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A New Common Fixed Point Theorem for Six Self-Mappings in Complete Metric Spaces[1]

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Abstract : Before, people extend some common fixed point theorems from the commuting mappings to the compatible mappings, so that fixed point theory has been further developed. However, these results are mostly for the two mappings, the three mappings, or four mappings given the circumstances. In this paper, we introduce a new contractive condition for six self-mappings, by using the compatible and sub-compatible conditions of self-mapping pairs, the existence and uniqueness of common fixed point in complete metric spaces is discussed, a new common fixed point theorem is obtained, and we succeed in getting the conclusion that six self-mappings in complete metric space has only one common fixed point. Our result extends, generalized and improves some results of Cirić, Gu, Li, Iséki, Rhoades, etc.

Keywords : Compatible mapping pairs; Sub-compatible mapping pairs; Noncompatible mapping pairs; Common fixed point.

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1 Introduction

In 1976, Jungck [1] proved a common fixed point theorem of commuting mappings on a metric space. In 1982, Sessa [2] introduced the concept of weakly commuting mappings, which is a generalization of the concept of commuting mappings, and he and others proved some fixed point theorems for weakly commuting mappings (see, e.g., [2–5]). In 1986, Jungck [6] introduced more generalized commuting mappings, called compatible mappings, which are more general than commuting and weakly commuting mappings. These concept has been useful for obtaining more comprehensive fixed point theorems (see, e.g., [3–39]).

The purpose of this paper is to use the concept of compatible and sub-compatible mappings to discuss a new fixed point problem for the six self-mappings. Our results improves and develops the relevant results of Cirić [11], Gu et al. [16], Gu and Du [17], Iséki [19], Li and Gu [26], and Rhoades [35], and others.

2 Preliminaries

In this section we recall some definitions as follows.

Definition 2.1 ([6])**.** Self-mappings f and g on a metric space (X, d) are said to be *compatible* if and only if whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$$

for some $t \in X$, then

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$$

Definition 2.2 ([28])**.** Self-mappings f and g on a metric space (X, d) are said to be *sub-compatible* if and only if $\{t \in X : f(t) = g(t)\} \subset \{t \in X : fg(t) = gf(t)\}$.

Remark 2.3. *From the definitions, it is easily seen that if f and g are compatible, then they are sub-compatible. However, sub-compatible mappings do not necessarily compatible. Counter-examples can be seen in [28].*

3 Main Results

Theorem 3.1. *Let (X, d) be a complete metric space and let S, T, A, B, U and V are six mappings of X into itself. Suppose that $\phi(x, y)$ is symmetry and continuous function from $X \times X$ to $[0, \infty)$ and satisfy $\phi(x, x) = 0$ for all $x \in X$, if the following statements hold:*

(i) $S(X) \subset BV(X)$, $T(X) \subset AU(X)$;

(ii) $SU = US$, $AU = UA$, $TV = VT$, $BV = VB$;

(iii) $\forall x, y \in X$, $\phi(Sx, Ty) \leq \beta \max\{\phi(AUx, BVy), \phi(AUx, Sx), \phi(BVy, Ty)\}$;

(iv) $\forall x, y \in X$,

$$d(Sx, Ty) \le \alpha \max \left\{ \begin{array}{c} d(AUx, BVy), d(AUx, Sx), d(BVy, Ty), \\ \frac{d(Sx, BVy) + d(AUx, Ty)}{2} \end{array} \right\} + \phi(AUx, BVy),$$

where α, $\beta \in [0, 1)$. If one of the following conditions is satisfied, then S, T, A, B, U and V have a unique common fixed point z in X, further, z is a unique common fixed point of mapping pairs (S, AU) and (T, BV).

(1) Either S or AU is continuous, the pair (S, AU) is compatible, the pair (T, BV) is sub-compatible;

(2) Either T or BV is continuous, the pair (T, BV) is compatible, the pair (S, AU) sub-compatible;

(3) Either AU or BV is surjective, both (S, AU) and (T, BV) are sub-compatible.

