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A New Application of Homotopy Analysis Method

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Abstract : In this paper, we focus on linear fuzzy Fredholm integral equations of the second kind and propose a new method for solving it, namely "homotopy analysis method" (HAM). It is found that the HAM provides us with a simple way to adjust and control the convergence region of solution series by introducing an auxiliary parameter $\hbar$. The results illustrate the validity and the great potential of the HAM in solving fuzzy integral equations.

Keywords : Fuzzy functions; Fuzzy numbers; Fuzzy integral equations; Homotopy analysis method.
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1 Introduction

The topics of fuzzy integral equations which growing interest for some time, in particular in relation to fuzzy control, have been rapidly developed in recent years. We know that solving fuzzy integral equations requires appropriate and applicable definitions of fuzzy function and fuzzy integral of a fuzzy function. The fuzzy mapping function was introduced by Chang and Zadeh [1]. Later, Dubois and Prade [2] presented an elementary fuzzy calculus based on the extension principle

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[3]. The concept of integration of fuzzy functions was first introduced by Dubois and Prade [2]. Alternative approaches were later suggested by Goetschel and Voxman [4], Kaleva [5], Nanda [6] and others.

In 1992, Liao [7] employed the basic idea of the homotopy in topology to propose a general analytic method for nonlinear problems, namely "homotopy analysis method" (HAM), [8–10]. The homotopy analysis method always provides us with a family of solution expressions in the auxiliary parameter $\hbar$, the convergence region and rate of each solution might be determined conveniently by the auxiliary parameter $\hbar$. This method has been successfully applied to solve many types of nonlinear problems [11–13].

The aim of this paper is to apply for the first time the homotopy analysis method (HAM) to obtain approximate solutions of the linear fuzzy Fredholm integral equations of the second kind.

2 Preliminaries

In this section, we review the fundamental notations of fuzzy set theory to be used throughout this paper.

Definition 2.1. A fuzzy number u is a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(r)$, $\overline{u}(r)$; $0 \leqslant r \leqslant 1$ which satisfying the following requirements:

i. $\underline{u}(r)$ is a bounded left-continuous non-decreasing function over $[0, 1]$,

ii. $\overline{u}(r)$ is a bounded left-continuous non-increasing function over $[0, 1]$,

iii. $\underline{u}(r) \leqslant \overline{u}(r)$, $0 \leqslant r \leqslant 1$.

A crisp number α is simply represented by $\underline{u}(r) = \overline{u}(r) = \alpha, \quad 0 \leqslant r \leqslant 1$. The set of all fuzzy numbers (as given by Definition 2.1) is denoted by E^1 [5].

For arbitrary fuzzy numbers $u = (\underline{u}, \overline{u})$, $v = (\underline{v}, \overline{v})$ and an arbitrary crisp number k, we define fuzzy addition and scalar multiplication as

1. $(\underline{u+v})(r) = (\underline{u}(r) + \underline{v}(r))$,
2. $(\overline{u+v})(r) = (\overline{u}(r) + \overline{v}(r))$,
3. $(\underline{ku})(r) = k\underline{u}(r)$, $(\overline{ku})(r) = k\overline{u}(r), \quad k \geqslant 0$,
4. $(\underline{ku})(r) = k\overline{u}(r)$, $(\overline{ku})(r) = k\underline{u}(r), \quad k < 0$.

We will next define the fuzzy function notation and a metric D in E^1 [4].

Definition 2.2. For arbitrary fuzzy numbers $u = (\underline{u}, \overline{u})$ and $v = (\underline{v}, \overline{v})$ the quantity

$$D(u, v) = \max\left\{ \sup_{0 \leqslant r \leqslant 1} |\underline{u}(r) - \underline{v}(r)|, \sup_{0 \leqslant r \leqslant 1} |\overline{u}(r) - \overline{v}(r)| \right\}, \tag{2.1}$$

is the distance between u and v.

This metric is equivalent to the one used by Puri and Ralescu [14] and Kaleva [5]. It is shown [15] that (E^1, D) is a complete metric space.

Definition 2.3. A function $f : \mathbb{R}^1 \to E^1$ is called a fuzzy function. If for arbitrary fixed $x_0 \in \mathbb{R}^1$ and $\varepsilon > 0$, a $\xi > 0$ such that

$$| x - x_0 |< \xi \Longrightarrow D[f(x), f(x_0)] < \varepsilon, \tag{2.2}$$

exists, f is said to be continuous.

Throughout this work we also consider fuzzy functions which are defined only over a finite interval $[a, b]$ (we simply replace $\mathbb{R}^1$ by $[a, b]$ in Definition 2.3).

We now follow Goetschel and Voxman [4] and define the integral of a fuzzy function using the Riemann integral concept.

