Thai Journal of Mathematics
Volume 10 (2012) Number 1 : 137–145
**www.math.science.cmu.ac.th/thaijournal**
Online ISSN 1686-0209

# Completely Slightly Compressible Modules

**Cesim Çelik**

Department of Mathematics, Faculty of Science
Abant Izzet Baysal University, Bolu, Turkey
e-mail : cesimcelik@gmail.com

**Abstract :** In this note we introduce and investigate completely slightly compressible modules as a generalization of compressible modules. It is shown that if $M$ is completely slightly compressible module and for any $0 \neq x \in M$, $xR$ is not isomorphic to any submodule of itself, then (1) every nonzero submodule of $M$ contains a nonzero simple direct summand and (2) $M$ has a decomposition $M = M_1 \oplus M_2$ where $M_1$ is a semisimple submodule, $M_2$ is a submodule which has an essential socle. Every simple submodule of a completely slightly compressible is a direct summand. Furthermore, an artinian completely slightly compressible ring is semisimple.

**Keywords :** Compressible modules; Completely slightly compressible modules; Slightly compressible modules.
**2010 Mathematics Subject Classification :** 13E10; 16D60.

## 1 Introduction

All rings are associative with identity and all modules are unitary right modules. Let $R$ be a ring $M$ a $R$-module and $N$ be a submodule of $M$, by $N \leq M(N \leq_e M, N \leq_d M)$ we usually mean that $N$ is a submodule (essential submodule, direct summand respectively) of $M$. The injective hull of $M$ is denoted by $E(M)$. The $R$-module $M$ is called *completely slightly compressible (CSC-,* for short*)module* if for each nonzero submodule $N$ of $M$, there exists a nonzero homomorphism $f{:}M \to N$ such that $Kerf \cap N = 0$. The ring $R$ is called *completely slightly compressible*(CSC-*,* for short*)ring* if $R_R$ is a completely slightly compressible module. Following [1] and [2] a right $R$-module $M$ is called *compressible* *(essentially compressible* (EC)) if for each nonzero submodule (essential submodule) $N$ of $M$, there exists a monomorphism $f : M \to N$. The right $R$-module $M$ is called *slightly compressible* or *retractable as in* [3](SC), if for each nonzero submodule $N$ of $M$, there exists a nonzero homomorphism $f{:}M \to N$. This modules class have studied in [2–5]. Every semisimple module is a CSC-module but it need not be a compressible module. A nonzero semisimple module $M$ is compressible if and only if it is simple [6]. In [2, 3], it has shown that any direct sum of slightly compressible (essentially compressible) $R$-modules is slightly compressible (essentially compressible). In [3] semi-essentially compressible modules are investigated. It is called in [3] that an R-module $M$ is *semi-essentially compressible* ((SEC), briefly) if for each essential submodule $N$ of $M$, $M$ can be embedded in a direct sum $N^{(I)}$ of copies of $N$ for some set $I$. So every essentially compressible module is semi-essentially compressible, and every semi-essentially compressible module is slightly compressible. For any module $M$, we have the following implications:

compressible ⇒ completely slightly compressible ⇒ essentially compressible ⇒
semi essentially compressible ⇒ slightly. compressible

However, it is not known whether any direct sum of completely slightly compressible modules is a completely slightly compressible module. An $R$-module $M$ is called *subisomorphic* to an $R$-module $M'$ if there exist monomorphisms $f : M \to M'$ and $g : M' \to M$.

## 2 Completely Slightly Compressible Modules

**Example 2.1.** *Let $M$ be a semisimple (not simple) right $R$-module and $N$ be a nonzero simple submodule of $M$. It is clear that $M$ is a completely slightly compressible module. However, $M$ is not a compressible module. Suppose that $M$ is compressible. Then there exists a monomorphism $f{:}M \to N$. Since $N$ is simple, $M$ is simple, a contradiction.*

**Example 2.2.** *Let $R = (\mathbb{Z}_4, +, .)$. $A = 2R = \{\bar{0}, \bar{2}\}$ is a unique proper submodule of $R_R$. $f : R \to A; f(\bar{1}) = \bar{2}$ is a unique nonzero homomorphism and $Ker f \cap A \neq 0$. Hence $R_R$ is slightly compressible module, but not a completely slightly compressible module.*