Proof. Let x_0 in X be arbitrary, since $S(X) \subset BV(X)$, $T(X) \subset AU(X)$, there exists the sequences $\{x_n\}$ and $\{y_n\}$ in X, such that

$$y_{2n} = Sx_{2n} = BVx_{2n+1},\ y_{2n+1} = Tx_{2n+1} = AUx_{2n+2}, \quad for \ \ n = 1, 2, 3, \cdots$$

We now prove $\{y_n\}$ is a Cauchy sequence in X. Actually, using the condition (iv) we have:

$$\begin{aligned} & d(y_{2n}, y_{2n+1}) = d(Sx_{2n}, Tx_{2n+1}) \\ & \le \alpha \max \left\{ \begin{array}{c} d(AUx_{2n}, BVx_{2n+1}), d(AUx_{2n}, Sx_{2n}), d(BVx_{2n+1}, Tx_{2n+1}), \\ \frac{d(Sx_{2n}, BVx_{2n+1}) + d(AUx_{2n}, Tx_{2n+1})}{2} \end{array} \right\} \\ & \quad + \phi(AUx_{2n}, BVx_{2n+1}) \\ & = \alpha \max \left\{ \begin{array}{c} d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), \\ \frac{d(y_{2n}, y_{2n}) + d(y_{2n-1}, y_{2n+1})}{2} \end{array} \right\} + \phi(y_{2n-1}, y_{2n}) \\ & = \alpha \max\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1})\} + \phi(y_{2n-1}, y_{2n}) \\ & = \left\{ \begin{array}{l} \alpha d(y_{2n-1}, y_{2n}) + \phi(y_{2n-1}, y_{2n}),\ d(y_{2n-1}, y_{2n}) \ge d(y_{2n}, y_{2n+1}) \\ \alpha d(y_{2n}, y_{2n+1}) + \phi(y_{2n-1}, y_{2n}),\ d(y_{2n-1}, y_{2n}) < d(y_{2n}, y_{2n+1}) \end{array} \right. . \end{aligned}$$

Since $\alpha \in [0, 1)$, so we have

$$d(y_{2n}, y_{2n+1}) \le \alpha d(y_{2n-1}, y_{2n}) + \frac{1}{1-\alpha} \phi(y_{2n-1}, y_{2n}).$$

Similarly, we can be proved that

$$d(y_{2n+1}, y_{2n+2}) \le \alpha d(y_{2n}, y_{2n+1}) + \frac{1}{1-\alpha} \phi(y_{2n}, y_{2n+1}).$$

Therefore, for all $n \geq 2$, we have:

$$d(y_n, y_{n+1}) \leq \alpha d(y_{n-1}, y_n) + \frac{1}{1-\alpha}\phi(y_{n-1}, y_n) \tag{3.1}$$

Using the condition (iii) we have

$$\begin{aligned}\phi(y_{2n}, y_{2n+1}) &= \phi(Sx_{2n}, Tx_{2n+1})\\ &\leq \beta \max\{\phi(AUx_{2n}, BVx_{2n+1}), \phi(AUx_{2n}, Sx_{2n}), \phi(BVx_{2n+1}, Tx_{2n+1})\}\\ &= \beta \max\{\phi(y_{2n-1}, y_{2n}), \phi(y_{2n-1}, y_{2n}), \phi(y_{2n}, y_{2n+1})\}\\ &= \beta \max\{\phi(y_{2n-1}, y_{2n}), \phi(y_{2n}, y_{2n+1})\}.\end{aligned}$$

Since $\beta \in [0, 1)$, so that $\phi(y_{2n}, y_{2n+1}) \leq \beta\phi(y_{2n-1}, y_{2n})$. Similarly, we can be proved that $\phi(y_{2n+1}, y_{2n+2}) \leq \beta\phi(y_{2n}, y_{2n+1})$. Hence, for all $n \geq 2$, we have

$$\phi(y_n, y_{n+1}) \leq \beta\phi(y_{n-1}, y_n). \tag{3.2}$$

By (3.1) and (3.2) we have

$$\begin{aligned}d(y_n, y_{n+1}) &\leq d(y_{n-1}, y_n) + \frac{1}{1-\alpha}\phi(y_{n-1}, y_n)\\ &\vdots\\ &\leq \alpha^n d(y_0, y_1) + \frac{1}{1-\alpha}(\alpha^{n-1} + \alpha^{n-2}\beta + \cdots + \alpha\beta^{n-2} + \beta^{n-1})\phi(y_0, y_1)\\ &= \begin{cases} \alpha^n d(y_0, y_1) + \frac{1}{1-\alpha}\frac{\alpha^n-\beta^n}{\alpha-\beta}, & \alpha \neq \beta\\ \alpha^n d(y_0, y_1) + \frac{n\alpha^{n-1}}{1-\alpha}\phi(y_0, y_1), & \alpha = \beta.\end{cases}\end{aligned} \tag{3.3}$$

