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A Note on Heat Type Polynomials Suggested by Jacobi Polynomials

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Abstract : In the present paper an attempt has been made to investigate a new class of heat type polynomials suggested by Jacobi polynomials. Results analogous to Jacobi polynomials obtained by Srivastava [H.M. Srivastava, Note on certain generating functions for Jacobi and Laguerre polynomials, Publ. Inst. Math. (Beograd) (N.S.) 17 (31) (1974) 149–154] and Varma [V.K. Varma, Appell's double hypergeometric function as a generating function of the Jacobi polynomials, Riv. Mat. Univ. Parma 8 (1967) 305–308] have also been obtained for this new class of heat type polynomials.

Keywords : Heat polynomials; Jacobi polynomials; Appell's functios; Kampé de Fériet's function; Generating function.
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1 Introduction

A heat polynomials $v_n(x,t)$ of degree n is defined as a coefficient of $\frac{z^n}{n!}$ in the power series expansion e^{zx+x^2t}:

$$e^{zx+x^2t} = \sum_{n=0}^{\infty} v_n(x,t)\frac{z^n}{n!} \tag{1.1}$$

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It is clearly a solution of one dimensional heat equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \tag{1.2}$$

for all values of the variables.

The generalized heat polynomials $P_{n,\nu}(x,u)$ defined by (Haimo [1], p.736, eq. (2.1), Bragg [2], p.272, eq. (2.3)) is as follows:

$$\begin{aligned} P_{n,\nu}(x,u) &= \sum_{k=0}^{n} 2^{2k} \binom{n}{k} \frac{\Gamma(\nu+n+\frac{1}{2})}{\Gamma(\nu+n-k+\frac{1}{2})} x^{2n-2k} u^k && (1.3) \\ &= (4u)^n \; n! \; L_n^{(\nu-\frac{1}{2})}\left(-\frac{x^2}{4u}\right) && (1.4) \end{aligned}$$

where $L_n^{(\alpha)}(x)$ denotes the Laguerre polynomials defined by (see, Rainville [3, pp. 200]), as given below:

$$L_n^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!} \; {}_1F_1\left[-n; 1+\alpha; x\right]. \tag{1.5}$$

The heat type polynomials suggested by Jacobi polynomials are denoted by, $P_{n,\lambda,\mu}(x,u)$ and defined as

$$\begin{aligned} P_{n,\lambda,\mu}(x,u) &= (4u)^n \left(\lambda + \frac{1}{2}\right)_n {}_2F_1\left[\begin{array}{cc} -n, \lambda+\mu+n; & -\dfrac{x^2}{4u} \\ \lambda+\frac{1}{2} \quad ; & \end{array}\right] && (1.6) \\ &= (4u)^n \; n! \; P_n^{(\lambda-\frac{1}{2},\mu-\frac{1}{2})}\left(-\frac{x^2}{4u}\right) && (1.7) \end{aligned}$$

where $P_n^{(\alpha,\beta)}(x)$ denotes the Jacobi polynomials defined by ([3, pp. 254]), as given below:

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!} {}_2F_1\left[\begin{array}{cc} -n, 1+\alpha+\beta+n; & \dfrac{1-x}{2} \\ 1+\alpha; & \end{array}\right]. \tag{1.8}$$

In the form similar to Jacobi polynomials due to Srivastava [4], (1.6) can be written as

$$P_{n,\lambda,\mu}(x,u) = \sum_{k=0}^{n} \binom{\lambda+n-\frac{1}{2}}{n-k} \binom{\mu+n-\frac{1}{2}}{k} n! \; x^{2k}(x^2+4u)^{n-k}. \tag{1.9}$$

In view of the relation ([3, Theorem 20, pp. 60]),

$$F\left[\begin{array}{cc} a, b; & z \\ c \;\; ; & \end{array}\right] = (1-z)^{-a} \; F\left[\begin{array}{cc} a, c-b; & \dfrac{-z}{1-z} \\ c; & \end{array}\right] \tag{1.10}$$

a relation (1.6), can be written in the elegant form

$$P_{n,\lambda,\mu}(x,u) = \left(\lambda + \frac{1}{2}\right)_n (x^2+4u)^n {}_2F_1\left[\begin{matrix} -n, \frac{1}{2}-\mu-n; \\ \lambda+\frac{1}{2} \end{matrix}; \frac{x^2}{x^2+4u}\right] \tag{1.11}$$