**Lemma 2.3.** *Every submodule of a completely slightly compressible module $M$ is completely slightly compressible.*

**Lemma 2.4.** *Let $M$ be an essential compressible module such that every nonzero submodule $N$ of $M$ is invariant under monomorphism $\alpha$ of $M$ and $\alpha^{-1}(N) \subseteq N$. Then $M$ and every factor module of $M$ are completely slightly compressible modules.*

*Proof.* Let $L$ and $N$ be two nonzero submodules of $M$ such that $N \leq L$. Then there exists a submodule $K$ of $M$ such that $K \cap L = 0$ and $K \oplus L \leq_e M$. By hypothesis, there exists a monomorphism $\alpha{:}M \to K \oplus L$ such that $\alpha(L) \subseteq L$. $\beta : M \to L$; $\beta{=}\pi\alpha$ is a nonzero homomorphism where $\pi{:}K \oplus L \to L$ is a natural projection map. Since $\alpha$ is a monomorphism, $Ker\beta \cap L = 0$ and so $M$ is a completely slightly compressible module.

Define $\bar{\beta} : \frac{M}{N} \to \frac{L}{N}$; $\bar{\beta}(m + N) = \beta(m) + N$ for each $m \in M$ Let $x \in L - N$. Then $0 \neq x + N \in \frac{L}{N} \leq \frac{M}{N}$. If $x + N \in Ker\bar{\beta}$, then $0 = \bar{\beta}(x + N) = \beta(x) + N = \pi(\alpha(x)) + N = \alpha(x) + N$ and so $\alpha(x) \in N$. Since $\alpha^{-1}(N) \subseteq N$, $x \in \alpha^{-1}(N) \subseteq N$, a contradiction. Hence $Ker\bar{\beta} \cap \frac{L}{N} = 0$. □

**Proposition 2.5.** *Let $M$ be an $R$-module. Then the following statements are equivalent.*

(i) *$M$ is a completely slightly compressible module.*

(ii) *$M$ is subisomorphic to a completely slightly compressible module.*

*Proof.* (i)⇒(ii): $M$ has an essential submodule $N$ such that $M \cong N$. By Lemma 2.3, $N$ is completely slightly compressible module.

(ii)⇒(i): Let $M'$ be a completely slightly compressible module and $M$ be subisomorphic to $M'$. Then there exist $\alpha : M \to M'$ and $\beta : M' \to M$ monomorphisms. Let $N$ be a nonzero submodule of $M$. Then there exists a nonzero submodule $L$ of $M'$ such that $L = \alpha(N) \leq M'$. Since $M'$ is completely slightly compressible module, there exists a homomorphism $g : M' \to L$ such that $Kerg \cap L = 0$. Define $\gamma : M \to N$ by $\gamma = fg\alpha$ where $f : L \to N$ is an isomorphism. $\gamma$ is a nonzero homomorphism. Let $0 \neq x \in N$. Then $\alpha(x) \neq 0$ and $\alpha(x) \in L$. This implies that $g(\alpha(x)) \neq 0$ and also $f(g(\alpha(x))) \neq 0$. Thus, $Ker\gamma \cap N = 0$ and so $M$ is completely slightly compressible module. □

**Lemma 2.6.** *Let $M$ be a completely slightly compressible $R$-module and $A = ann_R(M)$. Let $I, J$ be two proper ideals of $R$ such that $JI \subseteq A$ and $J \subsetneqq A$. Then either $I \subseteq A$ or $MI \cap MJ = 0$.*

*Proof.* Since $J \subsetneqq A$, $MJ$ is a nonzero submodule of $M$. By hypothesis, there exists a homomorphism $f : M \to MJ$ such that $Kerf \cap MJ = 0$. $f(MI) = f(M)I \subseteq (MJ)I = (M)JI = 0$ implies $MI \subseteq Kerf$. If $Kerf = 0$, then $MI = 0$ and so $I \subseteq A$. Assume that $Kerf \neq 0$. Then $MI \cap MJ = 0$. □