Note that progressions $\sum_{n=1}^{\infty} \alpha^n$, $\sum_{n=1}^{\infty} \beta^n$, $\sum_{n=1}^{\infty} n\alpha^{n-1}$ $(0 \leq \alpha, \beta < 1)$ are all convergence. Therefore, for any positive integer m and k, we have

$$d(y_m, y_{m+k}) \leq \sum_{i=1}^{k} d(y_{m+i-1}, y_{m+i}). \tag{3.4}$$

By (3.3) and (3.4) we know $\{y_n\}$ is a Cauchy sequence in X, since X is complete, there exists a point $z \in X$ such that $y_n \to z (n \to \infty)$.

Since the sequences $\{Sx_{2n}\} = \{BVx_{2n+1}\} = \{y_{2n}\}$ and $\{Tx_{2n-1}\} = \{AUx_{2n}\} = \{y_{2n-1}\}$ are all subsequences of $\{y_n\}$, then they all converge to z.

$$y_{2n} = Sx_{2n} = BVx_{2n+1} \to z, y_{2n-1} = Tx_{2n-1} = AUx_{2n} \to z \ (n \to \infty). \tag{3.5}$$

(1) Either S or AU is continuous, the pair (S, AU) is compatible, the pair (T, BV) is sub-compatible.

First we prove $z = Sz = Tz = AUz = BVz$. By (2.5) and compatibility of mapping pair (S, AU) we have

$$d(SAUx_{2n}, AUSx_{2n}) \to 0 \ (n \to \infty). \tag{3.6}$$

Suppose AU is continuous mapping, then $AUSx_{2n} \to AUz(n \to \infty)$. By (3.6) we know $SAUx_{2n} \to AUz(n \to \infty)$. Using the condition (iv) we know

$$\begin{aligned} &d(SAUx_{2n}, Tx_{2n+1}) \\ &\quad \leq \alpha \max \left\{ \begin{array}{c} d((AU)^2x_{2n}, BVx_{2n+1}), d((AU)^2x_{2n}, SAUx_{2n}), d(BVx_{2n+1}, Tx_{2n+1}), \\ \frac{d(SAUx_{2n}, BVx_{2n+1}) + d((AU)^2x_{2n}, Tx_{2n+1})}{2} \end{array} \right\} \\ &\qquad + \phi((AU)^2x_{2n}, BVx_{2n+1}). \end{aligned}$$

Letting $n \to \infty$ we have

$$\begin{aligned} d(AUz, z) &\leq \alpha \max \left\{ d(AUz, z), d(AUz, AUz), d(z, z), \frac{d(AUz, z) + d(AUz, z)}{2} \right\} \\ &\quad + \phi(AUz, z) \\ &= \alpha d(AUz, z) + \phi(AUz, z). \end{aligned} \tag{3.7}$$

Using the condition (iii) we have

$$\begin{aligned} &\phi(SAUx_{2n}, Tx_{2n+1}) \\ &\quad \leq \beta \max\{\phi((AU)^2x_{2n}, BVx_{2n+1}), \phi((AU)^2x_{2n}, SAUx_{2n}), \phi(BVx_{2n+1}, Tx_{2n+1})\}. \end{aligned}$$

Letting $n \to \infty$ we have

$$\phi(AUz, z) \leq \beta \max\{\phi(AUz, z), \phi(AUz, AUz), \phi(z, z)\} = \beta\phi(AUz, z).$$

