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Application of $H(\cdot,\cdot)$-Cocoercive Operators for Solving a System of Variational Inclusions[1]

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Abstract : In this paper, we apply $H(\cdot,\cdot)$-cocoercive operators for solving a system of variational inclusions. By using the resolvent operator technique associated with $H(\cdot,\cdot)$-cocoercive operators, we define an iterative algorithm for solving a system of variational inclusions. Convergence criteria is also discussed. Some examples are given in support the definition of $H(\cdot,\cdot)$-cocoercive operators.

Keywords : Algorithm; System; Variational inclusion; Operator; Convergence; Cocoercive.
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1 Introduction

Various noble and efficient methods have been studied to find solutions of variational inclusions. The method based on resolvent operator technique is a generalization of projection method. The resolvent operator method is an important and useful tool to study approximation solvability of nonlinear variational inequalities and inclusions, which are providing mathematical models to solve problems arising in optimization and control, economics and engineering sciences, etc.. By using this method, many variational inclusions and systems of variational inclusions have

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been studied by Lan [1], Ding and Fang [2], Peng and Zhu [3], Zeng [4] and Ding and Wang [5].

Fang and Huang [6] introduced H-monotone mappings for solving a system of variational inclusions involving a combination of H-monotone and strongly monotone mappings based on the resolvent operator technique. The notion of H-monotonicity has revitalized the theory of maximal monotone mappings in many directions. Verma [7] introduced A-monotone mappings with applications to solve systems of nonlinear variational inclusions. Zou and Huang [8, 9] introduced and studied $H(\cdot,\cdot)$-accretive operators and applied them to solve variational inclusions and systems of variational inclusions.

Very recently, Ahmad et al. [10] introduced and studied $H(\cdot,\cdot)$-cocoercive operators and applied them for solving set-valued variational inclusions in Hilbert spaces. $H(\cdot,\cdot)$-cocoercive operators provide a unified frame work for existing H-monotone, $H(\cdot,\cdot)$-monotone operators in Hilbert spaces and H-accretive and $H(\cdot,\cdot)$-accretive operators in Banach spaces.

Inspired and motivated by the excellent work going in the area, in this paper, we apply $H(\cdot,\cdot)$-cocoercive operators for solving a system of variational inclusions. By using the resolvent operator technique associated with $H(\cdot,\cdot)$-cocoercive operators due to Ahmad et al.[10] , we prove the existence of solutions of system considered. No doubt, the results of this paper are new and improve many known results. Some examples are given.

2 Preliminaries

Throughout the paper, we suppose that X is a real Hilbert space endowed with a norm $\|\cdot\|$ and an inner product $\langle\cdot,\cdot\rangle$. We recall some definitions needed in the sequel.

Definition 2.1. A mapping $T : X \to X$ is said to be

(i) *Lipschitz continuous*, if there exists a constant $\lambda_T > 0$ such that

$$\|T(x) - T(y)\| \leq \lambda_T \|x - y\|, \ \ \forall\, x, y \in X;$$

(ii) *monotone,* if

$$\langle T(x) - T(y),\ x - y\rangle \geq 0, \ \ \forall\, x, y \in X;$$

(iii) *strongly monotone*, if there exists a constant $\delta_T > 0$ such that

$$\langle T(x) - T(y),\ x - y\rangle \geq \delta_T \|x - y\|^2, \ \ \forall\, x, y \in X;$$

(iv) *α-expansive*, if there exists a constant $\alpha > 0$ such that

$$\|T(x) - T(y)\| \geq \alpha \|x - y\|, \ \ \forall\, x, y \in X,$$

if $\alpha = 1$, then it is expansive.

Definition 2.2. A mapping $S : X \to X$ is said to be *cocoercive*, if there exists a constant $\mu_1 > 0$ such that

$$\langle Sx - Sy,\ x - y\rangle \geq \mu_1 \|Sx - Sy\|^2, \quad \forall\ x, y \in X.$$

If $\mu_1 = 0$, then S is a monotone mapping.