**Theorem 2.7.** *Let $M$ be an $R$-module such that it contains no infinite direct sum of submodules. If $E(M)$ is completely slightly compressible module, then $M$ is an injective module.*

*Proof.* Since $E(M)$ is completely slightly compressible module and $M \leq_e E(M)$, there exists a monomorphism $f : E(M) \to M$. This implies that $M$ has a nonzero submodule $M_1$ such that $E(M) \cong f(E(M)) = M_1 \leq M$. $M_1$ is an injective submodule of $M$ and so $E(M) = M_1 \oplus M_2$ and $M = M_1 \oplus (M \cap M_2)$ for some $M_2 \leq E(M)$. If $M \cap M_2 = 0$, then $M_2 = 0$ and so $E(M) = M_1 = M$. Suppose that $M \cap M_2 \neq 0$. Again by the completely slightly compressible condition on

$E(M)$, there exists a nonzero homomorphism $f_1 : E(M) \to M \cap M_2$ such that $Ker f_1 \cap (M \cap M_2) = 0$. $Ker f_1 \cap (M \cap M_2) = (Ker f_1 \cap M_2) \cap M = 0$ implies $Ker f_1 \cap M_2 = 0$. Therefore $f_2 = f_1 |_{M_2}: M_2 \to M \cap M_2$ is a monomorphism. Thus, $M_2 \cong f_2(M_2) = M_3 \leq M \cap M_2$ and so $M_2 = M_3 \oplus M_4$, $M = M_1 \oplus M_3 \oplus (M \cap M_4)$ for some $M_4 \leq M_2$. After finite step, we have

$$M = M_1 \oplus M_3 \oplus \cdots \oplus M_{2n-1} \oplus (M \cap M_{2n})$$

where $M_i$ is an injective submodules of $M$ $(i = 1, 3, ..., 2n-1,)$, $M_{2n} \leq_d E(M)$ and $n$ is a positive integer. We claim that $M \cap M_{2n}$ must be a uniform injective submodule of $M$ for a positive integer $n$. In fact, if $M \cap M_{2n}$ is not uniform, then by the completely slightly compressible condition on $E(M)$, there exists a homomorphism $g : E(M) \to M \cap M_{2n}$ such that $0 = (Ker g) \cap (M \cap M_{2n}) = Ker g \cap M_{2n}$. This implies that $g_1 = g |_{M_{2n}}: M_{2n} \to M \cap M_{2n}$ is a monomorphism. $M_{2n} \cong g_1(M_{2n}) = M_{2n+1} \leq M \cap M_{2n}$. Hence $M_{2n+1}$ is an injective submodule of $M_{2n}$ and also $M_{2n+1} \leq_d M \cap M_{2n}$. This proceed can be repeated infinitely many times. So $M$ contains infinite direct sum of submodules. This is a contradiction. Since $M \cap M_{2n}$ is uniform, $g_1$ is an isomorphism and so $M_{2n+1} = M \cap M_{2n}$. Thus,

$$\begin{aligned} M &= M_1 \oplus M_3 \oplus \cdots \oplus M_{2n-1} \oplus (M \cap M_{2n}) \\ &= M_1 \oplus M_3 \oplus \cdots \oplus M_{2n-1} \oplus M_{2n+1} \end{aligned}$$

is injective. □

**Proposition 2.8.** *Let $M$ be a completely slightly compressible module. Then every simple submodule of $M$ is a direct summand of $M$.*

*Proof.* Let $N$ be a simple submodule of $M$. Then there exists a nonzero homomorphism $f : M \to N$ such that $Ker f \cap N = 0$. Two possibilities arise here. If $Ker f = 0$, then $M$ is isomorphic to simple module $N$. Hence the simple module $M$ is a direct summand of itself. Assume $Ker f \neq 0$. Then $M/Ker f$ is isomorphic to $N$ and so $Ker f$ is a maximal submodule of $M$. Since $Ker f \cap N = 0$, we have $M = Ker f \oplus N$. □

**Lemma 2.9.** *Let $M$ be a module such that for any $0 \neq x \in M$, $xR$ is not isomorphic to any submodule of itself. If $M$ is a completely slightly compressible module, then every nonzero submodule of $M$ has a nonzero simple direct summand of $M$.*