Since $\beta \in [0, 1)$, we have $\phi(AUz, z) = 0$, then by (3.7) we have $d(AUz, z) \leq \alpha d(AUz, z)$. Note that $\alpha \in [0, 1)$, we have $d(AUz, z) = 0$, and so that $AUz = z$. Using the condition (iv) we have

$$\begin{aligned} d(Sz, Tx_{2n+1}) \leq \alpha \max &\left\{ d(AUz, BVx_{2n+1}), d(AUz, Sz), d(BVx_{2n+1}, Tx_{2n+1}), \right. \\ &\left. \frac{d(Sz, BVx_{2n+1}) + d(AUz, Tx_{2n+1})}{2} \right\} + \phi(AUz, BVx_{2n+1}). \end{aligned}$$

Letting $n \to \infty$ we have

$$d(Sz, z) \leq \alpha \max \left\{ d(z, z), d(z, Sz), d(z, z), \frac{d(Sz, z) + d(z, z)}{2} \right\} + \phi(z, z) = \alpha\phi(Sz, z).$$

Note that $\alpha \in [0, 1)$, we have $d(Sz, z) = 0$, and so that $Sz = z$. Using the condition (i) we have $z = Sz \in S(X) \subset BV(X)$, there exists $u \in X$ such that $BVu = z$.

Using the condition (iv) we have

$$\begin{aligned}
d(z,Tu) &= d(Sz,Tu)\\
&\le \alpha \max\left\{d(AUz,BVu), d(AUz,Sz), d(BVu,Tu), \frac{d(Sz,BVu)+d(AUz,Tu)}{2}\right\}\\
&\quad + \phi(AUz,BVu)\\
&\le \alpha \max\left\{d(z,z), d(z,z), d(z,Tu), \frac{d(z,z)+d(z,Tu)}{2}\right\} + \phi(z,z)\\
&\le \alpha d(z,Tu).
\end{aligned}$$

Note that $\alpha \in [0,1)$, we have $d(z,Tu) = 0$, henge we have $z = Tu$, and so $z = Tu = BVu$.

Since T and BV are compatible mappings, then they are sub-compatible mappings, therefore $TBVu = BVTu$, so we have $Tz = BVz$. So by the condition (iv) we have

$$\begin{aligned}
d(z,Tz) &= d(Sz,Tz)\\
&\le \alpha \max\left\{d(AUz,BVz), d(AUz,Sz), d(BVz,Tz), \frac{d(Sz,BVz)+d(AUz,Tz)}{2}\right\}\\
&\quad + \phi(AUz,BVz)\\
&= \alpha \max\left\{d(z,Tz), d(z,z), d(Tz,Tz), \frac{d(z,Tz)+d(z,Tz)}{2}\right\} + \phi(AUz,BVz)\\
&= \alpha d(z,Tz) + \phi(AUz,BVz).
\end{aligned}$$

Using the condition (iii) and $AUz = Sz$, $BVz = Tz$ we have

$$\begin{aligned}
\phi(AUz,BVz) &= \phi(Sz,Tz)\\
&\le \beta \max\{\phi(AUz,BVz), \phi(AUz,Sz), \phi(BVz,Tz)\}\\
&= \beta\phi(AUz,BVz).
\end{aligned}$$

Note that $\beta \in [0,1)$, so $\phi(AUz,BVz) = 0$. Again, on applying $\phi(AUz,BVz) = 0$ to $d(z,Tz) \le \alpha d(z,Tz) + \phi(AUz,BVz)$, it follows that $d(z,Tz) \le \alpha d(z,Tz)$, so we have $d(z,Tz) = 0$, then we know that $Tz = z = BVz$. Consequently, $z = Sz = Tz = AUz = BVz$, so z is a common fixed point of S, T, AU, BV.