and in the finite series form (1.6) and (1.11) becomes,

$$P_{n,\lambda,\mu}(x,u) = \sum_{k=0}^{n} \binom{n}{k} \frac{\left(\lambda+\frac{1}{2}\right)_n (\lambda+\mu)_{n+k}}{\left(\lambda+\frac{1}{2}\right)_k (\lambda+\mu)_n} (4u)^n \left(\frac{x^2}{4u}\right)^k \tag{1.12}$$

$$P_{n,\lambda,\mu}(x,u) = \sum_{k=0}^{n} \binom{n}{k} \frac{\left(\lambda+\frac{1}{2}\right)_n \left(\mu+\frac{1}{2}\right)_n}{\left(\lambda+\frac{1}{2}\right)_k \left(\mu+\frac{1}{2}\right)_{n-k}} x^{2k}(x^2+4u)^{n-k} \tag{1.13}$$

Also by reversing the order of summation, (1.6) and (1.11) can be written as

$$P_{n,\lambda,\mu}(x,u) = \frac{(\lambda+\mu)_{2n}}{(\lambda+\mu)_n} x^{2n} {}_2F_1\left[\begin{matrix} -n, \frac{1}{2}-\lambda-n; \\ 1-\lambda-\mu-2n; \end{matrix} -\frac{4u}{x^2}\right] \tag{1.14}$$

$$P_{n,\lambda,\mu}(x,u) = \left(\mu + \frac{1}{2}\right)_n x^{2n} {}_2F_1\left[\begin{matrix} -n, \frac{1}{2}-\lambda-n; \\ \mu+\frac{1}{2}; \end{matrix} \frac{x^2+4u}{x^2}\right] \tag{1.15}$$

In 1989, Fitouchi [5] generalized the theory of heat polynomials introduced by Rosenbloom and Widder for a more general class of singular differential operator $0,\infty$. The heat polynomials associated with the Bessel operator and studied by Haimo appear as a particular case in this paper.

In 2001, Hile and Stanoyevitch [6], demonstrated polynomials solutions analogous to the heat polynomials for higher order linear homogenous evolution equations with coefficient depending on the time variable. Further parallels with the heat polynomials were established when the equation is parabolic with constant coefficients and only highest order terms.

Recently, in 2004, Hile and Stanoyevitch [7], generalized the heat polynomials for the heat equation to more general partial differential equations, of higher order with respect to both time variable and space variables. Whereas the heat equation requires only one family of polynomials, for an equation of lth order with respect to time they introduce l families of polynomials.These families correspond to the l initial conditions specified by the Cauchy problem.

Some of the definition and notations used in this paper are as follows (see [8]).

The Binomial coefficient is expressed as

$$\binom{\lambda}{n} = \frac{\lambda(\lambda-1)\cdots(\lambda-n+1)}{n!} = \frac{(-1)^n(-\lambda)_n}{n!}, \tag{1.16}$$

$$(1-z)^{-a} = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{n!}. \tag{1.17}$$

Appell's four functions of two variables are given by (see [9]).

$$F_1[a,b,b';c;x,y] = \sum_{n,k=0}^{\infty} \frac{(a)_{n+k}(b)_n(b')_k}{n!\ k!\ (c)_{n+k}} x^n y^k \tag{1.18}$$

$$F_2[a,b,b';c,c';x,y] = \sum_{n,k=0}^{\infty} \frac{(a)_{n+k}(b)_n(b')_k}{n!\ k!\ (c)_n(c')_k} x^n y^k \tag{1.19}$$

$$F_3[a,a',b,b';c;x,y] = \sum_{n,k=0}^{\infty} \frac{(a)_n(a')_k(b)_n(b')_k}{n!\ k!\ (c)_{n+k}} x^n y^k \tag{1.20}$$

$$F_4[a,b;c,c';x,y] = \sum_{n,k=0}^{\infty} \frac{(a)_{n+k}(b)_{n+k}}{n!\ k!\ (c)_n(c')_k} x^n y^k. \tag{1.21}$$

