{n-1}}x_{n-1}\| \\
&\le \|\frac{r_{n-1}}{r_n}x_n + (1-\frac{r_{n-1}}{r_n})J_{r_n}x_n - x_{n-1}\| \\
&= \|x_n - x_{n-1} + (1-\frac{r_{n-1}}{r_n})(J_{r_n}x_n - x_n)\| \\
&\le \|x_n - x_{n-1}\| + |1-\frac{r_{n-1}}{r_n}|\|J_{r_n}x_n - x_n\|. \qquad (4.10)
\end{aligned}$$

Combining (4.8), (4.9) and (4.10) gives

$$\begin{aligned}
\|x_{n+1} - x_n\| &\le |\alpha_n - \alpha_{n-1}|\|u - y_{n-1}\| + (1-\alpha_n)\|y_n - y_{n-1}\| \\
&\le |\alpha_n - \alpha_{n-1}|\|u - y_{n-1}\| + (1-\alpha_n)\big(\|a_n(x_n - x_{n-1}) \\
&\quad + (a_n - a_{n-1})(x_{n-1} - z_{n-1}) + (1-a_n)(z_n - z_{n-1})\|\big) \\
&\le |\alpha_n - \alpha_{n-1}|\|u - y_{n-1}\| + (1-\alpha_n)a_n\|x_n - x_{n-1}\| \\
&\quad + (1-\alpha_n)|a_n - a_{n-1}|\|x_{n-1} - z_{n-1}\| \\
&\quad +(1-\alpha_n)(1-a_n)\|z_n - z_{n-1})\| \\
&\le |\alpha_n - \alpha_{n-1}|\|u - y_{n-1}\| + (1-\alpha_n)a_n\|x_n - x_{n-1}\| \\
&\quad + |a_n - a_{n-1}|\|x_{n-1} - z_{n-1}\| + (1-\alpha_n)(1-a_n)\|z_n - z_{n-1})\| \\
&\le |\alpha_n - \alpha_{n-1}|\|u - y_{n-1}\| + (1-\alpha_n)a_n\|x_n - x_{n-1}\| \\
&\quad + |a_n - a_{n-1}|\|x_{n-1} - z_{n-1}\| + (1-\alpha_n)(1-a_n)\|x_n - x_{n-1}\| \\
&\quad + (1-\alpha_n)(1-a_n)|1-\frac{r_{n-1}}{r_n}|\|J_{r_n}x_n - x_n\| \\
&\le (1-\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|u - y_{n-1}\| \\
&\quad +|a_n - a_{n-1}|\|x_{n-1} - z_{n-1}\| + |1-\frac{r_{n-1}}{r_n}|\|J_{r_n}x_n - x_n\|.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|x_{n+1} - x_n\| &\le (1-\alpha_n)\|x_n - x_{n-1}\| \\
&\qquad + M\left(|\alpha_n - \alpha_{n-1}| + |a_n - a_{n-1}| + |1-\frac{r_{n-1}}{r_n}|\right), \qquad (4.11)
\end{aligned}$$

where M is a constant such that

$$M \ge \max\Big\{\|u - y_{n-1}\|,\ \|x_{n-1} - z_{n-1}\|,\ \|J_{r_n}x_n - x_n\|\Big\}$$

for all $n \ge 0$ and $r_n > 0$.
By assumption (i)-(iii) in the theorem, we have that

$$\lim_{n\to\infty}\alpha_n = 0,\ \sum_{n=0}^{\infty}\alpha_n = \infty,\ \text{ and }\ \sum_{n=0}^{\infty}(|\alpha_n - \alpha_{n-1}| + |a_n - a_{n-1}| + |1-\frac{r_{n-1}}{r_n}|) < \infty.$$