Next we prove $z = Az = Bz = Uz = Vz$. Actually, by the condition (iv) and $SU = US, AU = UA$ we have

$$\begin{aligned}
d(Uz,z) &= d(USz,Tz) = d(SUz,Tz)\\
&\le \alpha \max\left\{\begin{array}{c} d((AU)Uz,BVz), d((AU)Uz,SUz), d(BVz,Tz),\\ \frac{d(SUz,BVz)+d((AU)Uz,Tz)}{2}\end{array}\right\}\\
&\quad + \phi((AU)Uz,BVz).
\end{aligned}$$

Since $SU = US, AU = UA$ so we have $SUz = USz = Uz, (AU)Uz = U(AU)z = Uz$, then we know

$$d(Uz,z) \leq \alpha \max\left\{d(Uz,z), d(Uz,Uz), d(z,z), \frac{d(Uz,z)+d(Uz,z)}{2}\right\} + \phi(Uz,z)$$
$$= \alpha d(Uz,z) + \phi(Uz,z). \tag{3.8}$$

Using the condition (iii) we have

$$\begin{aligned}\phi(Uz,z) &= \phi(USz,Tz) = \phi(SUz,Tz)\\ &\leq \beta \max\{\phi((AU)Uz,BVz), \phi((AU)Uz,SUz), \phi(BVz,Tz)\}\\ &= \beta \max\{\phi(Uz,z), \phi(Uz,Uz), \phi(z,z)\}\\ &= \beta\phi(Uz,z).\end{aligned}$$

Since $\beta \in [0,1)$, so we have $\phi(Uz,z) = 0$, on applying $\phi(Uz,z) = 0$ to (3.8) we know that $d(Uz,z) \leq \alpha d(Uz,z)$. Note that $\alpha \in [0,1)$, so $d(Uz,z) = 0$, so $Uz = z$. For $AUz = z$, then we have $Az = z$, then we know that $Az = Uz = z$. Again by the condition (iv) and $TV = VT, BV = VB$ we have

$$\begin{aligned}d(z,Vz) &= d(Sz,VTz) = d(Sz,TVz)\\ &\leq \alpha \max\left\{\begin{array}{c} d(AUz,(BV)Vz), d(AUz,Sz), d((BV)Vz,Tz),\\ \frac{d(Sz,(BV)Vz)+d(AUz,TVz)}{2}\end{array}\right\}\\ &\quad + \phi(AUz,(BV)Vz).\end{aligned}$$

Note that $TV = VT, BV = VB$, so $TVz = VTz = Vz, (BV)Vz = V(BV)z = Vz$, so we have

$$d(z,Vz) \leq \alpha \max\left\{d(z,Vz), d(z,z), d(Vz,Vz), \frac{d(z,Vz)+d(z,Vz)}{2}\right\} + \phi(z,Vz)$$
$$= \alpha d(z,Vz) + \phi(z,Vz). \tag{3.9}$$

Using the condition (iii) we have

$$\begin{aligned}\phi(z,Vz) &= \phi(Sz,VTz) = \phi(Sz,TVz)\\ &\leq \beta \max\{\phi(AUz,(BV)Vz), \phi(AUz,SVz), \phi((BV)Vz,TVz)\}\\ &= \beta \max\{\phi(z,Vz), \phi(z,Vz), \phi(Vz,Vz)\}\\ &= \beta\phi(z,Vz).\end{aligned}$$

Since $\beta \in [0,1)$, so we have $\phi(z,Vz) = 0$, on applying $\phi(z,Vz) = 0$ to (3.9) we know that $d(z,Vz) \leq \alpha d(z,Vz)$. Note that $\alpha \in [0,1)$, so $d(z,Vz) = 0$, so $z = Vz$. For $BVz = z$, then we have $Bz = z$, then we know that $Bz = Vz = z$. Consequently, we can obtain $z = Sz = Tz = Az = Bz = Uz = Vz$, so z is a common fixed point of S, T, A, B, U, V.

We now suppose S is continuous mapping, then $SAUx_{2n} \to Sz (n \to \infty)$, then by (3.6) we have $AUSx_{2n} \to Sz (n \to \infty)$. Using the condition (iv) we have

$$\begin{aligned} & d(S^2x_{2n}, Tx_{2n+1}) \\ & \leq \alpha \max \left\{ \begin{array}{c} d(AUSx_{2n}, BVx_{2n+1}), d(AUSx_{2n}, S^2x_{2n}), d(BVx_{2n+1}, Tx_{2n+1}), \\ \frac{d(S^2x_{2n}, BVx_{2n+1}) + d(AUSx_{2n}, Tx_{2n+1})}{2} \end{array} \right\} \\ & \quad + \phi(AUSx_{2n}, BVx_{2n+1}). \end{aligned}$$