Further, Kampé de Fériet's type general double hypergeometric series (cf. Srivastava and Karlsson [10]) is defined as

$$F^{p:q;k}_{l:m;n}\left[\begin{array}{l} (a_p):(b_q);(c_k); \\ (\alpha_l):(\beta_m);(\gamma_n) \end{array} x,y\right] = \sum_{r,s=0}^{\infty} \frac{\prod\limits_{j=1}^{p}(a_j)_{r+s}\ \prod\limits_{j=1}^{q}(b_j)_r\ \prod\limits_{j=1}^{k}(c_j)_s}{\prod\limits_{j=1}^{l}(\alpha_j)_{r+s}\ \prod\limits_{j=1}^{m}(\beta_j)_r\ \prod\limits_{j=1}^{n}(\gamma_j)_s}. \tag{1.22}$$

2 Generating Functions

The polynomial $P_{n,\lambda,\mu}(x,u)$ admit the following generating relation.

$$\sum_{n=0}^{\infty} \frac{(\lambda+\mu)_n P_{n,\lambda,\mu}(x,u)}{\left(\lambda+\frac{1}{2}\right)_n\ n!} t^n = (1-4ut)^{-\lambda-\mu}\,{}_2F_1\left[\begin{array}{c} \frac{\lambda+\mu}{2}, \frac{\lambda+\mu+1}{2}; \\ \lambda+\frac{1}{2}; \end{array} \frac{4x^2t}{(1-4ut)^2}\right]. \tag{2.1}$$

Another generating function exist by using (1.11), which is similar of Bateman's generating function of Jacobi polynomials (see, Rainville [3, pp. 256]) and is as given below:

$$\sum_{n=0}^{\infty} \frac{P_{n,\lambda,\mu}(x,u)}{\left(\lambda+\frac{1}{2}\right)_n\left(\mu+\frac{1}{2}\right)_n n!} t^n = {}_0F_1\left[\begin{array}{c} \text{———}; \\ \lambda+\frac{1}{2}; \end{array} x^2t\right]{}_0F_1\left[\begin{array}{c} \text{———}; \\ \mu+\frac{1}{2}; \end{array} (x^2+4u)t\right]. \tag{2.2}$$

Also,

$$\sum_{n=0}^{\infty} \frac{(\gamma)_n P_{n,\lambda,\mu}(x,u)}{\left(\lambda+\frac{1}{2}\right)_n\left(\mu+\frac{1}{2}\right)_n n!} t^n = \psi_2\left[\gamma;\lambda+\frac{1}{2},\mu+\frac{1}{2};x^2t,(x^2+4u)t\right] \tag{2.3}$$

where ψ_2 is a (Humbert's) confluent hypergeometric function of two variables defined by [9].

$$\psi_2\left[\alpha;\gamma,\gamma';x,y\right]=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}}{(\gamma)_n(\gamma')_n}\frac{x^m}{m!}\frac{y^n}{n!}. \tag{2.4}$$

With the aid of (1.12), we find that

$$\sum_{n=0}^{\infty}\frac{(\gamma)_n(\delta)_n P_{n,\lambda,\mu}(x,u)}{\left(\lambda+\frac{1}{2}\right)_n\left(\mu+\frac{1}{2}\right)_n n!}t^n=F_4\left[\gamma,\delta,;\lambda+\frac{1}{2},\mu+\frac{1}{2};x^2t,(x^2+4u)t\right] \tag{2.5}$$

where F_4 denotes the fourth type of Appell's hypergeometric function defined by eq. (1.21).

By using, $\delta=\lambda+\mu-\gamma$ in (2.5) and in the conjunction with the theorem (Bailey's [11, pp. 81]; Rainville [3, pp. 269])

$$F_4\left[a,b;c,1-c+a+b;-\frac{u}{(1-u)(1-v)},-\frac{v}{(1-u)(1-v)}\right]$$
$$={}_2F_1\left[\begin{array}{ll}a,\ b; & \\ & -\dfrac{u}{1-u}\\ c\ ; & \end{array}\right]{}_2F_1\left[\begin{array}{ll}a\ ,\ b\ ; & \\ & -\dfrac{v}{1-v}\\ 1-c+a+b; & \end{array}\right] \tag{2.6}$$

a new generating relation is obtained as follows:

$$\sum_{n=0}^{\infty}\frac{(\gamma)_n(\lambda+\mu-\gamma)_n P_{n,\lambda,\mu}(x,u)}{\left(\lambda+\frac{1}{2}\right)_n\left(\mu+\frac{1}{2}\right)_n n!}t^n$$
$$={}_2F_1\left[\begin{array}{ll}\gamma,\lambda+\mu-\gamma; & \\ & \dfrac{1}{2}(1-4ut-\rho)\\ \lambda+\frac{1}{2}\quad ; & \end{array}\right]{}_2F_1\left[\begin{array}{ll}\gamma,\lambda+\mu-\gamma; & \\ & \dfrac{1}{2}(1+4ut-\rho)\\ \mu+\frac{1}{2}\quad ; & \end{array}\right]. \tag{2.7}$$

