Thai Journal of Mathematics
Volume 10 (2012) Number 1 : 59–73
**www.math.science.cmu.ac.th/thaijournal**
Online ISSN 1686-0209

# Iterative Algorithms for Pseudocontraction Semigroups in Banach Spaces[1]

**Rabian Wangkeeree[2] and Pakkapon Preechasilp**

Department of Mathematics, Faculty of Science
Naresuan University, Phitsanulok 65000, Thailand
e-mail : rabianw@nu.ac.th (R. Wangkeeree)
preechasilpp@gmail.com (P. Preechasilp)

**Abstract :** In this paper, a Moudafi's viscosity approximation method with continuous strong pseudocontractions for a pseudocontraction semigroup is considered. A strong convergence theorem is established in the framework of a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm. We also propose the modified implicit iteration for a pseudocontraction semigroup and prove the strong convergence theorem. The results presented in this paper mainly improved and extended the corresponding results announced by Song and Xu [Y. Song, S. Xu, Strong convergence theorems for nonexpansive semigroup in Banach spaces, J. Math. Anal. Appl. 338 (2008) 152–161] and Song and Chen [Y. Song, R. Chen, Convergence theorems of iterative algorithms for continuous pseudocontractive mappings, Nonlinear Anal. 67 (2007) 486–497].

**Keywords :** Pseudocontraction semigroup; Viscosity approximation method; Variational inequality; Fixed point; Banach space.

**2010 Mathematics Subject Classification :** 47H09; 47H10; 47H17.

---

[1]This research was supported by Naresuan university
[2]Corresponding author email: rabianw@nu.ac.th (R. Wangkeeree)

# 1 Introduction

Let $E$ be a Banach space with norm $\|\cdot\|$ and let $J$ be the normalized duality mapping from $E$ into $2^{E^*}$ given by

$$Jx = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|\|x^*\|, \|x\| = \|x^*\|\}, \forall x \in C,$$

where $E^*$ denotes the dual space of $E$ and $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing between $E$ and $E^*$. In what follows, we denote a single valued normalized duality mapping by $j$.

Let $C$ be a nonempty closed convex subset of a Banach space $E$ and $T$ a nonlinear mapping. From now on, we use $F(T)$ to denote the fixed point set of $T$. Now we recall the following :

A mapping $T$ is said to be *pseudocontractive* if there exists some $j(x-y) \in J(x-y)$ such that $\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2, \forall x, y \in C$.

$T$ is said to be *strongly pseudocontractive* if there exists a constant $\alpha \in (0,1)$ such that $\langle Tx-Ty, j(x-y)\rangle \leq \alpha\|x-y\|^2, \forall x, y \in C$ for some $j(x-y) \in J(x-y)$.

$T$ is said to be *Lipschitz* if there exists a constant $L > 0$ such that $\|Tx-Ty\| \leq L\|x-y\|, \forall x, y \in C$.

If $L = 1$, then $T$ is said to be *nonexpansive*.

The class of pseudocontractions is one of most important classes of mappings among nonlinear mappings. Many authors have been devoted to the existence and convergence of fixed points for pseudocontractions. In 1974, Deimling [1] proved the following existence result for continuous strong pseudocontractions in Banach spaces.

**Theorem 1.1** ([1]). *Let $E$ be a Banach space, $C$ be a nonempty closed convex subset of $E$ and $T : C \to C$ be a continuous and strong pseudocontraction. Then $T$ has a unique fixed point in $C$.*

A pseudocontraction semigroup is a family $\mathcal{F} = \{T(t) : t \geq 0\}$ of self-mapping of $C$ such that

(i) $T(0)x = x$ for all $x \in C$;

(ii) $T(s+t) = T(s)T(t)$ for all $s, t > 0$;

(iii) $\lim_{t\to 0^+} T(t)x = x$ for all $x \in C$;

(iv) for each $t > 0, T(t)$ is pseudocontractive; that is,

$$\langle T(t)x - T(t)y, j(x-y)\rangle \leq \|x-y\|^2, \forall x, y \in C.$$

We use $\Omega$ to denote the set of common fixed points of $\mathcal{F}$; that is,

$$\Omega := \{x \in C : T(t)x = x, t > 0\} = \cap_{t>0} F(T(t)).$$

Note that the class of pseudocontractive semigroups includes the class of nonexpansive semigroups as a special case. One classical way to study nonexpansive mappings is to use contractions to approximate a nonexpansive mappings. More precisely, take $t \in (0,1)$ and define a contraction $T_t : C \to C$ by

$$T_t = tu + (1-t)Tx, \ \ \forall x \in C,$$

where $u \in C$ is a fixed point. Banach's Contraction Mapping Principle guarantees that $T_t$ has a unique fixed point $x_t$ in $C$. It is unclear, in general, what the behavior of $x_t$ is as $t \to 0$, even if $T$ has a fixed point. However, in the case of $T$ having a fixed point, Browder [2] proved if $E$ is a Hilbert spaces, then $x_t$ converges strongly to a fixed point of $T$ that is nearest to $u$. Reich [3] extended Browder's result to setting of Banach spaces and proved that if $E$ is a uniformly smooth Banach space, then $x_t$ converges strongly to a fixed point of $T$ and the limit defines the (unique)sunny nonexpansive retraction from $C$ onto $F(T)$. It is an interesting problem to extend Browder's and Reich's results to the contraction semigroup case. In 2003, Suzuki [4] is the first to introduce in a Hilbert space the following iteration process:

$$x_n = \alpha_n u + (1-\alpha_n)T(t_n)x_n, \ \ \forall n \geq 1, \tag{1.1}$$

where $\{T(t) : t \geq 0\}$ is a strongly continuous semigroup of nonexpansive mappings on $C$ such that $\cap_{t\geq 0} F(T(t)) \neq \emptyset$ and $\{\alpha_n\}$ and $\{t_n\}$ are appropriate sequences of real numbers. Xu [5] extended Suzuki's results from Hilbert spaces to uniformly convex Banach spaces. In 2002, Benavides et al. [6] in a uniformly smooth Banach space, showed that if $\mathcal{F}$ satisfies an asymptotic regularity condition and $\{\alpha_n\}$ fulfills the control conditions $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 0$, then $\{x_n\}$ generated by (1.1) converges to a point of $\Omega$. Using Moudafi's viscosity approximation methods, Song and Xu [7] introduced the following iteration process:

$$x_n = \alpha_n f(x_n) + (1-\alpha_n)T(t_n)x_n, \ \ \forall n \geq 1,$$

where $\{T(t) : t \geq 0\}$ is a nonexpansive semigroup from $C$ into itself which satisfies an asymptotic regularity condition and $\cap_{t>0} F(T(t)) \neq \emptyset$, $f : C \to C$ is a fixed contraction with the coefficient $\alpha \in (0,1)$ and $\{\alpha_n\}$ and $\{t_n\}$ are sequences of real numbers such that $0 < \alpha_n < 1, t_n > 0$ and $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} t_n = \infty$. It is proved in [7] that $\{x_n\}$ converges strongly to $x^* \in \Omega$ with solves the following variational inequality:

$$\langle (I-f)x^*, j(x^*-x)\rangle \leq 0, \ \ \forall x \in \cap_{t>0} F(T(t)).$$

Furthermore, Moudafi's viscosity approximation methods have been recently studies by many authors; see the well known results in [8, 9]. However, the involved mapping $f$ is usually considered as a contraction. Note that Suzuki [10] proved the equivalence between Moudafi's viscosity approximation with contractions and Browder-type iterative processes (Halpern-type iterative processes); see [10] for more details.

In [11], Song and Chen considered the following iterative algorithm for a continuous pseudocontractive mapping $T$ on $C$ in a real reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm:

$$\begin{cases} y_n = \beta_n f(x_{n-1}) + (1-\beta_n)x_{n-1}, \\ x_n = \alpha_n y_n + (1-\alpha_n)Tx_n, \text{ for all } n \geq 1. \end{cases} \tag{1.2}$$

where $f : C \to C$ is a fixed contractive mapping, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers such that $\{\alpha_n\}, \{\beta_n\} \subseteq (0,1], t_n > 0$ with $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \beta_n = \infty$. Then $\{x_n\}$ converges strongly to $x^* \in \Omega$ with solves the following variational inequality:

$$\langle (I-f)x^*, j(x^*-x)\rangle \leq 0, \ \ \forall x \in F(T).$$

The purpose of this paper is to consider a pseudocontraction semigroup based on Moudafi's viscosity approximation with continuous strong pseudocontractions in the framework of a reflexive strictly convex Banach space with a uniformly Gâteaux differentiable norm. We also propose the modified implicit iteration for a pseudocontraction semigroup and prove the strong convergence theorem. The results presented in this paper mainly improved and extended the corresponding results announced by Song and Xu [7] and Song and Chen [11] and many others.

# 2 Preliminaries

Throughout this paper, let $E$ be a real Banach space and $E^*$ its dual space. We write $x_n \rightharpoonup x$ (respectively $x_n \rightharpoonup^* x$) to indicate that the sequence $\{x_n\}$ weakly (respectively weak*) converges to $x$; as usual $x_n \to x$ will symbolize strong convergence. Let $S(E) = \{x \in E : \|x\| = 1\}$ denote the unit sphere of a Banach space $E$. A Banach space $E$ is said to have *a Gâteaux differentiable norm* (we also say that E is smooth), if the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.1}$$

exists for each $x, y \in S(E)$. A Banach space $E$ is said to have *a uniformly Gâteaux differentiable norm* if for each $y$ in $S(E)$, the limit (2.1) is uniformly attained for $x \in S(E)$; *a Fréchet differentiable norm* if for each $x \in S(E)$, the limit (2.1) is attained uniformly for $y \in S(E)$; *a uniformly Fréchet differentiable norm* (we also say that $E$ is uniformly smooth) if the limit (2.1) is attained uniformly for $(x,y) \in S(E) \times S(E)$. A Banach space $E$ is said to *strictly convex* if $\frac{\|x+y\|}{2} < 1$ for $x, y \in S(E)$, $x \neq y$; *uniformly convex* if, for any $\varepsilon \in (0,2]$, there exists $\delta > 0$ such that, for any $x, y \in S(E)$, $\|x-y\| \geq \varepsilon$ implies $\|\frac{x+y}{2}\| \leq 1-\delta$. It is well known that the normalized duality mapping $J$ in a Banach space $E$ with a uniformly Gâteaux differentiable norm is single-valued and strong-weak* uniformly continuous on any bounded subset of $E$; each uniformly convex Banach space $E$

is reflexive and strictly convex and has fixed point property for nonexpansive self-mappings. Further, every uniformly smooth Banach space $E$ is a reflexive Banach space with a uniformly Gâteaux differentiable norm and has fixed point property for nonexpansive self-mappings (see [12, 13]).

Now, we present the concept of uniformly asymptotically regular semigroup (also see [14, 15]). Let $C$ be a nonempty closed convex subset of a Banach space $E$, $\mathcal{F} = \{T(t) : 0 \leq t < \infty\}$ a continuous operator semigroup on $C$. Then $\mathcal{F}$ is said to be *uniformly asymptotically regular* (in short, u.a.r.) on $C$ if for all $h \geq 0$ and any bounded subset $B$ of $C$,

$$\lim_{t\to\infty} \sup_{x\in B} \|T(h)(T(t)x) - T(t)x\| = 0.$$

The examples of u.a.r. operator semigroup can be found in [14, Examples 17, 18] and [16, Lemma 2.7].