Letting $n \to \infty$ we have

$$\begin{aligned} d(Sz, z) & \leq \alpha \max \left\{ d(Sz, z), d(Sz, Sz), d(z, z), \frac{d(Sz, z) + d(Sz, z)}{2} \right\} + \phi(Sz, z) \\ & = \alpha d(Sz, z) + \phi(Sz, z). \end{aligned} \tag{3.10}$$

Using the condition (iii) we have

$$\begin{aligned} & \phi(S^2x_{2n}, Tx_{2n+1}) \\ & \quad \leq \alpha \max\{\phi(AUSx_{2n}, BVx_{2n+1}), \phi(AUSx_{2n}, S^2x_{2n}), \phi(BVx_{2n+1}, Tx_{2n+1})\}. \end{aligned}$$

Letting $n \to \infty$ we have $\phi(Sz, z) \leq \beta\phi(Sz, z)$, so we have $\phi(Sz, z) = 0$, on applying $\phi(Sz, z) = 0$ to (3.9), it follows that $d(Sz, z) \leq \alpha d(Sz, z)$. So we have $d(Sz, z) = 0$, and so $Sz = z$.

By (i) we know that $z = Sz \in S(X) \subset BV(X)$, there exists $v \in X$ and contents $BVv = z$. Using the condition (iv) we have

$$\begin{aligned} d(S^2x_{2n}, Tv) \leq \alpha \max & \left\{ \begin{array}{c} d(AUSx_{2n}, BVv), d(AUSx_{2n}, S^2x_{2n}), d(BVv, Tv), \\ \frac{d(S^2x_{2n}, Bv) + d(AUSx_{2n}, Tv)}{2} \end{array} \right\} \\ & + \phi(AUSx_{2n}, BVv). \end{aligned}$$

Letting $n \to \infty$ we have

$$d(z, Tv) \leq \alpha \max \left\{ d(z, z), d(z, z), d(z, Tv), \frac{d(z, z) + d(z, Tv)}{2} \right\} + \phi(z, z) = \alpha d(z, Tv).$$

It follows that $d(z, Tv) = 0$, so we have $z = Tv$, and so $z = Tv = BVv$. Since T and BV are compatible mappings, then they are sub-compatible mappings, so $TBVv = BVTv$, so we have $Tz = BVz$. So by the condition (iv) we have

$$\begin{aligned} d(Sx_{2n}, Tz) \leq \alpha \max & \left\{ \begin{array}{c} d(AUx_{2n}, BVz), d(AUx_{2n}, Sx_{2n}), d(BVz, Tz), \\ \frac{d(Sx_{2n}, BVz) + d(AUx_{2n}, Tz)}{2} \end{array} \right\} \\ & + \phi(AUx_{2n}, BVz). \end{aligned}$$

Letting $n \to \infty$ we have

$$\begin{aligned} d(z, Tz) & \leq \alpha \max \left\{ d(z, Tz), d(z, z), d(Tz, Tz), \frac{d(z, Tz) + d(z, Tz)}{2} \right\} + \phi(z, Tz) \\ & = \alpha d(z, Tz) + \phi(z, Tz). \end{aligned} \tag{3.11}$$

Using the condition (iii) we have

$$\phi(Sx_{2n}, Tz) \le \beta \max\{\phi(AUx_{2n}, BVz), \phi(AUx_{2n}, Sx_{2n}), \phi(BVz, Tz)\}.$$

Letting $n \to \infty$ we have

$$\begin{aligned}\phi(z, Tz) &\le \beta \max\{\phi(z, BVz), \phi(z, z), \phi(BVz, Tz)\}\\ &= \beta \max\{\phi(z, Tz), \phi(z, z), \phi(Tz, Tz)\}\\ &= \beta\phi(z, Tz).\end{aligned}$$

So we can know $\phi(z, Tz) = 0$, on applying $\phi(z, Tz) = 0$ to (3.11), it follows that $d(z, Tz) \le \alpha d(z, Tz)$, so we have $d(z, Tz) = 0$, and so $Tz = z$.