Put, $\gamma=\lambda+\frac{1}{2}$ and $\delta=\mu+\frac{1}{2}$ in (2.5) and appeal to the theorem (Rainville [3, pp. 268]).

$$F_4\left[a,b;b,a;-\frac{u}{(1-u)(1-v)},-\frac{v}{(1-u)(1-v)}\right]=(1-uv)^{-1}(1-u)^a(1-v)^b. \tag{2.8}$$

We thus obtain the generating relation,

$$\sum_{n=0}^{\infty}\frac{P_{n,\lambda,\mu}(x,u)}{n!}t^n=2^{\lambda+\mu-1}\rho^{-1}(1+4ut+\rho)^{-(\lambda-\frac{1}{2})}(1-4ut+\rho)^{-(\mu-\frac{1}{2})} \tag{2.9}$$

where, in (2.7) and (2.9), $\rho=[(1-4ut)^2-4x^2t]^{\frac{1}{2}}$.

If we rewrite (1.11) as

$$P_{n,\lambda,\mu-n}(x,u)=\frac{\Gamma(\lambda+n+\frac{1}{2})}{\Gamma(\lambda+\frac{1}{2})}(x^2+4u)^n{}_2F_1\left[\begin{array}{ll}-n,-(\mu-\frac{1}{2}); & \dfrac{x^2}{x^2+4u}\\ \lambda+\frac{1}{2}\quad ; & \end{array}\right] \tag{2.10}$$

and the Gaussian hypergeometric transformation (see [9, pp. 3])

$$ {}_2F_1[a,b;c;z] = (1-z)^{c-a-b}\,{}_2F_1[c-a,c-b;c;z], \qquad |z|<1. \tag{2.11}$$

We thus arrive to an elegant form of generating relation from (2.10) and (2.11) as

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n(\rho)_n P_{n,\lambda,\mu-n}(x,u)}{(\beta)_n\left(\lambda+\frac{1}{2}\right)_n n!} t^n$$

$$= (1-4ut)^{-\alpha} F^{1:1;2}_{1:-;1}\left[\begin{array}{l} \alpha : \beta-\rho; \rho, \lambda+\mu; \\ \beta : \;\; - \;\; ; \;\; \lambda+\frac{1}{2} \;\; ; \end{array} \frac{4ut}{4ut-1}, -\frac{x^2 t}{4ut-1}\right] \tag{2.12}$$

and

$$\sum_{n=0}^{\infty} \frac{(c-b)_n P_{n,\lambda,\mu-n}(x,u)}{(c)_n\; n!} t^n$$

$$= \left(\frac{x^2+4u}{4u}\right)^{-\lambda-\mu} F_2\left[\lambda+\frac{1}{2}, \lambda+\mu, c-b\;;\lambda+\frac{1}{2}, c\;; \frac{x^2}{x^2+4u}, 4ut\right]. \tag{2.13}$$

In view of the hypergeometric transformation ([9, pp. 30]),

$$F_2\left[\alpha,\beta,\beta'\;;\alpha,\gamma\;;x,y\right]$$

$$= (1-x)^{-\beta} F_1\left[\beta',\beta,\alpha-\beta\;;\gamma\;;\frac{y}{1-x},y\right]$$

$$= (1-x)^{-\beta}(1-y)^{-\beta'} F_1\left[\beta',\gamma-\alpha,\beta\;;\gamma\;;\frac{y}{y-1},\frac{xy}{(1-x)(1-y)}\right]. \tag{2.14}$$

The generating relation (2.13) is equivalent to the form

$$\sum_{n=0}^{\infty} \frac{(c-b)_n P_{n,\lambda,\mu-n}(x,u)}{(c)_n\; n!} t^n$$

$$= (1-4ut)^{b-c} F_1\left[c-b, c-\lambda-\frac{1}{2}, \lambda+\mu\;; c\;; \frac{4ut}{4ut-1}, -\frac{x^2 t}{4ut-1}\right] \tag{2.15}$$

where F_1 is the Appell's function of first kind defined by eq.(1.18).

