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Mathematical Modelling of Magnetometric Resistivity Sounding Earth Structures

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Abstract : In this paper, we derive the solutions of the steady state magnetic field due to a DC current source in three types of heterogeneous earth models. N-layered continuously stratified earth models such as an exponentially, a linearly, and a binomially varying conductivity earth structures are considered. These solutions are critical to interpret the magnetometric resistivity (MMR) data.

Our solutions are achieved by solving a boundary value problem in the wave number domain and then transforming back to spatial domain. The propagator matrix techniques are used. The simple cases are explored for 2-layered earth model. The curves of magnetic field are plotted to show the behavior of the field while some parameters are approximately given. To determine the conductivity parameters, the inverse problem is introduced via the use of optimization technique.

Keywords : Magnetometric resistivity, Magnetic field, MMR.

1 Introduction

The magnetometric resistivity method has recently become an additional electrical prospecting technique used for finding mineral resources. This technique is based on the measurement of low-level, low-frequency magnetic fields associated with non-inductive current flow in the ground.

In 1985, Edwards et al.[3] discussed a specific case where the upper half-space is conductive seawater, as encountered in the magnetometric offshore electrical sounding system. Edwards [2] and Edwards and Nabighian [4] concentrated the ratio of the magnetic fields below and above a known conductive layer to infer the basement resistivity.

Sezginer and Habashy [6] computed the static magnetic field due to an arbitrary current injected into a conducting uniform half-space. Inayat-Hussein [5] gave a new proof that the magnetic field outside the one dimension medium is independent of the electrical conductivity. Veitch et al. [7] pointed out to the general solution for the magnetic field within a layered earth due to a point source has not been fully explored. They, indirectly, derived the magnetic field by applying Stoke's theorem and Ampere's law to the electric potential. Unfortunately, these

works do not supply the amount of information about the magnetic field that is required for many current applications.

Chen and Oldenburg [1] derived the magnetic field directly from solving a boundary value problems which was similar to the approach used by Edwards [2] and then discussed in a homogeneous and a 2-layered earth model. The motivation of this study is to determine magnetometric resistivity method may have applications for salinity mapping in different parts of Thailand.

The continuous varying conductivity ground profiles used in this paper are exponentially, linearly and binomially with depth. The objective of this paper is to present a technique whereby the magnetic observations obtained from a horizontal loop source above the ground surface can be inverted to determine the parameter of the conductivity model which is a continuous function of depth.

The iterative technique using conjugate gradient method is conducted for constructing the conductivity model whose calculated responses are close to the observed values. A conductivity profile satisfying the data is constructed iteratively via successive perturbation of a starting model.

2 Magnetic Field due to a Semi-infinite Source in a 1-Dimensional Earth

A semi-infinite vertical wire carries an exciting current I and terminates at the electrode Q. The electrode Q is deliberately placed at the interface $z = z_s$ of layer s and layer $s+1$ where s is a positive integer less than $N-1$. Each layer has conductivity as a function of depth, $\sigma_j(z)$ with thickness h_j . The Maxwell's equations can be used to determine the magnetic field intensity $\overline{H}$ as

$$\nabla \times \overline{E} = \overline{0} \tag{1}$$

and

$$\nabla \times \overline{H} = \sigma \overline{E} \tag{2}$$

where $\overline{E}$ is the electric field intensity, $\overline{H}$ is the magnetic field intensity and σ is the conductivity of the medium. Using (1) and (2), we have

$$\nabla \times \frac{1}{\sigma} \nabla \times \overline{H} = \overline{0}. \tag{3}$$

Since the problem is axi-symmetric, and $\overline{H}$ has only an azimuthal component in cylindrical coordinate (r, ϕ, z). For simplicity, we use H to represent the azimuthal component in the following derivations. Expanding equation (3) yields

$$\frac{1}{\sigma}\frac{\partial^2 H}{\partial z^2} + \frac{\partial}{\partial z}\left(\frac{1}{\sigma}\right)\frac{\partial H}{\partial z} + \frac{1}{\sigma}\left(\frac{1}{r}\frac{\partial^2}{\partial r^2}(rH) + \frac{\partial}{\partial r}\left(\frac{1}{r}\right)\frac{\partial}{\partial r}(rH)\right) + \frac{\partial}{\partial r}\left(\frac{1}{\sigma}\right)\frac{1}{r}\frac{\partial}{\partial r}(rH) = 0. \tag{4}$$