Definition 2.3. A multi-valued mapping $M : X \to 2^X$ is said to be *cocoercive*, if there exists a constant $\mu_2 > 0$ such that

$$\langle u - v,\ x - y\rangle \geq \mu_2 \|u - v\|^2, \quad \forall\ x, y \in X,\ u \in M(x), v \in M(y).$$

Definition 2.4. A mapping $G : X \to X$ is said to be *relaxed cocoercive*, if there exists a constant $\gamma_1 > 0$ such that

$$\langle Gx - Gy,\ x - y\rangle \geq (-\gamma_1) \|Gx - Gy\|^2, \quad \forall\ x, y \in X.$$

Definition 2.5. Let $H : X \times X \to X$ and $A, B : X \to X$ and $F : X \times X \to X$ be the mappings.

(i) $H(A, \cdot)$ is said to be *cocoercive with respect to* A, if there exists a constant $\mu > 0$ such that

$$\langle H(Ax, u) - H(Ay, u),\ x - y\rangle \geq \mu \|Ax - Ay\|^2, \quad \forall\ x, y \in X;$$

(ii) $H(\cdot, B)$ is said to be *relaxed cocoercive with respect to* B, if there exists a constant $\gamma > 0$ such that

$$\langle H(u, Bx) - H(u, By),\ x - y\rangle \geq (-\gamma) \|Bx - By\|^2, \quad \forall\ x, y \in X;$$

(iii) $H(A, \cdot)$ is said to be r_1-*Lipschitz continuous with respect to* A, if there exists a constant $r_1 > 0$ such that

$$\|H(Ax, \cdot) - H(Ay, \cdot)\| \leq r_1 \|x - y\|, \quad \forall\ x, y \in X;$$

(iv) $H(\cdot, B)$ is said to be r_2-*Lipschitz continuous with respect to* B, if there exists a constant $r_2 > 0$ such that

$$\|H(\cdot, Bx) - H(\cdot, By)\| \leq r_2 \|x - y\|, \quad \forall\ x, y \in X.$$

(v) $F(x, \cdot)$ is said to be *strongly monotone with respect to* $H(A, B)$ *in the first argument*, if there exists a constant $m_1 > 0$ such that

$$\langle H(Ax, Bx) - H(Ay, By),\ F(x, z) - F(y, z)\rangle \geq m_1 \|x - y\|^2, \quad \forall x, y \in X.$$

Similarly we can define the strong monotonicity of F with respect to $H(A, B)$ in the second argument.

Definition 2.6 ([10])**.** Let $A, B : X \to X$, $H : X \times X \to X$ be three single-valued mappings. Let $M : X \to 2^X$ be a multi-valued mapping. M is said to be *$H(\cdot,\cdot)$-cocoercive with respect to mappings A and B*, if M is cocoercive and $(H(A,B) + \lambda M)(X) = X$, for any $\lambda > 0$.

Example 2.7. *Let $X = \mathbb{R}^2$ with usual inner product. Let $A, B : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by*

$$Ax = (2x_1 - 2x_2, -2x_1 + 4x_2), \quad By = (-y_1 + y_2, \ -y_2), \quad \forall\ x, y \in \mathbb{R}^2.$$

Suppose that $H(A,B) : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$H(Ax, By) = Ax + By, \ \forall\ x, y \in \mathbb{R}^2.$$

Then it is easy to check that $H(A,B)$ is $\frac{1}{6}$-cocoercive with respect to A and $\frac{1}{2}$-relaxed cocoercive with respect to B.

Let $M = I$, where I is the identity mapping. Then M is $H(\cdot,\cdot)$-cocoercive mapping with respect to A and B.

Example 2.8. *Let $X = \mathbb{S}^2$, where $\mathbb{S}^2$ denotes the space of all 2×2 real symmetric matrices. Let $H(Ax, By) = x^2 - y$, for all $x, y \in \mathbb{S}^2$ and $M = I$. Then for $\lambda = 1$, we have*

$$(H(A,B) + M)(x) = x^2 - x + x = x^2,$$

but

$$\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \notin (H(A,B) + M)(\mathbb{S}^2),$$

because $\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$ *is not the square of any 2×2 real symmetric matrix. Hence M is not $H(\cdot,\cdot)$-cocoercive with respect to A and B.*