Definition 2.4. Let $f : [a, b] \to E^1$. For each partition $p = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$ and for arbitrary $\xi_i : x_{i-1} \leqslant \xi_i \leqslant x_i$, $1 \leqslant i \leqslant n$ let

$$R_p = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}). \tag{2.3}$$

The definite integral of $f(x)$ over $[a, b]$ is

$$\int_a^b f(x)dx = \lim R_p, \quad \max_{1\leqslant i\leqslant n} \{|x_i - x_{i-1}|\} \to 0, \tag{2.4}$$

provided that this limit exists in the metric D.

If the fuzzy function $f(x)$ is continuous in the metric D, its definite integral exists [4]. Furthermore,

$$\left(\underline{\int_a^b f(x;r)dx}\right) = \int_a^b \underline{f}(x;r)dx, \quad \left(\overline{\int_a^b f(x;r)dx}\right) = \int_a^b \overline{f}(x;r)dx. \tag{2.5}$$

It should be noted that the fuzzy integral can be also defined using the Lebesgue-type approach [5]. However, if $f(x)$ is continuous, both approaches yield the same value. Moreover, the representation of the fuzzy integral using Eq. (2.5) is more convenient for numerical calculations. More details about the properties of the fuzzy integral are given in [4, 5].

3 Fuzzy Integral Equation

The integral equations which are discussed in this section are the Fredholm equations of the second kind. The Fredholm integral equation of the second kind is [16]

$$F(x) = f(x) + \lambda \int_a^b K(x,t)F(t)dt, \tag{3.1}$$

where $\lambda > 0$, $K(x,t)$ is an arbitrary kernel function over the square $a \leqslant x, t \leqslant b$ and $f(x)$ is a function of $x : a \leqslant x \leqslant b$. If $f(x)$ is a crisp function then the solutions of Eq. (3.1) are crisp as well. However, if $f(x)$ is a fuzzy function this equation may only possess fuzzy solution. Sufficient conditions for the existence of a unique solution to the fuzzy Fredholm integral equation of the second kind(FFIE-2), i.e. to Eq. (3.1) where $f(x)$ is a fuzzy function, are given in [17].

Now, we introduce parametric form of a FFIE-2 with respect to Definition 2.1. Let $(\underline{f}(x;r), \overline{f}(x;r))$ and $(\underline{F}(x;r), \overline{F}(x;r))$, $0 \leqslant r \leqslant 1$ and $x \in [a,b]$ are parametric forms of $f(x)$ and $F(x)$, respectively. Then, parametric form of FFIE-2 is as follows:

$$\begin{cases} \underline{F}(x;r) = \underline{f}(x;r) + \lambda \int_a^b \underline{K(x,t)F(t;r)}\, dt, \\ \\ \overline{F}(x;r) = \overline{f}(x;r) + \lambda \int_a^b \overline{K(x,t)F(t;r)}\, dt, \end{cases} \tag{3.2}$$

where

$$\underline{K(x,t)F(t;r)} = \begin{cases} K(x,t)\underline{F}(t;r), & K(x,t) \geqslant 0, \\ K(x,t)\overline{F}(t;r), & K(x,t) < 0, \end{cases} \tag{3.3}$$

and

$$\overline{K(x,t)F(t;r)} = \begin{cases} K(x,t)\overline{F}(t;r), & K(x,t) \geqslant 0, \\ K(x,t)\underline{F}(t;r), & K(x,t) < 0, \end{cases} \tag{3.4}$$

for each $0 \leqslant r \leqslant 1$ and $a \leqslant x \leqslant b$. We can see that (3.2) is a system of linear Fredholm integral equations in crisp case for each $0 \leqslant r \leqslant 1$.

4 Homotopy Analysis Method

In this section, we apply homotopy analysis method (HAM) for the system (3.2) and obtain a recursive scheme for it.

Prior to applying HAM for the system (3.2), we suppose that the kernel $K(x,t)$ is non-negative for $a \leqslant t \leqslant c$ and non-positive for $c \leqslant t \leqslant b$. Therefore, we rewrite system (3.2) in the following form

$$\begin{cases} \underline{F}(x;r) = \underline{f}(x;r) + \lambda \int_a^c K(x,t)\underline{F}(t;r)\, dt + \lambda \int_c^b K(x,t)\overline{F}(t;r)\, dt, \\ \\ \overline{F}(x;r) = \overline{f}(x;r) + \lambda \int_a^c K(x,t)\overline{F}(t;r)\, dt + \lambda \int_c^b K(x,t)\underline{F}(t;r)\, dt. \end{cases} \tag{4.1}$$