The polynomial $P_{n+m,\lambda,\mu-n}(x,u)$ can be expressed, by applying (2.11) to the second member of (2.10) as

$$P_{n+m,\lambda,\mu-n}(x,u) = \frac{\Gamma(\lambda+n+m+\frac{1}{2})}{\Gamma(\lambda+\frac{1}{2})}(4u)^{n+m}\left(\frac{x^2+4u}{4u}\right)^{-\lambda-\mu-m}$$

$$\times\, {}_2F_1\left[\begin{array}{c} \lambda+n+m+\frac{1}{2}, \lambda+\mu+m; \\ \lambda+\frac{1}{2} \qquad\qquad ; \end{array} \frac{x^2}{x^2+4u}\right] \tag{2.16}$$

for every integer $m \geq 0$.

Next we rewrite (1.19) in the form

$$F_2\left[a, b, b'\ ; c, c'\ ; x, y\right] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!\ (c)_n}\ {}_2F_1\left[a+n, b'\ ; c';\ y\right] x^n \tag{2.17}$$

which in conjunction with (2.16) would leads at once to a generalization of (2.13) given by

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n P_{n+m,\lambda,\mu-n}(x,u)}{(\beta)_n\ n!} t^n = \frac{\Gamma(\lambda+m+\frac{1}{2})}{\Gamma(\lambda+\frac{1}{2})}(4u)^m \left(\frac{x^2+4u}{4u}\right)^{-\lambda-\mu-m} \times F_2\left[\lambda+m+\frac{1}{2}, \lambda+\mu+m, \alpha\ ; \lambda+\frac{1}{2}, \beta\ ; \frac{x^2}{x^2+4u}, 4ut\right]. \tag{2.18}$$

Evidently (2.13) would follow from (2.18) in the special case $m = 0$.

By using the hypergeometric transformation (Erdélyi et al. [12, vol. I, pp. 240, eq. (6)]).

$$F_2[a, b, b'\ ; c, c'\ ; x, y] = (1-x)^{-a} F_2\left[a, c-b, b'\ ; c, c'\ ; \frac{x}{x-1}, \frac{y}{1-x}\right] \tag{2.19}$$

the generating relation (2.18) is equivalent to the form

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n P_{n+m,\lambda,\mu-n}(x,u)}{(\beta)_n\ n!} t^n = \frac{\Gamma(\lambda+m+\frac{1}{2})}{\Gamma(\lambda+\frac{1}{2})}(4u)^m \left(\frac{x^2+4u}{4u}\right)^{-(\mu-\frac{1}{2})} \times F_2\left[\lambda+m+\frac{1}{2}, -\left(\mu+m-\frac{1}{2}\right), \alpha\ ; \lambda+\frac{1}{2}, \beta\ ; -\frac{x^2}{4u}, (x^2+4u)t\right]. \tag{2.20}$$

The polynomial $P_{n,\lambda-n,\mu-n}(x,u)$ given by the form

$$P_{n,\lambda-n,\mu-n}(x,u) = \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda-n+\frac{1}{2})}(x^2+4u)^n {}_2F_1\left[\begin{array}{ll} -n\ , -(\mu-\frac{1}{2})\ ; & \dfrac{x^2}{x^2+4u} \\ \lambda-n+\frac{1}{2}\quad ; & \end{array}\right] \tag{2.21}$$

admits the following generating relations

$$\sum_{n=0}^{\infty} \frac{P_{n,\lambda-n,\mu-n}(x,u)}{n!} t^n = \left[1+(x^2+4u)t\right]^{-(\lambda-\frac{1}{2})} \left[1+x^2 t\right]^{-(\mu-\frac{1}{2})}, \tag{2.22}$$

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n P_{n,\lambda-n,\mu-n}(x,u)}{(\beta)_n \ n!} t^n$$
$$= F_1\left[\alpha, -\left(\lambda - \frac{1}{2}\right), -\left(\mu - \frac{1}{2}\right);\ \beta\ ; -(x^2+4u)t, -x^2t\right]. \tag{2.23}$$

For $\alpha = \beta$, the generating relation (2.23) reduces immediately to (2.22).