Theorem 2.9 ([10])**.** *Let $H(A,B)$ be a μ-cocoercive with respect to A and γ-relaxed cocoercive with respect to B, A is α-expansive and B is β-Lipschitz continuous, $\mu > \gamma$ and $\alpha > \beta$. Let M be an $H(\cdot,\cdot)$-cocoercive operator with respect to A and B. Then the operator $(H(A,B) + \lambda M)^{-1}$ is single-valued.*

Definition 2.10 ([10])**.** Let $H(A,B)$ be μ-cocoercive with respect to A and γ-relaxed cocoercive with respect to B, A is α-expansive and B is β-Lipschitz continuous, and $\mu > \gamma, \alpha > \beta$. Let M be an $H(\cdot,\cdot)$-cocoercive operator with respect to A and B. The resolvent operator $R^{H(\cdot,\cdot)}_{\lambda,M} : X \to X$ is defined by

$$R^{H(\cdot,\cdot)}_{\lambda,M}(u) = (H(A,B) + \lambda M)^{-1}(u), \quad \forall\ u \in X.$$

Theorem 2.11 ([10])**.** *Let $H(A,B)$ be μ-cocoercive with respect to A, γ-relaxed cocoercive with respect to B, A is α-expansive and B is β-Lipschitz continuous and $\mu > \gamma, \alpha > \beta$. Let M be an $H(\cdot,\cdot)$-cocoercive operator with respect to A and*

B. Then the resolvent operator $R_{\lambda,M}^{H(\cdot,\cdot)} : X \to X$ *is* $\frac{1}{\mu\alpha^2-\gamma\beta^2}$*-Lipschitz continuous, that is*

$$\|R_{\lambda,M}^{H(\cdot,\cdot)}(u) - R_{\lambda,M}^{H(\cdot,\cdot)}(v)\| \leq \frac{1}{\mu\alpha^2 - \gamma\beta^2}\|u - v\|, \quad \forall\ u, v \in X.$$

Proof. Let u and v be any given points in X. It follows that

$$R_{\lambda,M}^{H(\cdot,\cdot)}(u) = (H(A,B) + \lambda M)^{-1}(u),$$

and

$$R_{\lambda,M}^{H(\cdot,\cdot)}(v) = (H(A,B) + \lambda M)^{-1}(v).$$

This implies that

$$\frac{1}{\lambda}(u - H(A(R_{\lambda,M}^{H(\cdot,\cdot)}(u)),\ B(R_{\lambda,M}^{H(\cdot,\cdot)}(u)))) \in M(R_{\lambda,M}^{H(\cdot,\cdot)}(u)),$$

and

$$\frac{1}{\lambda}(v - H(A(R_{\lambda,M}^{H(\cdot,\cdot)}(v)),\ B(R_{\lambda,M}^{H(\cdot,\cdot)}(v)))) \in M(R_{\lambda,M}^{H(\cdot,\cdot)}(v)).$$

For the sake of clarity, we take

$$Pu = R_{\lambda,M}^{H(\cdot,\cdot)}(u), \quad Pv = R_{\lambda,M}^{H(\cdot,\cdot)}(v).$$

Since M is cocoercive (hence monotone), we have

$$\frac{1}{\lambda}\langle u - H(A(Pu), B(Pu)) - (v - H(A(Pv), B(Pv))),\ Pu - Pv\rangle \geq 0$$
$$\frac{1}{\lambda}\langle u - v - H(A(Pu), B(Pv)) + H(A(Pv), B(Pv)),\ Pu - Pv\rangle \geq 0,$$