Eq. (4.1) is a system of linear Fredholm integral equations in crisp case for each $0 \leqslant r \leqslant 1$. For solving system (4.1) by homotopy analysis method, we construct the zero-order deformation equation

$$\begin{cases} (1-p)\ \mathcal{L}[\,\underline{U}(x,p;r) - \underline{w}_0(x;r)\,] \\ = p\,\hbar\,[\,\underline{U}(x,p;r) - \underline{f}(x;r) - \lambda \int_a^c K(x,t)\underline{U}(t,p;r)\, dt - \lambda \int_c^b K(x,t)\overline{U}(t,p;r)\, dt\,], \\ \\ (1-p)\ \mathcal{L}[\,\overline{U}(x,p;r) - \overline{w}_0(x;r)\,] \\ = p\,\hbar\,[\,\overline{U}(x,p;r) - \overline{f}(x;r) - \lambda \int_a^c K(x,t)\overline{U}(t,p;r)\, dt - \lambda \int_c^b K(x,t)\underline{U}(t,p;r)\, dt\,], \end{cases} \tag{4.2}$$

where $p \in [0,1]$ is the embedding parameter, $\hbar$ is non-zero auxiliary parameter, $\mathcal{L}$ is an auxiliary linear operator, $\underline{w}_0(x;r)$ and $\overline{w}_0(x;r)$ are initial guesses of $\underline{F}(x;r)$ and $\overline{F}(x;r)$, respectively and $\underline{U}(x,p;r)$ and $\overline{U}(x,p;r)$ are unknown function on independent variable p.

Using the above zero-order deformation equation, with assumption $\mathcal{L}[u]=u$, we have

$$\begin{cases} (1-p)[\,\underline{U}(x,p;r)-\underline{w}_0(x;r)\,] \\ = p\,\hbar\,[\,\underline{U}(x,p;r)-\underline{f}(x;r)-\lambda\int_a^c K(x,t)\underline{U}(t,p;r)\,dt-\lambda\int_c^b K(x,t)\overline{U}(t,p;r)\,dt\,], \\ (1-p)[\,\overline{U}(x,p;r)-\overline{w}_0(x;r)\,] \\ = p\,\hbar\,[\,\overline{U}(x,p;r)-\overline{f}(x;r)-\lambda\int_a^c K(x,t)\overline{U}(t,p;r)\,dt-\lambda\int_c^b K(x,t)\underline{U}(t,p;r)\,dt\,]. \end{cases} \tag{4.3}$$

Obviously, when $p=0$ and $p=1$, it holds

$$\begin{cases} \underline{U}(x,0;r)=\underline{w}_0(x;r), \\ \overline{U}(x,0;r)=\overline{w}_0(x;r), \end{cases} \tag{4.4}$$

and

$$\begin{cases} \underline{U}(x,1;r)=\underline{f}(x;r)+\lambda\int_a^c K(x,t)\underline{U}(t,1;r)\,dt+\lambda\int_c^b K(x,t)\overline{U}(t,1;r)\,dt, \\ \overline{U}(x,1;r)=\overline{f}(x;r)+\lambda\int_a^c K(x,t)\overline{U}(t,1;r)\,dt+\lambda\int_c^b K(x,t)\underline{U}(t,1;r)\,dt, \end{cases} \tag{4.5}$$

respectively. Thus, as p increases from 0 to 1, the solution $(\underline{U}(x,p;r),\overline{U}(x,p;r))$ varies from the initial guess $(\underline{w}_0(x;r),\overline{w}_0(x;r))$ to the solution $(\underline{F}(x;r),\overline{F}(x;r))$. Expanding $\underline{U}(x,p;r)$ and $\overline{U}(x,p;r)$ in Taylor series with respect to p, we have

$$\begin{cases} \underline{U}(x,p;r)=\underline{w}_0(x;r)+\sum_{m=1}^{\infty}\underline{u}_m(x;r)\,p^m, \\ \overline{U}(x,p;r)=\overline{w}_0(x;r)+\sum_{m=1}^{\infty}\overline{u}_m(x;r)\,p^m, \end{cases} \tag{4.6}$$

where

$$\begin{cases} \underline{u}_m(x;r)=\frac{1}{m!}\frac{\partial^m\underline{U}(x,p;r)}{\partial p^m}|_{p=0}, \\ \overline{u}_m(x;r)=\frac{1}{m!}\frac{\partial^m\overline{U}(x,p;r)}{\partial p^m}|_{p=0}. \end{cases} \tag{4.7}$$

It should be noted that $\underline{U}(x,0;r)=\underline{w}_0(x;r)$ and $\overline{U}(x,0;r)=\overline{w}_0(x;r)$. Differentiating the zero-order deformation equation (4.3) m times with respect to the embedding parameter p and then setting $p=0$ and finally dividing them by $m!$, we have

$$\begin{cases} \underline{u}_m(x;r) = \alpha_m\,\underline{u}_{m-1}(x;r) + \hbar\,[\,\underline{u}_{m-1}(x;r) - \beta_m \underline{f}(x;r) \\ \qquad -\lambda \int_a^c K(x,t)\underline{u}_{m-1}(t;r)\,dt \\ \qquad -\lambda \int_c^b K(x,t)\overline{u}_{m-1}(t;r)\,dt\,], \\ \overline{u}_m(x;r) = \alpha_m\,\overline{u}_{m-1}(x;r) + \hbar\,[\,\overline{u}_{m-1}(x;r) - \beta_m \overline{f}(x;r) \\ \qquad -\lambda \int_a^c K(x,t)\overline{u}_{m-1}(t;r)\,dt \\ \qquad -\lambda \int_c^b K(x,t)\underline{u}_{m-1}(t;r)\,dt\,], \end{cases} \tag{4.8}$$

where $m \geqslant 1$ and

$$\alpha_m = \begin{cases} 0, & m = 1, \\ 1, & m \neq 1, \end{cases} \qquad \beta_m = \begin{cases} 1, & m = 1, \\ 0, & m \neq 1, \end{cases}$$

and $\underline{u}_0(x;r) = \underline{w}_0(x;r)$ and $\overline{u}_0(x;r) = \overline{w}_0(x;r)$.