Hence, Lemma 2.4 is applicable to (4.11) and we conclude that $\|x_{n+1} - x_n\| \to 0$. Take a fixed number r such that $\epsilon > r > 0$. Again from the resolvent identity (4.1) we find

$$\begin{aligned}
\|y_n - J_r x_n\| &= \|a_n x_n + (1-a_n)J_{r_n}x_n - J_r x_n\| \\
&\leq a_n\|x_n - J_{r_n}x_n\| + \|J_{r_n}x_n - J_r x_n\| \\
&= a_n\|x_n - J_{r_n}x_n\| + \|J_r\big(\frac{r}{r_n}x_n + (1-\frac{r}{r_n})J_{r_n}x_n\big) - J_r x_n\| \\
&\leq a_n\|x_n - J_{r_n}x_n\| + \|\frac{r}{r_n}x_n + (1-\frac{r}{r_n})J_{r_n}x_n - x_n\| \\
&= a_n\|x_n - J_{r_n}x_n\| + \|(1-\frac{r}{r_n})(J_{r_n}x_n - x_n)\| \\
&\leq a_n\|x_n - J_{r_n}x_n\| + \|z_n - x_n\| \\
&\leq a_n\|x_n - J_{r_n}x_n\| + \|z_n - y_n\| + \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x_{n+1} - J_r x_{n+1}\| &= \|x_{n+1} - y_n + y_n - J_r x_n + J_r x_n - J_r x_{n+1}\| \\
&\leq \|x_{n+1} - y_n\| + \|y_n - J_r x_n\| + \|x_n - x_{n+1}\| \to 0.
\end{aligned}$$

Hence,

$$\|x_n - J_r x_n\| \to 0. \tag{4.12}$$

Since in a uniformly smooth Banach space, the sunny nonexpansive retract Q from X onto the fixed point set $F(J_r)(= F = A^{-1}(0))$ of J_r is unique, it must be obtained from Reich's theorem (Lemma 2.3). Namely,

$$Qu = s - \lim_{t\to 0} z_t, \qquad u \in X, \tag{4.13}$$

where $t \in (0,1)$ and $z_t \in X$ solves the fixed point equation

$$z_t = tu + (1-t)J_r z_t. \tag{4.14}$$

Since

$$\begin{aligned}
z_t - x_n &= tu + (1-t)J_r z_t - x_n \\
&= tu - tx_n + J_r z_t - x_n - tJ_r z_t + tx_n \\
&= t(u - x_n) + (1-t)(J_r z_t - x_n),
\end{aligned} \tag{4.15}$$

applying Lemma 2.2 we get

$$\begin{aligned}
\|z_t - x_n\|^2 &= \|t(u-x_n) + (1-t)(J_r z_t - x_n)\|^2 \\
&\leq (1-t)^2\|J_r z_t - x_n\|^2 + 2t\langle u - x_n, J(z_t - x_n)\rangle \\
&= (1-t)^2\|J_r z_t - x_n\|^2 + 2t\langle u - z_t, J(z_t - x_n)\rangle + 2t\|z_t - x_n\|^2 \\
&\leq (1-t)^2(\|J_r z_t - J_r x_n\| + \|J_r x_n - x_n\|)^2 \\
&\quad + 2t\langle u - z_t, J(z_t - x_n)\rangle + 2t\|z_t - x_n\|^2 \\
&\leq (1-t)^2\|z_t - x_n\|^2 + 2\|z_t - x_n\|\|J_r x_n - x_n\| + \|J_r x_n - x_n\|^2 \\
&\quad + 2t\langle u - z_t, J(z_t - x_n)\rangle + 2t\|z_t - x_n\|^2 \\
&= (1-t)^2\|z_t - x_n\|^2 + \big(2\|z_t - x_n\| + \|J_r x_n - x_n\|\big)\|J_r x_n - x_n\| \\
&\quad + 2t\langle u - z_t, J(z_t - x_n)\rangle + 2t\|z_t - x_n\|^2 \\
&= (1 - 2t + t^2)\|z_t - x_n\|^2 + a_n(t) \\
&\quad + 2t\langle u - z_t, J(z_t - x_n)\rangle + 2t\|z_t - x_n\|^2,
\end{aligned}$$

where

$$a_n(t) = (2\|z_t - x_n\| + \|J_r x_n - x_n\|\big)\|J_r x_n - x_n\| \to 0 \text{ by (4.13).} \tag{4.16}$$