which implies that

$$\begin{aligned}
\langle u - v,\ Pu - Pv\rangle &\geq \langle H(A(Pu), B(Pu)) - H(A(Pv), B(Pv)),\ Pu - Pv\rangle,\\
\|u - v\|\|Pu - Pv\| &\geq \langle u - v,\ Pu - Pv\rangle\\
&\geq \langle H(A(Pu), B(Pu)) - H(A(Pv), B(Pv)),\ Pu - Pv\rangle\\
&= \langle H(A(Pu), (B(Pu)) - H(A(Pv), B(Pu))\\
&\quad + H(A(Pv), B(Pu)) - H(A(Pv), B(Pv)),\ Pu - Pv\rangle\\
&= \langle H(A(Pu), (B(Pu)) - H(A(Pv), B(Pu)),\ Pu - Pv\rangle\\
&\quad + \langle H(A(Pv), B(Pu)) - H(A(Pv), B(Pv)),\ Pu - Pv\rangle\\
&\geq \mu\|A(Pu) - A(Pv)\|^2 - \gamma\|B(Pu) - B(Pv)\|^2\\
&\geq \mu\alpha^2\|Pu - Pv\|^2 - \gamma\beta^2\|Pu - Pv\|^2
\end{aligned}$$

and so

$$\|u - v\|\|Pu - Pv\| \geq (\mu\alpha^2 - \gamma\beta^2)\|Pu - Pv\|^2,$$

thus $\|Pu - Pv\| \leq \frac{1}{\mu\alpha^2-\gamma\beta^2}\|u - v\|$, i.e.

$$\|R_{\lambda,M}^{H(\cdot,\cdot)}(u) - R_{\lambda,M}^{H(\cdot,\cdot)}(v)\| \leq \frac{1}{\mu\alpha^2 - \gamma\beta^2}\|u - v\|, \quad \forall\ u, v \in X.$$

This completes the proof. □

3 System of Variational Inclusions and Iterative Algorithm

In this part, we formulate a system of variational inclusions in Hilbert spaces involving $H(\cdot,\cdot)$-cocoercive operators as follows:

Let X_1 and X_2 be two real Hilbert spaces and let $F : X_1 \times X_2 \to X_1$, $G : X_1 \times X_2 \to X_2$, $H_1 : X_1 \times X_1 \to X_1$, $H_2 : X_2 \times X_2 \to X_2$, $A_1, B_1 : X_1 \to X_1$, $A_2, B_2 : X_2 \to X_2$ be the single-valued mappings. Let $M : X_1 \to 2^{X_1}$ be a multi-valued, $H_1(A_1, B_1)$-cocoercive mapping and $N : X_2 \to 2^{X_2}$ be a multi-valued, $H_2(A_2, B_2)$-cocoercive mapping. Find $(a, b) \in X_1 \times X_2$ such that

$$0 \in F(a, b) + M(a), \;\; 0 \in G(a, b) + N(b). \tag{3.1}$$

Some examples of problem (3.1) are as follows.

(i) If $M : X_1 \to 2^{X_1}$ is (H_1, η)-monotone and $N : X_2 \to 2^{X_2}$ is (H_2, η)-monotone, then problem (3.1) includes the problem considered and studied by Fang et al. [11]. Find $(a, b) \in X_1 \times X_2$ such that

$$0 \in F(a, b) + M(a), \;\; 0 \in G(a, b) + N(b). \tag{3.2}$$

(ii) If X_1, X_2 are real Banach spaces and $M : X_1 \to 2^{X_1}$ is $H_1(A_1, B_1)$-accretive and $N : X_2 \to 2^{X_2}$ is $H_2(A_2, B_2)$-accretive, then problem (3.1) coincides with the problem introduced and studied by Zou and Huang [9]. Find $(a, b) \in X_1 \times X_2$ such that

$$0 \in F(a, b) + M(a), \;\; 0 \in G(a, b) + N(b). \tag{3.3}$$

For suitable choice of operators involved in the formulation of problem (3.1), one can obtain many systems of variational inequalities and variational inclusions exist in the literature.