*Proof.* Let $N$ be a nonzero submodule of $M$ and $0 \neq x \in N$. The submodule $xR$ is a cyclic submodule of $N$ and so it has a maximal submodule $Y$. Since $M$ is a completely slightly compressible module, by Lemma 2.3 there exists a nonzero homomorphism $f : xR \to Y$ such that $Ker f \cap Y = 0$. Then $K = Ker f \neq 0$. Otherwise $xR$ would be isomorphic to a submodule of itself. That would be a contradiction to the hypothesis. Hence $Ker f$ is nonzero and it implies that $xR = K \oplus Y$ since $Y$ is maximal. $K$ is a simple submodule of $xR$ and also it is a simple submodule of $M$. By the completely slightly compressible condition on $M$, there exists a nonzero homomorphism $g : M \to K$ such that $Kerg \cap K = 0$. Since $K$ is simple, $g$ is an epimorphism and so $M/Kerg \cong K$. Thus, $M = K \oplus Kerg$ and $K \leq N$. □

**Corollary 2.10.** *Let $M$ be a module such that for any $0 \neq x \in M$, $xR$ is not isomorphic to any submodule of itself. If $M$ is a completely slightly compressible module, then $Soc(M)$ is an essential submodule of $M$.*

**Corollary 2.11.** *Let $M$ be a module such that for any $0 \neq x \in M$, $xR$ is not isomorphic to any submodule of itself. If $M$ is a completely slightly compressible module, then $M$ has no nonzero small submodule.*

*Proof.* Suppose that $K$ be a nonzero small submodule of $M$. By Lemma 2.9, $K$ has a simple submodule which is a direct summand of $M$. This is a contradiction. □

**Theorem 2.12.** *Let $M$ be a module such that for any $0 \neq x \in M$, $xR$ is not isomorphic to any submodule of itself. If $M$ is a completely slightly compressible module, then $M = M_1 \oplus M_2$ where $M_1$ is a semisimple and $M_2$ has an essential socle.*

*Proof.* By Corollary 2.10, $Soc(M)$ is an essential submodule and by Corollary 2.11, $M$ has no nonzero small submodule. Then there exists a submodule $M_2$ of $M$ such that $M = Soc(M) + M_2$ and $N = Soc(M_2) = M_2 \cap Soc(M) \leq_e M_2$. $Soc(M) = M_1 \oplus N$ and so $M = M_1 \oplus N + M_2 = M_1 \oplus M_2$ for some $M_1 \leq_d Soc(M)$. □

**Theorem 2.13.** *Let $M$ be an $R$-module which has a finitely generated essential socle. Then $M$ is completely slightly compressible if and only if $M$ is semisimple.*

*Proof.* Suppose that $M$ is a completely slightly compressible module. Let $Soc(M) = S_1 \oplus S_2 \oplus \cdots \oplus S_n \leq_e M$. By Proposition 2.8, $S_i$ is direct summand of $M$ for all $i = 1, 2, ..., n$. We assume $n = 2$. $Soc(M) = S_1 \oplus S_2 \leq_e M$. Since $S_1$ is a direct summand of $M$, $M = S_1 \oplus L$ for some submodule $L$ of $M$ and $S = Soc(L)$ is simple. By Lemma 2.3, there exists a monomorphism $f : L \to S$. Being $S$ simple implies $L \cong S$ and also $L$ is simple. Thus, $M$ is semisimple. □

# 3 Completely Slightly Compressible Rings

In this section we investigate completely slightly compressible rings. We start with an example. As we have noted there are right essentially compressible rings but not right completely slightly compressible.