Since σ is a function of depth z only, the above equation becomes

$$\frac{\partial^2 H}{\partial z^2} + \sigma \frac{\partial}{\partial z}\left(\frac{1}{\sigma}\right)\frac{\partial H}{\partial z} + \left(\frac{\partial^2 H}{\partial r^2} + \left(\frac{1}{r}\right)\frac{\partial H}{\partial r}\right) - \frac{1}{r^2}H = 0. \tag{5}$$

Taking the Hankel transform defined by

$$\widetilde{H}(\lambda, z) = \int_0^\infty rH(r, z)J_1(\lambda r)dr$$

where J_1 is the Bessel function of the first kind of order one, to the equation (5) and we have

$$\frac{\partial^2 \widetilde{H}}{\partial z^2} + \sigma \frac{\partial}{\partial z}\left(\frac{1}{\sigma}\right)\frac{\partial \widetilde{H}}{\partial z} - \lambda \widetilde{H} = 0. \tag{6}$$

3 The Magnetic Field Response from the Exponential Conductivity Ground Profile

Soil salinity profiles frequently display monotonically increasing or decreasing salt concentrations with depth, z. This salt concentration is strongly correlated with the conductivity of the ground and frequently can be represented by an equation

$$\sigma(z) = ae^{nbz}$$

where a and b are greater than zero and $n \in \{-1, 0, 1\}$, and hence the equation (6) becomes

$$\frac{\partial^2 \widetilde{H}}{\partial z^2} + nb\frac{\partial \widetilde{H}}{\partial z} - \lambda^2 \widetilde{H} = 0. \tag{7}$$

The solution of (7) is

$$\widetilde{H}(\lambda, z) = Ae^{\alpha^- z} + Be^{\alpha^+ z} \tag{8}$$

where A and B are arbitrary constants which can be determined from the boundary conditions and

$$\alpha^- = \frac{nb - \sqrt{(nb)^2 + 4\lambda^2}}{2} \text{ and } \alpha^+ = \frac{nb + \sqrt{(nb)^2 + 4\lambda^2}}{2}.$$

For the N-layered stratified earths model, each layered has limit depth except for the lower most layered has infinite depth. A semi-infinite vertical wire carries an exciting current and terminates at the electrode Q. The electrode Q is deliberately placed at the interface $z = z_s$ of layer s and layer $s + 1$. Each layer has an exponentially varying conductivity profile defined as

$$\sigma_1(z) = ae^{n_1 b_1 z}, 0 \le z \le z_1$$

and for $2 \leq z \leq N-1$,

$$\sigma_k(z) = \sigma_{k-1}(z_{k-1})e^{n_k b_k(z-z_{k-1})}, \qquad z_{k-1} \leq z \leq z_k,$$
$$\sigma_N(z) = \sigma_{N-1}(z_{N-1})e^{n_N b_N(z-z_{N-1})}, \qquad z > z_{N-1}.$$

The magnetic fields in each layered are

$$\widetilde{H}_k(\lambda, z) = \frac{I}{2\pi\lambda} + A_k e^{\alpha_k^-(z-z_{k-1})} + B_k e^{\alpha_k^+(z-z_{k-1})}, \quad 1 \leq k \leq s \tag{9}$$

and

$$\widetilde{H}_k(\lambda, z) = A_k e^{\alpha_k^-(z-z_{k-1})} + B_k e^{\alpha_k^+(z-z_{k-1})}, \quad s+1 \leq k \leq N \tag{10}$$

where A_k and B_k are arbitrary constants which can be determined from the boundary conditions,