Lemma 3.1. *Let X_1 and X_2 be two real Hilbert spaces. Let $F : X_1 \times X_2 \to X_1$, $G : X_1 \times X_2 \to X_2$, $A_1, B_1 : X_1 \to X_1$, $A_2, B_2 : X_2 \to X_2$ be single-valued mappings. Let $H_1 : X_1 \times X_1 \to X_1$ be a single-valued mapping such that $H_1(A_1, B_1)$ is μ_1-cocoercive with respect to A_1 and γ_1-relaxed cocoercive with respect to B_1, A_1 is α_1-expansive and B_1 is β_1-Lipschitz continuous, $\alpha_1 > \beta_1$ and $\mu_1 > \gamma_1$. Let $H_2 : X_2 \times X_2 \to X_2$ be also a single-valued mapping such that $H_2(A_2, B_2)$ is μ_2-cocoercive with respect to A_2 and γ_2-relaxed cocoercive with respect to B_2, A_2 is α_2-expansive and B_2 is β_2-Lipschitz continuous, $\alpha_2 > \beta_2$ and $\mu_2 > \gamma_2$. Let $M : X_1 \to 2^{X_1}$ is $H_1(\cdot,\cdot)$-cocoercive, multi-valued mapping and $N : X_2 \to 2^{X_2}$ is $H_2(\cdot,\cdot)$-cocoercive, multi-valued mapping. Then for any $(a, b) \in X_1 \times X_2, (a, b)$ is a solution of problem (3.1) if and only if (a, b) satisfies*

$$a = R^{H_1(\cdot,\cdot)}_{\lambda,M}\big[H_1(A_1(a), B_1(a)) - \lambda F(a, b)\big],$$

$$b = R^{H_2(\cdot,\cdot)}_{\rho,N}\big[H_2(A_2(b), B_2(b)) - \rho G(a, b)\big],$$

where $\lambda > 0$ and $\rho > 0$ are two constants.

Proof. The conclusion can be obtained directly from the definition of resolvent operator. □

Based on Lemma 3.1, we now define an iterative algorithm for approximating a solution of problem (3.1).

Algorithm 3.1. Let $X_1, X_2, A_1, A_2, B_1, B_2, H_1, H_2, M, N, F$ and G are same as Lemma 3.1. For any given initial $(a_0, b_0) \in X_1 \times X_2$, we define the following iterative scheme:

$$a_{n+1} = R^{H_1(\cdot,\cdot)}_{\lambda,M}\big[H_1(A_1(a_n), B_1(a_n)) - \lambda F(a_n, b_n)\big],$$

$$b_{n+1} = R^{H_2(\cdot,\cdot)}_{\rho,N}\big[H_2(A_2(b_n), B_2(b_n)) - \rho G(a_n, b_n)\big],$$

for n (iteration number) $= 0, 1, 2, ...,$ where $\lambda > 0$ and $\rho > 0$ are two constants.

4 Existence and Convergence Result

In this section, we show the existence of solution of problem (3.1) and analyze the convergence of iterative algorithm.

Theorem 4.1. *Let X_1 and X_2 be two real Hilbert spaces. Let $A_1, B_1 : X_1 \to X_1$, $A_2, B_2 : X_2 \to X_2$ be single-valued mappings. Let $H_1 : X_1 \times X_1 \to X_1$ be a single-valued mapping such that $H_1(A_1, B_1)$ is μ_1-cocoercive with respect to A_1 and γ_1-relaxed cocoercive with respect to B_1, A_1 is α_1-expansive and B_1 is β_1-Lipschitz continuous, $\alpha_1 > \beta_1$ and $\mu_1 > \gamma_1$. Let $H_2 : X_2 \times X_2 \to X_2$ be also a single-valued mapping such that $H_2(A_2, B_2)$ is μ_2-cocoercive with respect to A_2 and γ_2-relaxed cocoercive with respect to B_2, A_2 is α_2-expansive and B_2 is β_2-Lipschitz continuous, $\alpha_2 > \beta_2$ and $\mu_2 > \gamma_2$. Let $M : X_1 \to 2^{X_1}$ is $H_1(\cdot,\cdot)$-cocoercive, multi-valued mapping and $N : X_2 \to 2^{X_2}$ is $H_2(\cdot,\cdot)$-cocoercive, multi-valued mapping. Assume that $H_1(A_1, B_1)$ is r_1-Lipschitz continuous with respect to A_1 and r_2-Lipschitz continuous with respect to B_1, $F : X_1 \times X_2 \to X_1$ is τ_1-Lipschitz continuous with respect to the first argument and τ_2-Lipschitz continuous with respect to the second argument, $H_2(A_2, B_2)$ is r_3-Lipschitz continuous with respect to A_2 and r_4-Lipschitz continuous with respect to B_2, $G : X_1 \times X_2 \to X_2$ is τ_1'-Lipschitz continuous with respect to first argument and τ_2'-Lipschitz continuous with respect to second argument. $F(x,\cdot)$ is m_1-strongly monotone with respect to $H_1(A_1, B_1)$ and $G(\cdot, y)$ is m_2-strongly monotone with respect to $H_2(A_2, B_2)$. If*