**Example 2.1** ([14, Example 17])**.** *Let $T$ be a linear firmly nonexpansive self-operator on a nonempty convex conpact subset $C$ of a Hilbert space $H$, let $G = \mathbb{N}$, and $\Gamma = \{T^n : n \in G\}$ be a semigroup of iterates of $T$. It is known [17] that if $C = -C$ and $T$ is odd, then $\{T^n x\}_{n\geq 0}$ converges strongly for all $x \in C$. Fix $\varepsilon > 0$. Then there exist $x_1, x_2, \ldots, x_k \in C$ such that $C \subset \bigcup_{i=1}^{k} B(x_i, \varepsilon)$, where*

$$B(x_i, \varepsilon) = \{x \in H : \|x - x_i\| < \varepsilon\},$$

*for all $i = 1, 2, \ldots, k$, and $n_0$ such that*

$$\|T^n x_i - T^m x_i\| < \varepsilon,$$

*for all $n, m > n_0, i = 1, 2, \ldots, k$, Take $x \in C$ and $x_i$ such that $\|x - x_i\| \leq \varepsilon$. Then*

$$\|T^n x - T^m x\| \leq \|T^n(x - x_i)\| + \|(T^n - T^m)x_i\| + \|T^m(x - x_i)\| \leq \varepsilon,$$

*for all $n, m > n_0$. That is, $\Gamma$ is a uniformly asymptotically regular semigroup of iterates of $T$.*

**Example 2.2** ([14, Example 18])**.** *Let the following assumptions hold. $C$ is a nonempty bounded closed convex subset of a Hilbert space $H$, $T : C \to C$ is a contraction operator with Lipschitz constant $k < 1, G = \mathbb{N}$, and $\Gamma = \{T^n : n \in G\}$ is a semigroup of iterates of $T$. For all $n, m \in G$ we have*

$$\begin{aligned}\|T^{m+n}x - T^n x\| &= \sum_{i=0}^{m-1} \|T^{n+i+1}x - T^{n+i}x\| \\ &= \sum_{i=0}^{m-1} k^{n+i}\|Tx - x\| \\ &= \frac{k^n}{1-k}\|Tx - x\|,\end{aligned}$$

*therefore,*

$$\lim_{n\to\infty}\left(\sup_{x\in C}\|T^{m+n}x - T^n x\|\right) = 0,$$

*uniformly for all* $m \in G$*. That is,* $\Gamma$ *is a uniformly asymptotically regular semigroup of iterates of* $T$*.*

In order to prove our main result, we need the following lemmas and definitions. Let $l^\infty$ be the Banach space of all bounded real-valued sequences. Let $\mu$ be a continuous linear functional on $l^\infty$ satisfying $\|\mu\| = 1 = \mu(1)$. Then we know that $\mu$ is mean on $\mathbb{N}$ if and only if

$$\inf\{a_n : n \in \mathbb{N}\} \le \mu(a) \le \sup\{a_n : n \in \mathbb{N}\}$$

for every $a = (a_1, a_2, \dots) \in l^\infty$. Occasionally, we shall use $\mu_n(a_n)$ instead of $\mu(a)$. A mean $\mu$ on $\mathbb{N}$ is called a Banach limit if

$$\mu_n(a_n) = \mu_n(a_{n+1})$$

for every $a = (a_1, a_2, \dots) \in l^\infty$. Using the Hahn-Banach theorem, or the Tychonoff fixed point theorem, we can prove the existence of a Banach limit. We know that if $\mu$ is a Banach limit, then

$$\liminf_{n\to\infty} a_n \le \mu_n(a_n) \le \limsup_{n\to\infty} a_n$$

for every $a = (a_1, a_2, \dots) \in l^\infty$. Subsequently, the following result was showed in [18, Lemma 1]

**Lemma 2.1** ([18])**.** *Let* $C$ *be a nonempty closed convex subset of a Banach space* $E$ *with uniformly Gâteaux differentiable norm. Let* $\{x_n\}$ *be a bounded sequence of* $E$ *and let* $\mu$ *be a mean on* $\mathbb{N}$*. Let* $z \in C$*. Then*

$$\mu_n\|x_n - z\|^2 = \min_{y\in C}\mu_n\|x_n - y\|^2 \Leftrightarrow \mu_n\langle y - z, j(x_n - z)\rangle \le 0, \forall y \in C.$$

**Proposition 2.2** ([7, Proposition 3.1])**.** *Let* $E$ *be a reflexive strictly convex Banach space with a uniformly Gâteaux differentiable norm, and* $C$ *a nonempty closed convex subset of* $E$*. Suppose* $\{x_n\}$ *is a bounded sequence in* $C$ *such that* $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$*, an approximation fixed point of nonexpansive self-mapping* $T$ *on* $C$*. Define the set*

$$K = \left\{x^* \in C : \mu_n\|x_n - x^*\|^2 = \inf_{y\in C}\mu_n\|x_n - y\|^2\right\}.$$

*If* $F(T) \neq \emptyset$*, then* $K \cap F(T) \neq \emptyset$*.*

**Lemma 2.3.** *Let* $C$ *be a nonempty closed convex subset of a real Banach space* $E$ *and* $T : C \to C$ *be a continuous pseudocontractive map. We denote* $A = (2I - T)^{-1}$*. Then* *(i) [19, Theorem 6] The map $A$ is a nonexpansive self-mapping on $C$.*

*(ii) [11, Lemma 1.1] If $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, then $\lim_{n\to\infty} \|x_n - Ax_n\| = 0$.*

In the following, we also need the following lemma that can be found in the existing literature [5, 9].

**Lemma 2.4** ([5, Lemma 2.1])**.** *Let $\{a_n\}$ be a sequence of non-negative real number satisfying the property*

$$a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n\beta_n, \quad n \geq 0,$$

*where $\{\gamma_n\} \subseteq (0,1)$ and $\{\beta_n\} \subseteq \mathbb{R}$ such that $\sum_{n=0}^{\infty} \gamma_n = \infty$ and $\limsup_{n\to\infty} \beta_n \leq 0$. Then $\{a_n\}$ converges to zero, as $n \to \infty$.*

# 3 Viscosity Iterative Algorithm

Now, we are a position to state and prove our main results.