If we take $\underline{w}_0(x;r) = \overline{w}_0(x;r) = 0$, then we have

$$\begin{cases} \underline{u}_1(x;r) = -\hbar\,\underline{f}(x;r), \\ \overline{u}_1(x;r) = -\hbar\,\overline{f}(x;r), \\ \underline{u}_m(x;r) \\ = (1+\hbar)\,\underline{u}_{m-1}(x;r) - \hbar\lambda\,[\,\int_a^c K(x,t)\underline{u}_{m-1}(t;r)\,dt + \int_c^b K(x,t)\overline{u}_{m-1}(t;r)\,dt\,], \\ \overline{u}_m(x;r) \\ = (1+\hbar)\,\overline{u}_{m-1}(x;r) - \hbar\lambda\,[\,\int_a^c K(x,t)\overline{u}_{m-1}(t;r)\,dt + \int_c^b K(x,t)\underline{u}_{m-1}(t;r)\,dt\,], \end{cases} \tag{4.9}$$

where $m \geq 2$.

Hence, the solution of Eq. (4.1) in series form is obtained as

$$\begin{cases} \underline{F}(x;r) = \lim_{p\to 1} \underline{U}(x,p;r) = \sum_{m=1}^{\infty} \underline{u}_m(x;r), \\ \overline{F}(x;r) = \lim_{p\to 1} \overline{U}(x,p;r) = \sum_{m=1}^{\infty} \overline{u}_m(x;r). \end{cases} \tag{4.10}$$

We denote the nth-order approximation to solution $\underline{F}(x;r)$ with

$$\underline{F}_n(x;r) = \sum_{m=1}^{n} \underline{u}_m(x;r),$$

and $\overline{F}(x;r)$ with

$$\overline{F}_n(x;r) = \sum_{m=1}^{n} \overline{u}_m(x;r).$$

Therefore, we have

$$\begin{cases} \underline{F}(x;r) = \lim\limits_{n\to\infty} \underline{F}_n(x;r), \\ \overline{F}(x;r) = \lim\limits_{n\to\infty} \overline{F}(x;r). \end{cases} \tag{4.11}$$

5 Text Examples

Example 5.1. *Consider the fuzzy Fredholm integral equation with*

$$\underline{f}(x;r) = \frac{1}{2}x^2(r+1),\ \ \overline{f}(x;r) = \frac{1}{2}x^2(3-r),$$

and kernel

$$K(x,t) = t\,x^2,\ \ -1 \leqslant x,t \leqslant 1,$$

and $a=-1$, $b=1$, $\lambda=1$. *The exact solution in this case is given by*

$$\underline{F}(x;r) = x^2 r,\ \ \overline{F}(x;r) = x^2(2-r).$$

In this example, $K(x,t) \leqslant 0$ for each $-1 \leqslant t \leqslant 0$ and $K(x,t) \geqslant 0$ for each $0 \leqslant t \leqslant 1$. By Eq. (4.9), we can see that, some first terms of HAM series are as follows:

$$\begin{aligned}
\underline{u}_1(x;r) &= -\frac{1}{2}\hbar x^2[\,(1+r)\,],\\
\underline{u}_2(x;r) &= -\frac{1}{4}\hbar x^2\,[\,2(r+1)+\hbar(r+3)\,],\\
\underline{u}_3(x;r) &= -\frac{1}{8}\hbar x^2\,[\,4(r+1)+4\hbar(r+3)+\hbar^2(r+7)\,],\\
\underline{u}_4(x;r) &= -\frac{1}{16}\hbar x^2\,[\,8(r+1)+12\hbar(r+3)+6\hbar^2(r+7)+\hbar^3(r+15)\,],\\
\underline{u}_5(x;r) &= -\frac{1}{32}\hbar x^2\,[\,16(r+1)+32\hbar(r+3)+24\hbar^2(r+7)+8\hbar^3(r+15)\\
&\qquad +\hbar^4(r+31)\,],\\
\underline{u}_6(x;r) &= -\frac{1}{64}\hbar x^2\,[\,32(r+1)+80\hbar(r+3)+80\hbar^2(r+7)+40\hbar^3(r+15)\\
&\qquad +10\hbar^4(r+31)+\hbar^5(r+63)\,],\\
\underline{u}_7(x;r) &= -\frac{1}{128}\hbar x^2\,[\,64(r+1)+192\hbar(r+3)+240\hbar^2(r+7)+160\hbar^3(r+15)\\
&\qquad +60\hbar^4(r+31)+12\hbar^5(r+63)+\hbar^6(r+127)\,],
\end{aligned}$$