$$\begin{aligned} 0 &< \frac{\sqrt{(r_1+r_2)^2 - 2\lambda m_1 + \lambda^2\tau_1^2}}{\mu_1\alpha_1^2 - \gamma_1\beta_1^2} + \frac{\rho\tau_1'}{\mu_2\alpha_2^2 - \gamma_2\beta_2^2} \ < \ 1, \\ 0 &< \frac{\sqrt{(r_3+r_4)^2 - 2\rho m_2 + \rho^2{\tau_2'}^2}}{\mu_2\alpha_2^2 - \gamma_2\beta_2^2} + \frac{\lambda\tau_2}{\mu_1\alpha_1^2 - \gamma_1\beta_1^2} \ < \ 1. \end{aligned} \tag{4.1}$$

Then the problem (3.1) admits a solution $(a,b) \in X_1 \times X_2$ and the sequence $\{(a_n,b_n)\}$ generated by Algorithm 3.1 converges strongly to a solution (a,b) of problem (3.1).

Proof. From Algorithm 3.1 and Theorem 2.11, we have

$$\begin{aligned}
&\|a_{n+1}-a_n\| \\
&\quad = \|R^{H_1(\cdot,\cdot)}_{\lambda,M}[H_1(A_1(a_n),B_1(a_n)) - \lambda F(a_n,b_n)] \\
&\qquad - R^{H_1(\cdot,\cdot)}_{\lambda,M}[H_1(A_1(a_{n-1}),B_1(a_{n-1})) - \lambda F(a_{n-1},b_{n-1})]\| \\
&\quad \leq \frac{1}{\mu_1\alpha_1^2-\gamma_1\beta_1^2}\|H_1(A_1(a_n),B_1(a_n)) - \lambda F(a_n,b_n) \\
&\qquad - [H_1(A_1(a_{n-1}),B_1(a_{n-1})) - \lambda F(a_{n-1},b_{n-1})]\| \\
&\quad = \frac{1}{\mu_1\alpha_1^2-\gamma_1\beta_1^2}\|[H_1(A_1(a_n),B_1(a_n)) - H_1(A_1(a_{n-1}),B_1(a_{n-1}))] \\
&\qquad - \lambda[F(a_n,b_n) - F(a_{n-1},b_n) + F(a_{n-1},b_n) - F(a_{n-1},b_{n-1})]\| \\
&\quad \leq \frac{1}{\mu_1\alpha_1^2-\gamma_1\beta_1^2}\|[H_1(A_1(a_n),B_1(a_n)) - H_1(A_1(a_{n-1}),B_1(a_{n-1}))] \\
&\qquad - \lambda[F(a_n,b_n) - F(a_{n-1},b_n)]\| + \frac{\lambda}{\mu_1\alpha_1^2-\gamma_1\beta_1^2}\|F(a_{n-1},b_n) - F(a_{n-1},b_{n-1})\|.
\end{aligned} \tag{4.2}$$

Further,

$$\begin{aligned}
&\|[H_1(A_1(a_n),B_1(a_n)) - H_1(A_1(a_{n-1}),B_1(a_{n-1}))] - \lambda[F(a_n,b_n) - F(a_{n-1},b_n)]\|^2 \\
&\leq \|H_1(A_1(a_n),B_1(a_n)) - H_1(A_1(a_{n-1}),B_1(a_{n-1}))\|^2 \\
&\quad - 2\lambda\langle H_1(A_1(a_n),B_1(a_n)) - H_1(A_1(a_{n-1}),B_1(a_{n-1})), F(a_n,b_n) - F(a_{n-1},b_n)\rangle \\
&\quad + \lambda^2\|F(a_n,b_n) - F(a_{n-1},b_n)\|^2.
\end{aligned} \tag{4.3}$$