**Theorem 3.1.** *Let $E$ be a real reflexive strictly convex Banach space with a uniformly Gâteaux differentiable norm, and $C$ a nonempty closed convex subset of $E$. Let $\{T(t) : t \geq 0\}$ be an u.a.r. continuous L-Lipschitz pseudocontraction semigroup on $C$ such that $\Omega \neq \emptyset$. Let $f : C \to C$ be a fixed Lipschitz strong pseudocontraction with pseudocontractive coefficient $\alpha \in (0,1)$. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers such that $0 < \alpha_n < 1, t_n > 0$, $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} t_n = \infty$. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$x_n = \alpha_n f(x_n) + (1-\alpha_n)T(t_n)x_n, \quad \forall n \geq 1. \tag{3.1}$$

*Then $\{x_n\}$ converges strongly to $x^* \in \Omega$ with solves the following variational inequality:*

$$\langle (I-f)x^*, j(x^* - x)\rangle \leq 0, \quad \forall x \in \Omega. \tag{3.2}$$

*Proof.* First, we show that the sequence $\{x_n\}$ generated in (3.1) is well defined. For any $n \geq 1$, define a mapping $T_n$ as follows

$$T_n x = \alpha_n f(x) + (1-\alpha_n)T(t_n)x, \quad \forall x \in C.$$

Notice that

$$\begin{aligned}
&\langle T_n x - T_n y, j(x-y)\rangle \\
&\quad = \langle \alpha_n f(x) + (1-\alpha_n)T(t_n)x - \alpha_n f(y) - (1-\alpha_n)T(t_n)y, j(x-y)\rangle \\
&\quad = \alpha_n \langle f(x) - f(y), j(x-y)\rangle + (1-\alpha_n)\langle T(t_n)x - T(t_n)y, j(x-y)\rangle \\
&\quad \leq \alpha_n \alpha \|x-y\|^2 + (1-\alpha_n)\|x-y\|^2 \\
&\quad = (1-\alpha_n(1-\alpha))\|x-y\|^2, \quad \forall x, y \in C.
\end{aligned}$$

Hence $T_n$ is continuous and strong pseudocontraction with the coefficient $1-\alpha_n(1-\alpha)$. From Theorem 1.1, one sees that $T_n$ has a unique fixed point, denoted as $x_n$, which uniquely solves the fixed point equation $x_n = \alpha_n f(x_n) + (1-\alpha_n)T(t_n)x_n$. That is, (3.1) is well defined. Next, we show that $\{x_n\}$ is bounded. Indeed, for any fixed $q \in \Omega$,

$$\begin{aligned}\|x_n - q\|^2 &= \alpha_n\langle f(x_n) - f(q), j(x_n - q)\rangle + \alpha_n\langle f(q) - q, j(x_n - q)\rangle \\ &\quad + (1-\alpha_n)\langle T(t_n)x_n - T(t_n)q, j(x_n - q)\rangle \\ &\le \alpha_n\alpha\|x_n - q\|^2 + \alpha_n\langle f(q) - q, j(x_n - q)\rangle + (1-\alpha_n)\|x_n - q\|^2.\end{aligned}$$

Therefore

$$\|x_n - q\|^2 \le \frac{1}{1-\alpha}\langle f(q) - q, j(x_n - q)\rangle \le \frac{1}{1-\alpha}\|f(q) - q\|\|x_n - q\|. \tag{3.3}$$

Thus $\|x_n - q\| \le \frac{1}{1-\alpha}\|f(q) - q\|$. This implies that $\{x_n\}$ is bounded. Since $f$ is Lipschitz, we also have $\{f(x_n)\}$ is bounded. Since $\lim_{n\to\infty}\alpha_n = 0$, there exists $N_0$ and $a \in (0,1)$ such that $\alpha \le a$ for all $n \ge N_0$. By $x_n = \alpha_n f(x_n) + (1-\alpha_n)T(t_n)x_n$, $\forall n \ge 1$, we obtain

$$T(t_n)x_n = \frac{1}{1-\alpha_n}x_n - \frac{\alpha_n}{1-\alpha_n}f(x_n).$$

Thus,

$$\begin{aligned}\|T(t_n)x_n\| &\le \frac{1}{1-\alpha_n}\|x_n\| - \frac{\alpha_n}{1-\alpha_n}\|f(x_n)\| \\ &\le \frac{1}{1-a}\|x_n\| - \frac{\alpha_n}{1-a}\|f(x_n)\|.\end{aligned}$$

Therefore, the set $\{T(t_n)x_n\}$ is bounded. This implies that

$$\lim_{n\to\infty}\|x_n - T(t_n)x_n\| = \lim_{n\to\infty}\alpha_n\|T(t_n)x_n - f(x_n)\| = 0.$$

Since $\{T(t)\}$ is u.a.r. L-Lipschitz semigroup and $\lim_{n\to\infty} t_n = \infty$, then for all $h > 0$,

$$\lim_{n\to\infty}\|T(h)(T(t_n)x_n) - T(t_n)x_n\| \le \lim_{n\to\infty}\sup_{x\in B}\|T(h)(T(t_n)x) - T(t_n)x\| = 0$$

where $B$ is any bounded subset of $C$ containing $\{x_n\}$. Hence

$$\begin{aligned}\|x_n - T(h)x_n\| &\le \|x_n - T(t_n)x_n\| + \|T(t_n)x_n - T(h)(T(t_n)x_n)\| \\ &\quad + \|T(h)(T(t_n)x_n) - T(h)x_n\| \\ &\le (1+L)\|T(t_n)x_n - x_n\| + \|T(t_n)x_n - T(h)(T(t_n)x_n)\|.\end{aligned}$$