and

$$
\begin{aligned}
\overline{u}_1(x;r) &= -\frac{1}{2}\hbar x^2[\,(3-r)\,],\\
\overline{u}_2(x;r) &= -\frac{1}{4}\hbar x^2\,[\,2(3-r)+\hbar(5-r)\,],\\
\overline{u}_3(x;r) &= -\frac{1}{8}\hbar x^2\,[\,4(3-r)+4\hbar(5-r)+\hbar^2(9-r)\,],\\
\overline{u}_4(x;r) &= -\frac{1}{16}\hbar x^2\,[\,8(3-r)+12\hbar(5-r)+6\hbar^2(9-r)+\hbar^3(17-r)\,],\\
\overline{u}_5(x;r) &= -\frac{1}{32}\hbar x^2\,[\,16(3-r)+32\hbar(5-r)+24\hbar^2(9-r)+8\hbar^3(17-r)\\
&\qquad +\hbar^4(33-r)\,],\\
\overline{u}_6(x;r) &= -\frac{1}{64}\hbar x^2\,[\,32(3-r)+80\hbar(5-r)+80\hbar^2(9-r)+40\hbar^3(17-r)\\
&\qquad +10\hbar^4(33-r)+\hbar^5(65-r)\,],\\
\overline{u}_7(x;r) &= -\frac{1}{128}\hbar x^2\,[\,64(3-r)+192\hbar(5-r)+240\hbar^2(9-r)+160\hbar^3(17-r)\\
&\qquad +60\hbar^4(33-r)+12\hbar^5(65-r)+\hbar^6(129-r)\,].
\end{aligned}
$$

Then we approximate $\underline{F}(x;r)$ with

$$
\begin{aligned}
\underline{F}_7(x;r) &= \sum_{m=1}^{7}\underline{u}_m(x;r)\\
&= -\frac{1}{128}\hbar x^2\,[\,448(r+1)+672\hbar(r+3)+560\hbar^2(r+7)+280\hbar^3(r+15)\\
&\qquad +84\hbar^4(r+31)+14\hbar^5(r+63)+\hbar^6(r+127)\,],
\end{aligned}
$$

and $\overline{F}(x;r)$ with

$$
\begin{aligned}
\overline{F}_7(x;r) &= \sum_{m=1}^{7}\overline{u}_m(x;r)\\
&= -\frac{1}{128}\hbar x^2\,[\,448(3-r)+672\hbar(5-r)+560\hbar^2(9-r)+280\hbar^3(17-r)\\
&\qquad +84\hbar^4(33-r)+14\hbar^5(65-r)+\hbar^6(129-r)\,].
\end{aligned}
$$

It has been proved that, as long as a series solution given by the homotopy analysis method converges, it must be one of exact solutions. So it is important to ensure that the solution series (4.10) is convergent. Note that the solution series (4.10) contains the auxiliary parameter $\hbar$, which provides us with a simple way to adjust and control the convergence of the solution series. In general, by means of the so-call $\hbar$-curve (a curve of a versus $\hbar$), it is straightforward to choose an appropriate range for $\hbar$ which ensures the convergence of the solution series. As pointed by Liao [8], the valid region of $\hbar$ is a horizontal line segment. In Figure

1, we plot the $\hbar$-curves of $\underline{F}(0.5;0.5)$ and $\overline{F}(0.5;0.5)$ given by 7th-order approximate solution, i.e., $\underline{F}_7(0.5;0.5)$ and $\overline{F}_7(0.5;0.5)$, respectively. From the Figure 1, we could find that if $\hbar$ is about in area $[-1.6,-0.4]$ the result is convergent. We compare results with exact solutions using metric of Definition 2.2 for different values of $\hbar$ by 7th-order approximate solution in Table 1. The results reveal that the homotopy analysis method can provide us with a convenient way to adjust and control the convergence region and rate of approximation series by introducing an auxiliary parameter $\hbar$.

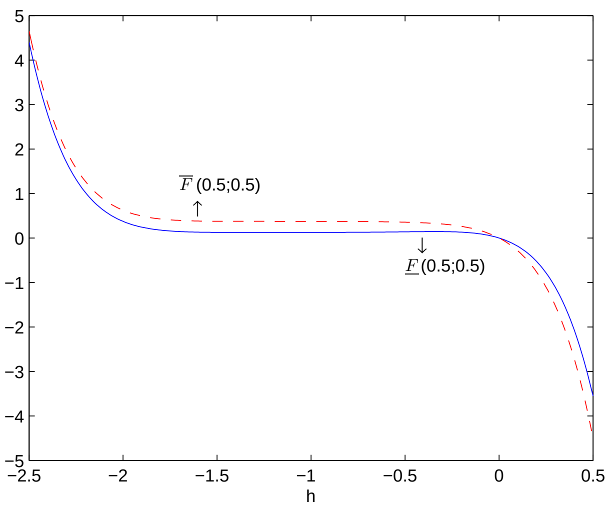

Figure 1: $\hbar$-curves of $\underline{F}(0.5;0.5)$ and $\overline{F}(0.5;0.5)$ given by the 7th-order approximate solution in Example 5.1.