It follows that

$$\langle z_t - u, J(z_t - x_n)\rangle \leq \frac{t}{2}\|z_t - x_n\|^2 + \frac{1}{2t}a_n(t). \tag{4.17}$$

Let $n \to \infty$ in (4.17) and noting (4.16) yields

$$\limsup_{n\to\infty}\langle z_t - u, J(z_t - x_n)\rangle \leq \frac{t}{2}M, \tag{4.18}$$

where $M > 0$ is a constant such that $M \geq \|z_t - x_n\|^2$ for all $t \in (0,1)$ and $n \geq 1$. Since the set $\{z_t - x_n\}$ is bounded, the duality map J is norm-to-norm uniformly continuous on bounded sets of X (Lemma 2.1), and z_t strongly converges to $Q(u)$. By letting $t \to 0$ in (4.18),

$$\limsup_{n\to\infty}\langle u - Q(u), J(x_n - Q(u))\rangle \leq 0. \tag{4.19}$$

Finally, we prove that $\{x_n\}$ strongly converges to $Q(u)$. Indeed, by (4.4) and using Lemma 2.2 again we obtain

$$\begin{aligned}
\|x_{n+1} - Q(u)\|^2 &= \|\alpha_n u + (1-\alpha_n)y_n - Q(u)\|^2 \\
&= \|\alpha_n(u - Q(u) + (1-\alpha_n)(y_n - Q(u))\|^2 \\
&\leq \|(1-\alpha_n)(y_n - Q(u))\|^2 + 2\langle\alpha_n(u - Q(u)), J(x_{n+1} - Q(u))\rangle \\
&\leq (1-\alpha_n)\|y_n - Q(u)\|^2 + 2\langle\alpha_n(u - Q(u)), J(x_{n+1} - Q(u))\rangle \\
&\leq (1-\alpha_n)\|y_n - Q(u)\|^2 + 2\langle\alpha_n(u - Q(u)), J(x_{n+1} - Q(u))\rangle \\
&\leq (1-\alpha_n)\|x_n - Q(u)\|^2 + 2\langle\alpha_n(u - Q(u)), J(x_{n+1} - Q(u))\rangle.
\end{aligned}$$

By Lemma 2.4 and (4.19), we obtain that $\|x_n - Q(u)\| \to 0$. Since

$$\|y_n - Q(u)\| \le \|x_n - Q(u)\| \text{ and } \|z_n - Q(u)\| \le \|x_n - Q(u)\|,$$

it follows that $y_n \to Q(u)$ and $z_n \to Q(u)$ as $n \to \infty$. □

If $a_n = 0$ for all $n \in \mathbb{N}$ in Theorem 4.2, we obtain the following result.

Theorem 4.3 ([10, Theorem 2]) *Assume that X is a uniformly smooth Banach space and A is an $m-$accretive operator in X such that $A^{-1}(0) \neq \emptyset$. Let $\{x_n\}$ be defined by*

$$\begin{cases} x_0 = x \in X, \\ y_n = J_{r_n} x_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n) y_n, \ \ n \ge 0. \end{cases}$$

Suppose $\{\alpha_n\}$ and $\{r_n\}$ satisfy the conditions :

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(iii) $r_n \ge \epsilon$ *for some* $\epsilon > 0$ *and for all* $n \ge 1$.

Also assume

$$\sum_{n=1}^{\infty} \left|1 - \frac{r_{n-1}}{r_n}\right| < \infty.$$

Then $\{x_n\}$ converges strongly to a zero of A.

Acknowledgement

The author would like to thank the Thailand Research Fund for their financial support.

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(Received 21 December 2005)

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