Therefore

$$\lim_{n\to\infty}\|x_n - T(h)x_n\| = 0 \text{ for all } h > 0. \tag{3.4}$$

By Lemma 2.3, we get that the mapping $A(h) := (2I - T(h))^{-1} : C \to C$ is nonexpansive such that $F(A(h)) = F(T(h))$ and so $\cap_{h>0}F(A(h)) = \cap_{h>0}F(T(h)) \neq \emptyset$ and $\lim_{n\to\infty} \|x_n - A(h)x_n\| = 0, \forall h > 0$, where $I$ denotes the identity operator. We claim that the set $\{x_n\}$ is sequentially compact. Indeed, define the set

$$K = \left\{x \in C : \mu_n\|x_n - x\|^2 = \inf_{y\in C} \mu_n\|x_n - y\|^2\right\}.$$

By Proposition 2.2, there exists $x^* \in K$ such that $A(h)x^* = x^*$ for all $h > 0$. It implies that $x^* \in \cap_{h>0}F(T(h))$. Thus $\mu_n\|x_n - x^*\|^2 = \inf_{y\in C} \mu_n\|x_n - y\|^2$. By Lemma 2.1,

$$\mu_n\langle y - x^*, j(x_n - x^*)\rangle \le 0, \text{ for all } y \in C.$$

From (3.3), we have $\mu_n\|x_n - x^*\|^2 \le \frac{1}{1-\alpha}\mu_n\langle f(x^*) - x^*, j(x_n - x^*)\rangle \le 0$. That is, $\mu_n\|x_n - x^*\| = 0$. Hence, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x^*$ as $k \to \infty$. Next, we show that $x^*$ is a solution in $\Omega$ to the variational inequality (3.2). In fact, for any fixed $x \in \Omega$, we have

$$\begin{aligned}
&\langle x_n - f(x_n), j(x_n - x)\rangle \\
&\qquad = (1 - \alpha_n)\langle T(t_n)x_n - x_n, j(x_n - x)\rangle \\
&\qquad = (1 - \alpha_n)[\langle T(t_n)x_n - x, j(x_n - x)\rangle - \langle x_n - x, j(x_n - x)\rangle] \\
&\qquad = (1 - \alpha_n)[\langle T(t_n)x_n - T(t_n)x, j(x_n - x)\rangle - \langle x_n - x, j(x_n - x)\rangle] \\
&\qquad \le (1 - \alpha_n)[\|x_n - x\|^2 - \|x_n - x\|^2] = 0.
\end{aligned} \tag{3.5}$$

In particular, we have

$$\langle x_{n_k} - f(x_{n_k}), j(x_{n_k} - x)\rangle \le 0. \tag{3.6}$$

Since $E$ has a uniformly Gâteaux differential norm, we know that $j$ is norm-to-weak$^*$ uniformly continuous on any bounded subset of $E$. Taking limit in (3.6), one can obtain that

$$\langle x^* - f(x^*), j(x^* - x)\rangle \le 0, \forall x \in \Omega. \tag{3.7}$$

Let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ such that $x_{n_j} \to y^*$. From (3.4), we have

$$\|T(h)x_{n_j} - y^*\| \le \|T(h)x_{n_j} - x_{n_j}\| + \|x_{n_j} - y^*\| \to 0.$$

That is, $y^* \in \Omega$. It follows from (3.7) that

$$\langle x^* - f(x^*), j(x^* - y^*)\rangle \le 0. \tag{3.8}$$

On the other hand, one sees from (3.5) that

$$\langle x_{n_j} - f(x_{n_j}), j(x_{n_j} - x^*)\rangle \le 0. \tag{3.9}$$

Taking limit in (3.9), one can obtain that

$$\langle y^* - f(y^*), j(y^* - x^*)\rangle \leq 0. \tag{3.10}$$

Adding up (3.8) and (3.10), one arrives at

$$\langle x^* - y^* + f(y^*) - f(x^*), j(x^* - y^*)\rangle \leq 0,$$

which yields that

$$\|x^* - y^*\|^2 \leq \langle f(x^*) - f(y^*), j(x^* - y^*)\rangle \leq \alpha\|x^* - y^*\|^2.$$

So $(1-\alpha)\|x^* - y^*\|^2 \leq 0$. Since $\alpha \in (0,1)$, we get that $\|x^* - y^*\| = 0$. Thus $x^* = y^*$. Hence $\{x_n\}$ converge strongly to $x^* \in \Omega$, which is the unique solution to the variational inequality (3.2). This completes the proof. □

If $\{T(t) : t \geq 0\}$ is an u.a.r. nonexpansive semigroup from $C$ into itself and $f : C \to C$ is a fixed contractive mapping, then we obtain the following result.

**Corollary 3.2** ([7, Theorem 3.2])**.** *Let $E$ be a real reflexive strictly convex Banach space with a uniformly Gâteaux differentiable norm, and $C$ a nonempty closed convex subset of $E$, and $\{T(t) : t \geq 0\}$ an u.a.r. nonexpansive semigroup from $C$ into itself such that $\Omega \neq \emptyset$ and $f : C \to C$ a fixed contractive mapping with contractive coefficient $k \in (0,1)$. Suppose $\lim_{n\to\infty} t_n = \infty$ and $\alpha_n \in (0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$. If $\{x_n\}$ is defined by*

$$x_n = \alpha_n f(x_n) + (1-\alpha_n)T(t_n)x_n, \quad \forall n \geq 1.$$

*Then as $n \to \infty$, $\{x_n\}$ converges strongly to some common fixed point $x^*$ of $\Omega$ which is the unique solution in $\Omega$ to the following variational inequality:*