**Example 3.1.** *Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$. Since $\mathbb{Z}$ is an integral domain, $\mathbb{Z}$ is a (right) essentially compressible ring. By [2, Corollary 5.7], $R$ is a (right) essentially compressible ring. Assume that $R$ is a right completely slightly compressible ring and we get a contradiction. Let $I = \begin{pmatrix} 0 & 2\mathbb{Z} \\ 0 & 0 \end{pmatrix}$. Then $I$ is a right ideal of $R$. By* *assumption there exists a nonzero homomorphism* $R \xrightarrow{f} I$ *such that* $Ker(f) \cap I = 0$. *But* $\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \in Ker(f) \cap I$. *This is the required contradiction.*

**Proposition 3.2.** *Let* $R$ *be a ring and* $I$ *a minimal right (left) ideal of* $R$. *If* $R$ *is a completely slightly compressible ring, then there exists* $e^2 = e \in R$ *such that* $I = eR (I = Re)$.

*Proof.* Let $I$ be a minimal right ideal of a completely slightly compressible ring $R$. Then $I = aR$ for some $a \in I$ and there exists a homomorphism $f : R \to aR$ such that $Kerf \cap aR = 0$. Let $f(1_R) = ar (r \in R)$. Then $f(ar) = f(1_R)ar = (ar)^2 \neq 0$. So $I^2 \neq 0$. This implies that there exists $e^2 = e \in I$ such that $I = eR$. □

**Corollary 3.3.** *Let* $R$ *be a completely slightly compressible ring. Then for any right (left) ideal* $I$ *of* $R$ *there exists a nonzero element* $x \in I$ *such that* $r.ann(x) \cap I = 0$ $(l.ann(x) \cap I = 0)$.

*Proof.* Let $I$ be a right (left) ideal of $R$. Since $R$ is a completely slightly compressible ring, there exists a homomorphism $f : R \to I$ such that $Kerf \cap I = 0$. Let $f(1_R) = x \in I$. For any $0 \neq y \in I$, $f(y) \neq 0$. Thus, $0 \neq f(y) = xy$ and $r.ann(x) \cap I = 0$. □

**Lemma 3.4.** *Let* $R$ *be a completely slightly compressible ring. Then*

*(1)* $R$ *has no nonzero nilpotent ideal.*

*(2) If* $R$ *is commutative, then* $R$ *has no nonzero nilpotent element.*

*Proof.* (1) : Let $I$ be a nilpotent ideal of $R$ and $I^n = 0$ for some positive integer $n$. If $I^{n-1} = 0$, there is nothing to do. Otherwise there exists a nonzero homomorphism $R \xrightarrow{f} I^{n-1}$. Let $f(1) = x \in I^{n-1}$. Then $f(x) = x^2 = 0$ since $x \in I^{n-1}$ and $(I^{n-1})^2 = 0$. Hence $x \in Kerf \cap (I^{n-1}) = 0$. Thus $I^{n-1} = 0$. By induction on $n$ we may have $I = 0$.

(2) : Let $a$ be a nilpotent element of $R$ with $a^n = 0$ for some positive integer $n$. There exists a homomorphism $f : R \to Ra$ such that $Kerf \cap Ra = 0$. Let $f(1) = ra$ for some $r \in R$. Then $f(a^{n-1}) = ra^n = 0$. Since $a^{n-1} \in Ra$ and $Kerf \cap Ra = 0$, $a^{n-1} \in Kerf \cap Ra = 0$. By induction on $n$ we may conclude that $a = 0$. Thus $R$ has no nonzero nilpotent elements. □

Recall that a module $M$ is semisimple if every submodule of $M$ is a direct summand. The ring $R$ is semisimple if the right $R$-module $R$ is semisimple, equivalently every $R$-module is semisimple. According to the Wedderburn-Artin Theorem a ring $R$ is semisimple if and only if it is isomorphic to a finite direct sum of full matrix rings over division rings.

**Theorem 3.5.** *Let* $R$ *be a right artinian ring. Then* $R$ *is a right completely slightly compressible ring if and only if* $R$ *is semisimple.*

*Proof.* ($\Rightarrow$:) Assume that $R$ is a right completely slightly compressible ring. By Lemma 3.4(1), $R$ does not has a nonzero nilpotent ideal. By hypothesis Jacobson radical $J(R)$ of $R$ is nilpotent. Hence $J(R) = 0$. So by Proposition 15.17 in [7], $R$ is semisimple. □

# 4 Retractability and Related concepts

In this section we investigate module properties under assumption of retractability. Let $M$ be a right $R$-module with $S = \mathrm{End}_R(M)$. The following results are proved in [8].