Table 1: The error in the solution obtained by HAM for various $\hbar$ by 7th-order approximate solution.

x	$\hbar=-1.4$	$\hbar=-1.3$	$\hbar=-1.2$	$\hbar=-1.1$	$\hbar=-1$
-1	$1.8571e-003$	$8.6209e-004$	$1.6512e-003$	$3.7368e-003$	$7.8125e-003$
-0.8	$1.1885e-003$	$5.5174e-004$	$1.0568e-003$	$2.3915e-003$	$5.0000e-003$
-0.6	$6.6856e-004$	$3.1035e-004$	$5.9443e-004$	$1.3452e-003$	$2.8125e-003$
-0.4	$2.9714e-004$	$1.3793e-004$	$2.6419e-004$	$5.9789e-004$	$1.2500e-003$
-0.2	$7.4284e-005$	$3.4484e-005$	$6.6048e-005$	$1.4947e-004$	$3.1250e-004$
0	0	0	0	0	0
0.2	$7.4284e-005$	$3.4484e-005$	$6.6048e-005$	$1.4947e-004$	$3.1250e-004$
0.4	$2.9714e-004$	$1.3793e-004$	$2.6419e-004$	$5.9789e-004$	$1.2500e-003$
0.6	$6.6856e-004$	$3.1035e-004$	$5.9443e-004$	$1.3452e-003$	$2.8125e-003$
0.8	$1.1885e-003$	$5.5174e-004$	$1.0568e-003$	$2.3915e-003$	$5.0000e-003$
1	$1.8571e-003$	$8.6209e-004$	$1.6512e-003$	$3.7368e-003$	$7.8125e-003$

Example 5.2 ([18]). *Consider the fuzzy Fredholm integral equation with*

$$\underline{f}(x;r) = \sin\left(\frac{x}{2}\right)\left[\frac{13}{15}(r^2+r) + \frac{2}{15}(4-r-r^3)\right],$$

$$\overline{f}(x;r) = \sin\left(\frac{x}{2}\right)\left[\frac{2}{15}(r^2+r) + \frac{13}{15}(4-r-r^3)\right],$$

and kernel

$$K(x,t) = 0.1\,\sin(t)\sin\left(\frac{x}{2}\right), \;\; 0 \leqslant x, t \leqslant 2\pi,$$

and $a = 0$, $b = 2\pi$, $\lambda = 1$. *The exact solution in this case is given by*

$$\underline{F}(x;r) = \sin\left(\frac{x}{2}\right)[r^2+r], \;\; \overline{F}(x;r) = \sin\left(\frac{x}{2}\right)[4-r-r^3].$$

In this example, $K(x,t) \geqslant 0$ for each $0 \leqslant t \leqslant \pi$ and $K(x,t) \leqslant 0$ for each $\pi \leqslant t \leqslant 2\pi$. By Eq. (4.9), Some first terms of HAM series are:

$$\begin{aligned}
\underline{u}_1(x;r) &= \frac{1}{15}\hbar\,\sin\left(\frac{x}{2}\right)[\,2r^3 - 13r^2 - 11r - 8\,],\\
\underline{u}_2(x;r) &= \frac{1}{225}\hbar\,\sin\left(\frac{x}{2}\right)[\,30r^3 - 195r^2 - 165r - 120\,]\\
&\quad + \frac{1}{225}\hbar^2\sin\left(\frac{x}{2}\right)[\,52r^3 - 173r^2 - 121r - 208\,],\\
\underline{u}_3(x;r) &= \frac{1}{3375}\hbar\,\sin\left(\frac{x}{2}\right)[\,450r^3 - 2925r^2 - 2475r - 1800\,]\\
&\quad + \frac{1}{3375}\hbar^2\sin\left(\frac{x}{2}\right)[\,1560r^3 - 5190r^2 - 3630r - 6240\,]\\
&\quad + \frac{1}{3375}\hbar^3\sin\left(\frac{x}{2}\right)[\,1022r^3 - 2353r^2 - 1331r - 4088\,],\\
\underline{u}_4(x;r) &= \frac{1}{50625}\hbar\,\sin\left(\frac{x}{2}\right)[\,6750r^3 - 43875r^2 - 37125r - 27000\,]\\
&\quad + \frac{1}{50625}\hbar^2\sin\left(\frac{x}{2}\right)[\,35100r^3 - 116775r^2 - 81675r - 140400\,]\\
&\quad + \frac{1}{50625}\hbar^3\sin\left(\frac{x}{2}\right)[\,45990r^3 - 105885r^2 - 59895r - 183960\,]\\
&\quad + \frac{1}{50625}\hbar^4\sin\left(\frac{x}{2}\right)[\,17992r^3 - 32633r^2 - 14641r - 71968\,],\\
\underline{u}_5(x;r) &= \frac{1}{759375}\hbar\,\sin\left(\frac{x}{2}\right)[\,101250r^3 - 658125r^2 - 556875r - 405000\,]\\
&\quad + \frac{1}{759375}\hbar^2\sin\left(\frac{x}{2}\right)[\,702000r^3 - 2335500r^2 - 1633500r - 2808000\,]\\
&\quad + \frac{1}{759375}\hbar^3\sin\left(\frac{x}{2}\right)[\,1379700r^3 - 3176550r^2 - 1796850r - 5518800\,]\\
&\quad + \frac{1}{759375}\hbar^4\sin\left(\frac{x}{2}\right)[\,1079520r^3 - 1957980r^2 - 878460r - 4318080\,]\\
&\quad + \frac{1}{759375}\hbar^5\sin\left(\frac{x}{2}\right)[\,299162r^3 - 460213r^2 - 161051r - 1196648\,],
\end{aligned}$$