$$\langle (I-f)x^*, j(x^* - x)\rangle \leq 0, \quad \forall x \in \Omega.$$

# 4 Modified Implicit Iteration Scheme

**Theorem 4.1.** *Let $E$ be a real reflexive strictly convex Banach space with a uniformly Gâteaux differentiable norm, and $C$ a nonempty closed convex subset of $E$. Let $\{T(t) : t \geq 0\}$ be an u.a.r. continuous L-Lipschitz pseudocontraction semigroup of $C$ into itself such that $\Omega \neq \emptyset$. Let $f : C \to C$ be a fixed contractive mapping with the coefficient $k \in (0,1)$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{t_n\}$ be sequences of real numbers such that $\{\alpha_n\}, \{\beta_n\} \subseteq (0,1], t_n > 0$ with $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} t_n = \infty$ and $\sum_{n=0}^{\infty} \beta_n = \infty$. For $x_0 \in C$, let the sequence $\{x_n\}$ be generated by:*

$$\begin{cases} y_n = \beta_n f(x_{n-1}) + (1-\beta_n)x_{n-1}, \\ x_n = \alpha_n y_n + (1-\alpha_n)T(t_n)x_n, \ \text{for all } n \geq 1. \end{cases} \tag{4.1}$$

*Then $\{x_n\}$ converges strongly to $x^* \in \Omega$ with solves the following variational inequality (3.2).*

*Proof.* Firstly, we prove that $\{x_n\}$ is well defined. In fact, for each $n \in \mathbb{N}$, define the mapping $G_n : C \to C$ by

$$G_n x = \alpha_n(\beta_n f(x_{n-1}) + (1-\beta_n)x_{n-1}) + (1-\alpha_n)T(t_n)x,$$

for all $x \in C$. Then, for any $y, z \in C$,

$$\begin{aligned}
&\langle G_n y - G_n z, j(y-z)\rangle \\
&\qquad = \langle \alpha_n(\beta_n f(x_{n-1}) + (1-\beta_n)x_{n-1}) + (1-\alpha_n)T(t_n)y \\
&\qquad\qquad - \alpha_n(\beta_n f(x_{n-1}) + (1-\beta_n)x_{n-1}) - (1-\alpha_n)T(t_n)z, j(y,z)\rangle \\
&\qquad = \langle (1-\alpha_n)(T(t_n)y - T(t_n)z), j(y-z)\rangle \le (1-\alpha_n)\|y-z\|^2.
\end{aligned}$$

Hence $G_n$ is continuous and strong pseudocontraction. From Theorem 1.1, there exists a unique fixed point, denoted as $x_n$, which uniquely solves the fixed point equation

$$x_n = \alpha_n(\beta_n f(x_{n-1}) + (1-\beta_n)x_{n-1}) + (1-\alpha_n)T(t_n)x_n, \text{ for all } n \ge 1.$$

That is, $\{x_n\}$ is well defined. Next, we show that $\{x_n\}$ is bounded. Let $q \in \Omega$, we have

$$\begin{aligned}
\|x_n - q\|^2 &= \langle \alpha_n y_n + (1-\alpha_n)T(t_n)x_n - q, j(x_n - q)\rangle \\
&= (1-\alpha_n)\langle T(t_n)x_n - T(t_n)q, j(x_n-q)\rangle + \alpha_n\langle y_n - q, j(x_n-q)\rangle \\
&\le (1-\alpha_n)\|x_n - q\|^2 + \alpha_n\|y_n - q\|\|x_n - q\|,
\end{aligned}$$

and hence $\|x_n - q\|^2 \le \|y_n - q\|\|x_n - q\|$. Therefore

$$\begin{aligned}
\|x_n - q\| \le \|y_n - q\| &\le \beta_n\|f(x_{n-1}) - q\| + (1-\beta_n)\|x_{n-1} - q\| \\
&\le \beta_n(\|f(x_{n-1}) - f(q)\| + \|f(q) - q\|) + (1-\beta_n)\|x_{n-1} - q\| \\
&= (1-(1-k)\beta_n)\|x_{n-1} - q\| + \beta_n\|f(q) - q\| \\
&\le \max\left\{\|x_{n-1} - q\|, \frac{1}{1-k}\|f(q) - q\|\right\}.
\end{aligned}$$

By induction, we get that

$$\|x_n - q\| \le \max\left\{\|x_0 - q\|, \frac{\|f(q) - q\|}{1-k}\right\}, \ \ \forall\, n \ge 0.$$

Hence $\{x_n\}$ is bounded, so are $\{y_n\}$, $\{f(x_n)\}$ and $\{T(t_n)x_n\}$. This implies that

$$\lim_{n\to\infty}\|x_n - T(t_n)x_n\| = \lim_{n\to\infty}\alpha_n\|y_n - T(t_n)x_n\| = 0.$$

Since $\{T(t)\}$ is u.a.r. and $\lim_{n\to\infty} t_n = \infty$, then for all $h > 0$,

$$\lim_{n\to\infty}\|T(h)(T(t_n)x_n) - T(t_n)x_n\| \le \lim_{n\to\infty}\sup_{x\in B}\|T(h)(T(t_n)x) - T(t_n)x\| = 0,$$

where $B$ is any bounded subset of $C$ containing $\{x_n\}$. Hence

$$\begin{aligned}\|x_n - T(h)x_n\| &\le \|x_n - T(t_n)x_n\| + \|T(t_n)x_n - T(h)(T(t_n)x_n)\| \\ &\quad + \|T(h)(T(t_n)x_n) - T(h)x_n\| \\ &\le (1+L)\|T(t_n)x_n - x_n\| + \|T(h)(T(t_n)x_n) - T(t_n)x_n\| \to 0. \end{aligned} \tag{4.2}$$