Since $H_1(A_1,B_1)$ is r_1-Lipschitz continuous with respect to A_1 and r_2-Lipschitz continuous with respect to B_1, we have

$$\begin{aligned}
\|H_1(A_1(a_n),B_1(a_n)) &- H_1(A_1(a_{n-1}),B_1(a_{n-1}))\| \\
&\leq \|H_1(A_1(a_n),B_1(a_n)) - H_1(A_1(a_{n-1}),B_1(a_n))\| \\
&\quad + \|H_1(A_1(a_{n-1}),B_1(a_n)) - H_1(A_1(a_{n-1}),B_1(a_{n-1}))\| \\
&\leq r_1\|a_n-a_{n-1}\| + r_2\|a_n-a_{n-1}\| \\
&= (r_1+r_2)\|a_n-a_{n-1}\|.
\end{aligned} \tag{4.4}$$

As $F(x,\cdot)$ is strongly monotone with respect to $H_1(A_1,B_1)$, we have

$$\begin{aligned}
-\langle H_1(A_1(a_n),B_1(a_n)) - H_1(A_1(a_{n-1}),B_1(a_{n-1})),\ F(a_n,b_n) - F(a_{n-1},b_n)\rangle& \\
\leq -m_1\|a_n-a_{n-1}\|^2.&
\end{aligned} \tag{4.5}$$

Using the τ_1-Lipschitz continuity of $F(\cdot,\cdot)$ in the first argument, we obtain

$$\|F(a_n,b_n)-F(a_{n-1},b_n)\| \le \tau_1\|a_n-a_{n-1}\|. \tag{4.6}$$

Combining (4.4) to (4.6) with (4.3), we obtain

$$\begin{aligned}\|[H_1(A_1(a_n),B_1(a_n))-H_1(A_1(a_{n-1}),B_1(a_{n-1}))]-\lambda[F(a_n,b_n)-F(a_{n-1},b_n)]\|^2\\ \le [(r_1+r_2)^2-2\lambda m_1+\lambda^2\tau_1^2]\|a_n-a_{n-1}\|^2,\end{aligned}$$

which implies that

$$\begin{aligned}\|[H_1(A_1(a_n),B_1(a_n))-H_1(A_1(a_{n-1}),B_1(a_{n-1}))]-\lambda[F(a_n,b_n)-F(a_{n-1},b_n)]\|\\ \le \sqrt{(r_1+r_2)^2-2\lambda m_1+\lambda^2\tau_1^2}\ \|a_n-a_{n-1}\|. \end{aligned}\tag{4.7}$$

Also as $F(\cdot,\cdot)$ is τ_2-Lipschitz continuous in the second argument, we have

$$\|F(a_{n-1},b_n)-F(a_{n-1},b_{n-1})\| \le \tau_2\|b_n-b_{n-1}\|. \tag{4.8}$$

Using (4.7) and (4.8), (4.2) becomes

$$\begin{aligned}\|a_{n+1}-a_n\| \le \frac{\sqrt{(r_1+r_2)^2-2\lambda m_1+\lambda^2\tau_1^2}}{\mu_1\alpha_1^2-\gamma_1\beta_1^2}\ \|a_n-a_{n-1}\|\\ +\frac{\lambda\tau_2}{\mu_1\alpha_1^2-\gamma_1\beta_1^2}\ \|b_n-b_{n-1}\|.\end{aligned}\tag{4.9}$$

In a similar way, we estimate

$$\begin{aligned}\|b_{n+1}-b_n\| &= \|R_{\rho,N}^{H_2(\cdot,\cdot)}\big[H_2(A_2(b_n),B_2(b_n))-\rho G(a_n,b_n)\big]\\ &\quad - R_{\rho,N}^{H_2(\cdot,\cdot)}[H_2(A_2(b_{n-1}),B_2(b_{n-1}))-\rho G(a_{n-1},b_{n-1})]\|\\ &\le \frac{1}{\mu_2\alpha_2^2-\gamma_2\beta_2^2}\|H_2(A_2(b_n),B_2(b_n))-H_2(A_2(b_{n-1}),B_2(b_{n-1}))]\\ &\quad -\rho[G(a_n,b_n)-G(a_n,b_{n-1})+G(a_n,b_{n-1})-G(a_{n-1},b_{n-1})]\|\\ &\le \frac{1}{\mu_2\alpha_2^2-\gamma_2\beta_2^2}\|[H_2(A_2(b_n),B_2(b_n))-H_2(A_2(b_{n-1}),B_2(b_{n-1}))]\\ &\quad -\rho[G(a_n,b_n)-G(a_n,b_{n-1})]\|\\ &\quad +\frac{\rho}{\mu_2\alpha_2^2-\gamma_2\beta_2^2}\|[G(a_n,b_{n-1})-G(a_{n-1},b_{n-1})]\|.\end{aligned}\tag{4.10}$$