and

$$\begin{aligned}
\overline{u}_1(x;r) &= \frac{1}{15}\hbar\,\sin\left(\frac{x}{2}\right)\,[\,13r^3-2r^2+11r-52\,],\\
\overline{u}_2(x;r) &= \frac{1}{225}\hbar\,\sin\left(\frac{x}{2}\right)\,[\,195r^3-30r^2+165r-780\,]\\
&\quad+\frac{1}{225}\hbar^2\sin\left(\frac{x}{2}\right)\,[\,173r^3-52r^2+121r-692\,],\\
\overline{u}_3(x;r) &= \frac{1}{3375}\hbar\,\sin\left(\frac{x}{2}\right)\,[\,2925r^3-450r^2+2475r-11700\,]\\
&\quad+\frac{1}{3375}\hbar^2\sin\left(\frac{x}{2}\right)\,[\,5190r^3-1560r^2+3630r-20760\,]\\
&\quad+\frac{1}{3375}\hbar^3\sin\left(\frac{x}{2}\right)\,[\,2353r^3-1022r^2+1331r-9412\,],\\
\overline{u}_4(x;r) &= \frac{1}{50625}\hbar\,\sin\left(\frac{x}{2}\right)\,[\,43875r^3-6750r^2+37125r-175500\,]\\
&\quad+\frac{1}{50625}\hbar^2\sin\left(\frac{x}{2}\right)\,[\,116775r^3-35100r^2+81675r-467100\,]\\
&\quad+\frac{1}{50625}\hbar^3\sin\left(\frac{x}{2}\right)\,[\,105885r^3-45990r^2+59895r-423540\,]\\
&\quad+\frac{1}{50625}\hbar^4\sin\left(\frac{x}{2}\right)\,[\,32633r^3-17992r^2+14641r-130532\,],\\
\overline{u}_5(x;r) &= \frac{1}{759375}\hbar\,\sin\left(\frac{x}{2}\right)\,[\,658125r^3-101250r^2+556875r-2632500\,]\\
&\quad+\frac{1}{759375}\hbar^2\sin\left(\frac{x}{2}\right)\,[\,2335500r^3-702000r^2+1633500r-9342000\,]\\
&\quad+\frac{1}{759375}\hbar^3\sin\left(\frac{x}{2}\right)\,[\,3176550r^3-1379700r^2+1796850r-12706200\,]\\
&\quad+\frac{1}{759375}\hbar^4\sin\left(\frac{x}{2}\right)\,[\,1957980r^3-1079520r^2+878460r-7831920\,]\\
&\quad+\frac{1}{759375}\hbar^5\sin\left(\frac{x}{2}\right)\,[\,460213r^3-299162r^2+161051r-1840852\,].
\end{aligned}$$

Then we approximate $\underline{F}(x;r)$ with

$$\begin{aligned}\underline{F}_5(x;r) &= \sum_{m=1}^{5} \underline{u}_m(x;r)\\ &= \frac{1}{759375}\hbar \sin\left(\frac{x}{2}\right) [\,506250r^3 - 3290625r^2 - 2784375r - 2025000\,]\\ &\quad + \frac{1}{759375}\hbar^2 \sin\left(\frac{x}{2}\right) [\,1755000r^3 - 5838750r^2 - 4083750r - 7020000\,]\\ &\quad + \frac{1}{759375}\hbar^3 \sin\left(\frac{x}{2}\right) [\,2299500r^3 - 5294250r^2 - 2994750r - 9198000\,]\\ &\quad + \frac{1}{759375}\hbar^4 \sin\left(\frac{x}{2}\right) [\,1349400r^3 - 2447475r^2 - 1098075r - 5397600\,]\\ &\quad + \frac{1}{759375}\hbar^5 \sin\left(\frac{x}{2}\right) [\,299162r^3 - 460213r^2 - 161051r - 1196648\,],\end{aligned}$$