For each $m \in \mathbb{N}$, putting $z_m = \alpha_m f(z_m) + (1-\alpha_m)T(t_m)z_m$, where $\{t_m\}$ and $\{\alpha_m\}$ satisfies the condition of Theorem 3.1. It follows from Theorem 3.1 that as $m \to \infty$, $\{z_m\}$ converges strongly to some fixed point $x^*$ in $\Omega$ which is the unique solution to the variational inequality (3.2). Next, we show that

$$\limsup_{n\to\infty} \langle f(x^*) - x^*, j(x_n - x^*)\rangle \le 0. \tag{4.3}$$

$$\begin{aligned}\|z_m - x_n\|^2 &= \langle \alpha_m(f(z_m) - x_n) + (1-\alpha_m)(T(t_m)z_m - x_n), j(z_m - x_n)\rangle \\ &= (1-\alpha_m)\langle T(t_m)z_m - x_n, j(z_m - x_n)\rangle + \alpha_m\langle f(z_m) - x_n, j(z_m - x_n)\rangle \\ &= (1-\alpha_m)\langle T(t_m)z_m - T(t_m)x_n, j(z_m - x_n)\rangle \\ &\quad + (1-\alpha_m)\langle T(t_m)x_n - x_n, j(z_m - x_n)\rangle \\ &\quad + \alpha_m\langle f(z_m) - z_m - (f(x^*) - x^*), j(z_m - x_n)\rangle \\ &\quad + \alpha_m\langle f(x^*) - x^*, j(z_m - x_n)\rangle + \alpha_m\langle z_m - x_n, j(z_m - x_n)\rangle \\ &\le \|x_n - z_m\|^2 + \|T(t_m)x_n - x_n\|M + \alpha_m\langle f(x^*) - x^*, j(z_m - x_n)\rangle \\ &\quad + M\alpha_m(\|f(z_m) - f(x^*)\| + \|z_m - x^*\|), \\ &\le \|x_n - z_m\|^2 + \|T(t_m)x_n - x_n\|M + \alpha_m\langle f(x^*) - x^*, j(z_m - x_n)\rangle \\ &\quad + M\alpha_m(1+k)\|z_m - x^*\|, \end{aligned}$$

and hence

$$\langle f(x^*) - x^*, j(x_n - z_m)\rangle \le \frac{\|T(t_m)x_n - x_n\|}{\alpha_m}M + M(1+k)\|z_m - x^*\|, \tag{4.4}$$

where $M$ is a constant satisfying $M \ge \|x_n - z_m\|$ for all $n, m \in \mathbb{N}$. Therefore, taking upper limit as $n \to \infty$ firstly, and then as $m \to \infty$ in (4.4), we have

$$\limsup_{m\to\infty}\limsup_{n\to\infty}\langle f(x^*) - x^*, j(x_n - z_m)\rangle \le 0$$

by the inequality (4.2). On the other hand, since $\lim_{m\to\infty} z_m = x^*$ due to the fact the duality mapping $J$ is norm-to-weak$^*$ uniformly continuous on bounded subset of $E$, it follows that as $m \to \infty$,

$$\langle f(x^*) - x^*, j(x_n - z_m)\rangle \to \langle f(x^*) - x^*, j(x_n - x^*)\rangle \text{ uniformly.}$$

Then for any given $\varepsilon > 0$, there exists a natural number $N$ such that for each $m \ge N$,

$$\langle f(x^*) - x^*, j(x_n - x^*)\rangle < \langle f(x^*) - x^*, j(x_n - z_m)\rangle + \varepsilon.$$

Taking upper limit as $n \to \infty$ firstly, and then as $m \to \infty$ in the last inequality, we have

$$\limsup_{n\to\infty}\langle f(x^*) - x^*, j(x_n - x^*)\rangle \le \limsup_{m\to\infty}\limsup_{n\to\infty}\langle f(x^*) - x^*, j(x_n - z_m)\rangle + \varepsilon \le \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, (4.3) is proved. Finally we show that $x_n \to x^*(n \to \infty)$. Indeed, we get that

$$\begin{aligned}\|x_n - x^*\|^2 &= (1-\alpha_n)\langle T(t_n)x_n - x^*, j(x_n - x^*)\rangle + \alpha_n\langle y_n - x^*, j(x_n - x^*)\rangle \\ &\le (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n(1-\beta_n)\langle x_{n-1} - x^*, j(x_n - x^*)\rangle \\ &\quad + \alpha_n\beta_n\langle f(x_{n-1}) - x^*, j(x_n - x^*)\rangle.\end{aligned}$$

Therefore,

$$\begin{aligned}\|x_n - x^*\|^2 &\le (1-\beta_n)\|x_{n-1} - x^*\|\|j(x_n - x^*)\| + \beta_n\langle f(x^*) - x^*, j(x_n - x^*)\rangle \\ &\quad + \beta_n\langle f(x_{n-1}) - f(x^*), j(x_n - x^*)\rangle \\ &\le (1-\beta_n)\|x_{n-1} - x^*\|\|x_n - x^*\| + \beta_n\langle f(x^*) - x^*, j(x_n - x^*)\rangle \\ &\quad + \beta_n\|f(x_{n-1}) - f(x^*)\|\|x_n - x^*\| \\ &\le (1-\beta_n)\frac{\|x_{n-1} - x^*\|^2 + \|x_n - x^*\|^2}{2} + \beta_n\langle f(x^*) - x^*, j(x_n - x^*)\rangle \\ &\quad + (1-\beta_n)\frac{k^2\|x_{n-1} - x^*\|^2 + \|x_n - x^*\|^2}{2}.\end{aligned}$$

Hence

$$\|x_n - x^*\|^2 \le (1-\gamma_n)\|x_{n-1} - x^*\|^2 + \gamma_n\theta_n, \tag{4.5}$$

where $\gamma_n = (1-k^2)\beta_n$ and $\theta_n = \frac{2}{1-k^2}\langle f(x^*) - x^*, j(x_n - x^*)\rangle$. Since $\sum_{n=0}^{\infty}\beta_n = \infty$ and inequality (4.3), we obtain that $\sum_{n=1}^{\infty}\gamma_n = +\infty$, $\limsup_{n\to\infty}\theta_n \le 0$. Now we apply Lemma 2.4 to (4.5), we have that $\lim_{n\to\infty}\|x_n - x^*\| = 0$. The proof is complete. □