Further, by the condition (i) we have $z = Tz \in T(X) \subset AU(X)$, there exists $w \in X$ and contents $AUw = Tz = z$. Since $Tz = BVz, Tz = z$, so by the condition (iv) we have

$$\begin{aligned}d(Sw, z) &= d(Sw, Tz)\\ &\le \alpha \max\left\{\begin{array}{c} d(AUw, BVz), d(AUw, Sw), d(BVz, Tz),\\ \frac{d(Sw,BVz)+d(AUw,Tz)}{2}\end{array}\right\}\\ &\quad + \phi(AUw, BVz)\\ &= \alpha \max\left\{d(z, BVz), d(z, Sw), d(BVz, Tz), \frac{d(Sw, BVz) + d(z, Tz)}{2}\right\}\\ &\quad + \phi(z, BVz)\\ &= \alpha \max\left\{d(z, z), d(z, Sw), d(z, z), \frac{d(Sw, z) + d(z, z)}{2}\right\} + \phi(z, z)\\ &= \alpha d(Sw, z).\end{aligned}$$

It follows that $d(Sw, z) = 0$, so we have $Sw = z = AUw$. Since S and AU are compatible mappings, then they are sub-compatible mappings, so $(AU)Sw = S(AU)w$, so we have $AUz = Sz = z$. Consequently, $z = Sz = Tz = AUz = BVz$, so z is a common fixed point of S, T, AU, and BV. Similarly, we can prove $z = Az = Bz = Uz = Vz$, so we proved z is a common fixed point of S, T, A, B, U, V.

Next, we prove z is a unique common fixed point of S, T, A, B, U, V, and z is a unique common fixed point of (S, AU) and (T, BV). Actually, suppose $y \ne z$, $y \in X$ is also a common fixed point of S and AU, then by the condition (iv) we have

$$\begin{aligned}d(y, z) &= d(Sy, Tz)\\ &\le \alpha \max\left\{\begin{array}{c} d(AUy, BVz), d(AUy, Sy), d(BVz, Tz),\\ \frac{d(Sy,BVz)+d(AUy,Tz)}{2}\end{array}\right\} + \phi(y, z)\\ &= \alpha d(y, z) + \phi(y, z).\end{aligned} \tag{3.12}$$

Using the condition (iv) we have

$$\begin{aligned}\phi(y,z) &= \phi(Sy,Tz)\\ &\le \beta\max\{\phi(AUy,BVz),\phi(AUy,Sy),\phi(BVz,Tz)\}\\ &= \beta\max\{\phi(y,z),\phi(y,y),\phi(z,z)\}\\ &= \beta\phi(y,z).\end{aligned}$$

So we can know $\phi(y,z)=0$, on applying $\phi(y,z)=0$ to (3.12), it follows that $d(y,z)\le \alpha d(y,z)$. So we have $d(y,z)=0$, and so $y=z$. So z is the unique common fixed point of (S,AU) in X, similarly, we can prove z is the unique common fixed point of (T,BV). By the condition (iv) we can similarly prove z is the unique common fixed point of S,T,A,B,U,V.

(2) When one of T,BV continuous, and (T,BV) compatible, (S,AU) sub-compatible, similar to (1) for the same reason we can prove the theorem.

(3) Suppose one of AU,BV is surjective and both (S,AU) and (T,BV) are sub-compatible.

If AU is surjective, then for $z\in X,\exists u\in X$, and satisfy $AUu=z$, by the condition (iv) we can know

$$\begin{aligned}d(Su,Tx_{2n+1}) \le{}& \alpha\max\left\{\begin{array}{c} d(AUu,BVx_{2n+1}),d(AUu,Su),d(BVx_{2n+1},Tx_{2n+1}),\\ \frac{d(Su,BVx_{2n+1})+d(AUu,Tx_{2n+1})}{2}\end{array}\right\}\\ &+\phi(AUu,BVx_{2n+1}).\end{aligned}\tag{3.13}$$

Letting $n\to\infty$ we have $d(Su,z)\le\alpha d(Su,z)+\phi(z,z)$, note $\alpha\in[0,1)$ we can have $d(Su,z)=0$, such that $Su=z$, so $Su=AUu=z$.