Using the same arguments as for (4.7), we have

$$\begin{aligned}\|H_2(A_2(b_n),B_2(b_n))-H_2(A_2(b_{n-1}),B_2(b_{n-1}))]-\rho[G(a_n,b_n)-G(a_n,b_{n-1})]\|\\ \le \sqrt{(r_3+r_4)^2-2\rho m_2+\rho^2{\tau'}_2^2}\ \|b_n-b_{n-1}\|.\end{aligned}\tag{4.11}$$

As $G(\cdot,\cdot)$ is τ_1'-Lipschitz continuous in the first argument, we have

$$\|G(a_n, b_{n-1}) - G(a_{n-1}, b_{n-1})\| \leq \tau_1' \|a_n - a_{n-1}\|. \tag{4.12}$$

Combining (4.11)-(4.12) with (4.10), we have

$$\begin{aligned} \|b_{n+1} - b_n\| \leq & \frac{\sqrt{(r_3 + r_4)^2 - 2\rho m_2 + \rho^2 {\tau'}_2^2}}{\mu_2 \alpha_2^2 - \gamma_2 \beta_2^2} \|b_n - b_{n-1}\| \\ & + \frac{\rho \tau_1'}{\mu_2 \alpha_2^2 - \gamma_2 \beta_2^2} \|a_n - a_{n-1}\|. \end{aligned} \tag{4.13}$$

Combining (4.9) and (4.13), we have

$$\begin{aligned} &\|a_{n+1} - a_n\| + \|b_{n+1} - b_n\| \\ &\quad \leq \left[\frac{\sqrt{(r_1 + r_2)^2 - 2\lambda m_1 + \lambda^2 {\tau_1}^2}}{\mu_1 \alpha_1^2 - \gamma_1 \beta_1^2} + \frac{\rho \tau_1'}{\mu_2 \alpha_2^2 - \gamma_2 \beta_2^2} \right] \|a_n - a_{n-1}\| \\ &\qquad + \left[\frac{\sqrt{(r_3 + r_4)^2 - 2\rho m_2 + \rho^2 {\tau_2'}^2}}{\mu_2 \alpha_2^2 - \gamma_2 \beta_2^2} + \frac{\lambda \tau_2}{\mu_1 \alpha_1^2 - \gamma_1 \beta_1^2} \right] \|b_n - b_{n-1}\| \\ &\quad \leq \theta \left[\|a_n - a_{n-1}\| + \|b_n - b_{n-1}\| \right], \end{aligned} \tag{4.14}$$

where

$$\theta = \max\left\{ \frac{\sqrt{(r_1+r_2)^2 - 2\lambda m_1 + \lambda^2 {\tau_1}^2}}{\mu_1 \alpha_1^2 - \gamma_1 \beta_1^2} + \frac{\rho \tau_1'}{\mu_2 \alpha_2^2 - \gamma_2 \beta_2^2}, \frac{\sqrt{(r_3+r_4)^2 - 2\rho m_2 + \rho^2 {\tau_2'}^2}}{\mu_2 \alpha_2^2 - \gamma_2 \beta_2^2} + \frac{\lambda \tau_2}{\mu_1 \alpha_1^2 - \gamma_1 \beta_1^2} \right\}.$$

By (4.1), $0 < \theta < 1$ and (4.14) implies that $\{a_n\}$ and $\{b_n\}$ both are Cauchy sequences. Therefore, $\{(a_n, b_n)\}$ converges to a solution (a, b) of problem (3.1). This completes the proof. □

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