**Corollary 4.2.** *Let $E$ be a real reflexive strictly convex Banach space with a uniformly Gâteaux differentiable norm, and $C$ a nonempty closed convex subset of $E$. Let $\{T(t) : t \ge 0\}$ be an u.a.r. nonexpansive semigroup on $C$ such that $\Omega \ne \emptyset$. Let $f : C \to C$ be a fixed contractive mapping with the coefficient $k \in (0,1)$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{t_n\}$ be sequences of real numbers such that $\{\alpha_n\}, \{\beta_n\} \subseteq (0,1], t_n > 0$ with $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}t_n = \infty$ and $\sum_{n=0}^{\infty}\beta_n = \infty$. For $x_0 \in C$, let the sequence $\{x_n\}$ be generated by:*

$$\begin{cases} y_n = \beta_n f(x_{n-1}) + (1-\beta_n)x_{n-1}, \\ x_n = \alpha_n y_n + (1-\alpha_n)T(t_n)x_n, \ \text{for all } n \ge 1. \end{cases} \tag{4.6}$$

*Then $\{x_n\}$ converges strongly to $x^* \in \Omega$ with solves the following variational inequality (3.2).*

For each $t \geq 0$, setting $T(t) := T$ a continuous pseudocontractive mapping in Theorem 4.1. Furthermore, the requirement that $\{T(t) : t \geq 0\}$ is uniformly asymptotically regular (u.a.r) and L-Lipschitz are not necessary. In fact, the following can be obtain from Theorem 4.1 immediately.

**Corollary 4.3** ([11, Thorem 3.1]). *Let $E$ be a real reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm, and $C$ a nonempty closed convex subset of $E$. Let $T$ be a continuous pseudocontraction semigroup on $C$ such that $F(T) \neq \emptyset$. Let $f : C \to C$ be a fixed contractive mapping with the coefficient $k \in (0,1)$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences of real numbers such that $\{\alpha_n\}, \{\beta_n\} \subseteq (0,1], t_n > 0$ with $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \beta_n = \infty$. For $x_0 \in C$, let the sequence $\{x_n\}$ be generated by:*

$$\begin{cases} y_n = \beta_n f(x_{n-1}) + (1-\beta_n)x_{n-1}, \\ x_n = \alpha_n y_n + (1-\alpha_n)Tx_n, \ \text{for all } n \geq 1. \end{cases} \tag{4.7}$$

*Then $\{x_n\}$ converges strongly to $x^* \in \Omega$ with solves the following variational inequality:*

$$\langle (I-f)x^*, j(x^*-x)\rangle \leq 0, \ \ \forall x \in F(T).$$

# References


[1] K. Diemling, Zeros of accretive operators', Manuscripta Math. 13 (1974) 365–374.

[2] F.E. Browder, Fixed point theorems for noncompact mappings in Hilbert spaces, Proc. Natl. Acad. Sci. USA 53 (1965) 1272–1276.

[3] S. Reich, Strong convergence theorems for resolvents of accretive operators in Banach spaces, J. Math. Anal. Appl. 75 (1980) 287–292.

[4] T. Suzuki, On strong convergence to common fixed points of nonexpansive semigroups in Hilbert spaces, Proc. Amer. Math. Soc. 131 (2003) 2133–2136.

[5] H.-K. Xu, An iterative approach to quadratic optimization, J. Optim. Theory Appl. 116 (2003) 659–678.

[6] T.D. Benavides, G.L. Acedo, H.K. Xu, Construction of sunny nonexpansive retractions in Banach spaces, Bull. Aust. Math. Soc. 66 (1) (2002) 9–16.

[7] Y. Song, S. Xu, Strong convergence theorems for nonexpansive semigroup in Banach spaces, J. Math. Anal. Appl. 338 (2008) 152–161.

[8] A. Moudafi, Viscosity approximation methods for fixed-points problems, J. Math. Anal. Appl. 241 (2000) 46–55.

[9] H.-K. Xu, Viscosity approximation methods for nonexpansive mappings, J. Math. Anal. Appl. 298 (2004) 279–291.

[10] T. Suzuki, Moudafi's viscosity approximations with Meir-Keeler contractions, J. Math. Aanl. Appl. 325 (2007) 342–352.

[11] Y. Song, R. Chen, Convergence theorems of iterative algorithms for continuous pseudocontractive mappings, Nonlinear Anal. 67 (2007) 486–497.

[12] W. Takahashi, Nonlinear Functional Analysis-Fixed Point Theory and its Applications, Yokohama Publishers Inc., Yokohama, 2000 (in Japanese).

[13] R.E. Megginson, An Introduction to Banach Space Theory, Springer-Verlag New York, Inc., 1998.

[14] A. Aleyner, Y. Censor, Best approximation to common fixed points of a semigroup of nonexpansive operators, J. Nonlinear Convex Anal. 6 (1) (2005) 137–151.

[15] A. Aleyner, S. Reich, An explicit construction of sunny nonexpansive retractions in Banach spaces, Fixed Point Theory Appl. 2005 (3) (2005) 295–305.

[16] R. Chen, Y. Song, Convergence to common fixed point of nonexpansive semigroups, J. Comput. Appl. Math. 200 (2007) 566–575.

[17] J.-B.Baillon, Quelques propriétés de convergence asymptotique pour les semigroupes de contractions impaires, Comptes Rendus de l'Académie des Sciences de Paris, Série A (Mathématique) 283 (1976) 75–78.

[18] W. Takahashi, Y. Ueda, On Reich's Strong convergence for resolvents of accretive operators, J. Math. Anal. Appl. 104 (1984) 546–553.

[19] R.H. Martin, Differential equations on closed subsets of a Banach spaces, Trans. Amer. Math. Soc. 179 (1973) 399–414.

(Received 1 December 2011)
(Accepted 23 January 2012)