and $\overline{F}(x;r)$ with

$$\begin{aligned}\overline{F}_5(x;r) &= \sum_{m=1}^{5} \overline{u}_m(x;r)\\ &= \frac{1}{759375}\hbar \sin\left(\frac{x}{2}\right) [\,3290625r^3 - 506250r^2 + 2784375r - 13162500\,]\\ &\quad + \frac{1}{759375}\hbar^2 \sin\left(\frac{x}{2}\right) [\,5838750r^3 - 1755000r^2 + 4083750r - 23355000\,]\\ &\quad + \frac{1}{759375}\hbar^3 \sin\left(\frac{x}{2}\right) [\,5294250r^3 - 2299500r^2 + 2994750r - 21177000\,]\\ &\quad + \frac{1}{759375}\hbar^4 \sin\left(\frac{x}{2}\right) [\,2447475r^3 - 1349400r^2 + 1098075r - 9789900\,]\\ &\quad + \frac{1}{759375}\hbar^5 \sin\left(\frac{x}{2}\right) [\,460213r^3 - 299162r^2 + 161051r - 1840852\,].\end{aligned}$$

As pointed out earlier that the convergence region and rate of approximation strongly depend on the choice of the values of the auxiliary parameter $\hbar$ for the HAM. We should therefore focus on the choice of $\hbar$ by plotting of $\hbar$-curves. Figure 2, shows the $\hbar$-curves of $\underline{F}(\pi;0.5)$ and $\overline{F}(\pi;0.5)$ given by 5th-order approximate solutions, i.e., $\underline{F}_5(\pi;0.5)$ and $\overline{F}_5(\pi;0.5)$, respectively. It is seen that convergent results can be obtained when $\hbar \in [-1.5,-0.6]$.

We compare results with exact solutions using metric of Definition 2.2 for different values of $\hbar$ by 5th-order approximate solution in Table 2. The results reveal that the HAM is very effective and simple.

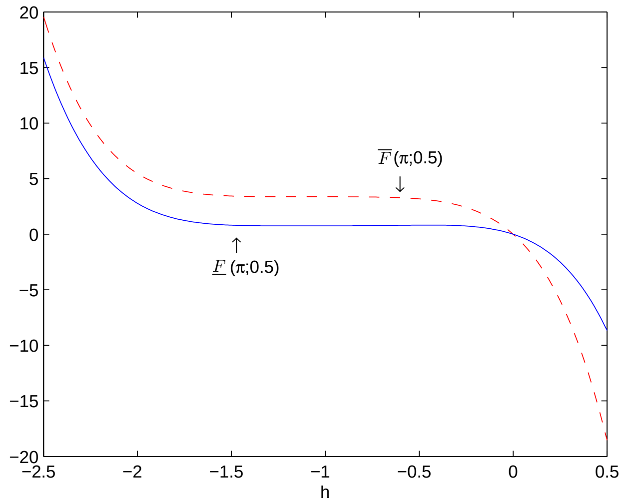

Figure 2: $\hbar$-curves of $\underline{F}(\pi;0.5)$ and $\overline{F}(\pi;0.5)$ given by the 5th-order approximate solution in Example 5.2.

Table 2: The error in the solution obtained by HAM for various $\hbar$ by 5th-order approximate solution.

x	$\hbar=-1.3$	$\hbar=-1.2$	$\hbar=-1.1$	$\hbar=-1$	$\hbar=-0.9$
0	0	0	0	0	0
$\pi/5$	$1.5575e-003$	$2.1408e-004$	$1.7311e-004$	$8.3340e-003$	$2.8142e-003$
$2\pi/5$	$2.9626e-003$	$4.0720e-004$	$3.2928e-004$	$1.5852e-003$	$5.3530e-003$
$3\pi/5$	$4.0776e-003$	$5.6046e-004$	$4.5322e-004$	$2.1819e-003$	$7.3678e-003$
$4\pi/5$	$4.7935e-003$	$6.5886e-004$	$5.3279e-004$	$2.5650e-003$	$8.6614e-003$
π	$5.0402e-003$	$6.9276e-004$	$5.6021e-004$	$2.6970e-003$	$9.1071e-003$
$6\pi/5$	$4.7935e-003$	$6.5886e-004$	$5.3279e-004$	$2.5650e-003$	$8.6614e-003$
$7\pi/5$	$4.0776e-003$	$5.6046e-004$	$4.5322e-004$	$2.1819e-003$	$7.3678e-003$
$8\pi/5$	$2.9626e-003$	$4.0720e-004$	$3.2928e-004$	$1.5852e-003$	$5.3530e-003$
$9\pi/5$	$1.5575e-003$	$2.1408e-004$	$1.7311e-004$	$8.3340e-004$	$2.8142e-003$
2π	$4.8986e-016$	$4.8986e-016$	$4.8986e-016$	$4.8986e-016$	$4.8986e-016$

6 Conclusion

In this paper, the homotopy analysis method (HAM) is successfully applied for solving linear fuzzy Fredholm integral equations of the second kind. It is illustrated that the HAM provides a convenient way to adjust and control the convergence of approximation series, which is a main advantage of this method. The results show the validity and the great potential of HAM in solving linear fuzzy Fredholm integral equations.

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