Another interesting special case of (2.23) would occurs, when we set $\beta = -\lambda - \mu + 1$ and appeal to the hypergeometric reduction formula (see, Erdélyi et al. [12, vol. I, pp. 238, eq. (1)])

$$F_1[a,b,b'\ ; b+b'\ ; x,y] = (1-y)^{-a} {}_2F_1\left[\begin{array}{ll} a\ ,\ b\ ; & \dfrac{x-y}{1-y} \\ b+b'\ ; & \end{array}\right]. \tag{2.24}$$

We thus obtain the generating function

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n P_{n,\lambda-n,\mu-n}(x,u)}{(-\lambda-\mu+1)_n \ n!} t^n = (1+x^2t)^{-\alpha} {}_2F_1\left[\begin{array}{ll} \alpha, -(\lambda-\frac{1}{2})\ ; & \dfrac{-4ut}{1+x^2t} \\ -\lambda-\mu+1\ ; & \end{array}\right] \tag{2.25}$$

or equivalently,

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n P_{n,\lambda-n,\mu-n}(x,u)}{(-\lambda-\mu+1)_n \ n!} t^n$$
$$= (1+(x^2+4u)t)^{-\alpha} {}_2F_1\left[\begin{array}{ll} \alpha, -(\mu-\frac{1}{2})\ ; & \dfrac{4ut}{1+(x^2+4u)t} \\ -\lambda-\mu+1\ ; & \end{array}\right] \tag{2.26}$$

where we have used Euler transformation defined by (1.10).

In view of the confluence principle exhibited by (see, Srivastava and Manocha [8]),

$$\phi_2[\beta,\beta'\ ;\gamma\ ;x,y] = \lim_{|\alpha|\longrightarrow\infty} F_1\left[\alpha,\beta,\beta'\ ;\gamma\ ; \frac{x}{\alpha}, \frac{y}{\alpha}\right] \tag{2.27}$$

and

$${}_1F_1[a\ ;c\ ;\ z] = \lim_{|b|\longrightarrow\infty} {}_2F_1\left[a,b\ ;c\ ;\frac{z}{b}\right]. \tag{2.28}$$

Some confluent form of generating function (2.23), (2.25) and (2.26) can be represented in the elegant form by replacing t in each of these results by $\frac{t}{\alpha}$ and letting $|\alpha| \longrightarrow \infty$.

$$\sum_{n=0}^{\infty} \frac{P_{n,\lambda-n,\mu-n}(x,u)}{(\beta)_n \ n!} t^n$$
$$= \phi_2\left[-\left(\lambda-\frac{1}{2}\right), -\left(\mu-\frac{1}{2}\right); \alpha; -(x^2+4u)t, -x^2t\right] \tag{2.29}$$

$$\sum_{n=0}^{\infty} \frac{P_{n,\lambda-n,\mu-n}(x,u)}{(-\lambda-\mu+1)_n\ n!} t^n = \exp(-x^2 t) {}_1F_1 \left[\begin{array}{c} -\left(\lambda-\frac{1}{2}\right)\ ; \\ -\lambda-\mu+1\ ; \end{array} -4ut \right]. \tag{2.30}$$

Similarly,

$$\sum_{n=0}^{\infty} \frac{P_{n,\lambda-n,\mu-n}(x,u)}{(-\lambda-\mu+1)_n\ n!} t^n = \exp(-(x^2+4ut)) {}_1F_1 \left[\begin{array}{c} -\left(\mu-\frac{1}{2}\right)\ ; \\ -\lambda-\mu+1\ ; \end{array} 4ut \right] \tag{2.31}$$

where ϕ_2 is a confluent hypergeometric function of two variables (see, Srivastava and Manocha [8, pp. 58]).

Yet, another generating function is obtained by using (2.21) which in conjunction with (2.11) would readily give us

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n (\rho)_n P_{n,\lambda-n,\mu-n}(x,u)}{(\beta)_n (-\lambda-\mu+1)_n\ n!} t^n$$

$$= (1+x^2 t)^{-\alpha} F^{1:1;2}_{1:-;2} \left[\begin{array}{l} \alpha : \beta-\rho;\ \rho\ ,\ -\lambda+\frac{1}{2}\ ; \\ \\ \beta :\ -\ ;\ -\lambda-\mu+1\ ; \end{array} -\frac{4ut}{1+x^2 t}, \frac{x^2 t}{1+x^2 t} \right]. \tag{2.32}$$

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