Further, since (S,AU) is sub-compatible mapping pair, so we can obtain $AUz=(AU)Su=S(AU)u=Sz$. Let z instead of u in (3.13) we have

$$\begin{aligned}d(Sz,Tx_{2n+1}) \le{}& \alpha\max\left\{\begin{array}{c} d(AUz,BVx_{2n+1}),d(AUz,Sz),d(BVx_{2n+1},Tx_{2n+1}),\\ \frac{d(Sz,BVx_{2n+1})+d(AUz,Tx_{2n+1})}{2}\end{array}\right\}\\ &+\phi(AUz,BVx_{2n+1}).\end{aligned}$$

Letting $n\to\infty$ and note that $AUz=Sz$ we have

$$d(Sz,z)\le\alpha d(Sz,z)+\phi(Sz,z).\tag{3.14}$$

By the condition (iii) we have

$$\phi(Sz,Tx_{2n+1})\le\beta\max\{\phi((AU)z,BVx_{2n+1}),\phi(AUz,Sz),\phi(BVx_{2n+1},Tx_{2n+1})\}.$$

Letting $n\to\infty$ we have $\phi(Sz,z)\le\beta\phi(Sz,z)$, so $\phi(Sz,z)=0$. On applying $\phi(Sz,z)=0$ to (3.14) we can know $d(Sz,z)\le\alpha d(Sz,z)$, so $d(Sz,z)=0$, such that $Sz=z$. So $AUz=Sz=z$. Similar to (1) we can prove z is a unique common fixed point of S,T,A,B,U,V, and z is a unique common fixed point of (S,AU) and (T,BV).

When BV is surjective, similarly we can prove z is a unique common fixed point of S,T,A,B,U,V, and z is a unique common fixed point of (S,AU) and (T,BV). This completes the proof of Theorem 3.1. □

Remark 3.2. *Take $A = B = U = V = I$ (I is identity mapping, the same below) in Theorem 3.1, the result is further improved of the corresponding result in [19].*

Remark 3.3. *Take $U = V = I$ in Theorem 3.1, the result is further improved of the corresponding result in [26]. In [26] need the two pairs of mappings are all weak commutative, but in Theorem 2.1 will it weakened into a pair of compatible and a pair of sub-compatible.*

In Theorem 3.1 let $\phi(x, y) \equiv 0$ we have the following theorem.

Theorem 3.4. *Let (X, d) be a complete metric space and let S, T, A, B, U and V are six mappings of X into itself. Suppose that (S, AU) and (T, BV) are compatible self-mappings, if the following statements hold:*

(i) $S(X) \subset BV(X)$, $T(X) \subset AU(X)$;

(ii) $SU = US$, $AU = UA$, $TV = VT$, $BV = VB$;

(iii) $\forall x, y \in X$,

$$d(Sx, Ty) \le \alpha \max \left\{ \begin{array}{c} d(AUx, BVy), d(AUx, Sx), d(BVy, Ty), \\ \frac{d(Sx, BVy) + d(AUx, Ty)}{2} \end{array} \right\}$$

where $\alpha \in [0, 1)$. If one of the following conditions satisfy, then S, T, A, B, U and V have a unique common fixed point z in X, further, z is a unique common fixed point of (S, AU) and (T, BV).

(1) Either S or AU is continuous, the pair (S, AU) is compatible, the pair (T, BV) is sub-compatible;

(2) Either T or BV is continuous, the pair (T, BV) is compatible, the pair (S, AU) sub-compatible;

(3) Either AU or BV is surjective, both (S, AU) and (T, BV) are sub-compatible.

Remark 3.5. *Take $A = B = U = V = I, S = T$ in Theorem 3.4, the result is further improved of the corresponding result in [11].*

Remark 3.6. *Take $A = B = U = V = I$ in Theorem 3.4, the result is further improved of the corresponding result in [35].*

Remark 3.7. *Take $U = V = I$ in Theorem 3.4, the result is improved of Theorem 1 in [17]. Reference [17] called for the two pairs of mappings are weak commutative, but where will it weakened into a pair of compatible and a pair of sub-compatible.*

Remark 3.8. *Take $U = V = I$ in Theorem 3.4, the result is further improved of the corresponding result in [16]. Reference [16] called for the two pairs of mappings are weak commutative, but where will it weakened into a pair of compatible and a pair of sub-compatible.*

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