1. If the module $M$ is retractable and has a semisimple endomorphism ring, then it is semisimple artinian.
2. If $M$ is a retractable $\mathcal{K}$-nonsingular, then S is right nonsingular.
3. If $M$ is a S-(quasi-)Baer module, then $S$ is a (quasi-)Baer ring. Converse holds if $M$ is retractable.

The module $M$ is called *principally S-Baer* if for any $m \in M$, $l_S(m) = Se$ (which is equal to $l_S(mR)$) for some $e^2 = e \in S$ [6]. An $R$- module $M$ is said to be *S-quasi-Baer* if the right annihilator in $M$ of any ideal of $S$ is generated by an idempotent of $S$(or equivalently, for all fully invariant $R$-submodules $N$ of $M$, $l_S(N) = Se$ with $e^2 = e \in S$), while $M$ is called a *principally S-quasi-Baer* module if the right annihilator in $M$ of any principal right ideal of $S$ is generated by an idempotent of $S$. A module $M$ is called *principally retractable* if for any nonzero cyclic submodule $N$ of $M$, there exists a nonzero $M \xrightarrow{f} N$(or, equivalently, for any nonzero $f \in S$ $f(M)$ is contained in a cyclic submodule of $M$). In [9], principally retractable module is called quasi-retractable.

**Proposition 4.1** ([9]). *If $M$ is principally S-quasi-Baer module, then $S$ is a right principally quasi-Baer ring. The converse holds if $M$ is a principally retractable module.*

Let $M$ be a right $R$-module with $S = \mathrm{End}_R(M)$. In [6], The module $M$ is called a *principally S-Baer* if for any $m \in M$, $l_S(m) = Se$ (which is equal to $l_S(mR)$) for some $e^2 = e \in S$. Note that a ring $R$ is called *right principally projective*(or *a right Rickart ring*) if every cyclic right ideal of $R$ is a projective right $R$-module(see namely [10]). Then the module $R_R$ is principally Baer if and only if the ring $R$ is left principally projective.

Let $M$ be a right $R$-module with $S = \mathrm{End}_R(M)$. In [11], the module $M$ is called *S-Rickart* if for any $f \in S$, $r_M(f) = eM$ for some $e^2 = e \in S$. Then the ring $R$ is right Rickart if and only if the module $R_R$ is Rickart. S-Rickart rings are studied in detail in [6]. Left S-Rickart rings are defined in a symmetric way.

**Theorem 4.2.** *If $M$ is a S-Rickart module, then $S$ is a right Rickart ring. The converse is true if $M$ is principally retractable.*

*Proof.* Let $M$ be a S-Rickart module. In [6], it is proved that $S$ is a right Rickart ring. Assume now that $M$ is principally retractable and $S$ is a right Rickart ring. Let $f \in S$. We prove $r_M(f) = eM$ for some $e^2 = e \in S$. There exists $e^2 = e \in S$ such that $r_S(f) = eS$. So $fe = 0$ and $feM = 0$. It follows that $eM \leq r_M(f)$. For the inverse inclusion let $m \in r_M(f)$. By modularity $r_M(f) = eM \oplus ((1-e)M) \cap (r_M(f))$. Write $m = em_1 + (1-e)m_2$ where $m_1, m_2 \in M$ with $(1-e)m_2 \in r_M(f)$. Assume that $(1-e)m_2$ is nonzero and get a contradiction. Now by hypothesis there exists a nonzero homomorphism $M \xrightarrow{g} (1-e)m_2R$. Then we have $gM \leq (1-e)m_2R$ and it implies $g = (1-e)g \in (1-e)S$. Since $(1-e)m_2 \in r_M(f)$, $fg = 0$. Hence $g \in r_S(f) = eS$, and then $eg = g \in eS$. Thus $g \in (eS) \cap ((1-e)S) = 0$. This is the required contradiction. □

**Acknowledgement :** I would like to thank the referee(s) and Prof. Dr. Abdullah Harmanci for his comments and suggestions on the manuscript.

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(Received 29 November 2010)
(Accepted 28 October 2011)