$$\alpha_k^- = \frac{n_k b_k - \sqrt{(n_k b_k)^2 + 4\lambda^2}}{2}$$

and

$$\alpha_k^+ = \frac{n_k b_k + \sqrt{(n_k b_k)^2 + 4\lambda^2}}{2}.$$

4 The Magnetic Field Response from the Linearly Conductivity Ground Profile

For the linearly varying conductivity ground profile, the equation represented the variation is denoted by

$$\sigma(z) = a(1 + \overleftrightarrow{n} bz)^m$$

where $a, b > 0$, $\overleftrightarrow{n} \in \{-1, 1\}$ and m$\in \{0, 1\}$. Putting $\sigma(z)$ to the equation (6), we now have

$$\frac{\partial^2 \widetilde{H}}{\partial z^2} - \frac{\overleftrightarrow{n} mb}{1 + \overleftrightarrow{n} bz}\frac{\partial \widetilde{H}}{\partial z} - \lambda^2 \widetilde{H} = 0$$

and the solution is

$$\widetilde{H}(\lambda, z) = \alpha^p(z)\left[EI_{-p}\left(\lambda\alpha(z)/b\right) + FK_{-p}\left(\lambda\alpha(z)/b\right)\right]$$

where E and F are arbitrary constants which can be determined from the boundary conditions. I_p and K_p are the Modified Bessel functions of the first and second kind of order p. $\alpha(z) = 1 + \overleftrightarrow{n} bz$, $p = (1+m)/2$.

For the N-layered stratified earths model, each layered has limit depth except for the lower most layered has infinite depth as mention in the previous section. The difference of this sections is in the varying conductivity profile which is denoted by

$$\sigma_1(z) = a(1 + \overleftrightarrow{n}_1 b_1 z)^{m_1}, \; 0 \leq z \leq z_1$$

and for $2 \leq z \leq N-1$,

$$\sigma_k(z) = \sigma_{k-1}(z_{k-1})(1 + \overleftrightarrow{n}_k b_k(z - z_{k-1}))^{m_k}, \qquad z_{k-1} \leq z \leq z_k,$$
$$\sigma_N(z) = \sigma_{N-1}(z_{N-1})(1 + \overleftrightarrow{n}_N b_N(z - z_{N-1}))^{m_N}, \qquad z > z_{N-1}.$$

The magnetic fields in each layered are

$$\widetilde{H}_k(\lambda, z) = \frac{I}{2\pi\lambda} + \alpha_k^{p_k}(z - z_{k-1})\Big[E_k I_{-p_k}(\lambda\alpha_k(z - z_{k-1})/b_k) + F_k K_{-p_k}(\lambda\alpha_k(z - z_{k-1})/b_k)\Big], \quad 1 \leq k \leq s, \tag{11}$$

and

$$\widetilde{H}_k(\lambda, z) = \alpha_k^{p_k}(z - z_{k-1})\Big[E_k I_{-p_k}(\lambda\alpha_k(z - z_{k-1})/b_k) + F_k K_{-p_k}(\lambda\alpha_k(z - z_{k-1})/b_k)\Big], \quad s+1 \leq k \leq N \tag{12}$$

where E_k and F_k are arbitrary constants which can be determined from the boundary conditions, $\alpha_k(z) = 1 + \overleftrightarrow{n}_k b_k z$, $p_k = (1 + m_k)/2$, $b_N > 0$, $n_N = 1$ and $m_N = 0$.

5 The Magnetic Field Response from the Binomially Conductivity Ground Profile

For the binomially varying conductivity ground profile, the equation represented the variation is denoted by

$$\sigma(z) = a(1 + bz)^{\overline{n}}$$

where $a, b > 0$, $\overline{n}$ is an integer. Putting $\sigma(z)$ to the equation (6), we now have

$$\frac{\partial^2 \widetilde{H}}{\partial z^2} - \frac{\overline{n}b}{1 + bz}\frac{\partial \widetilde{H}}{\partial z} - \lambda^2 \widetilde{H} = 0$$

and the solution is

$$\widetilde{H}(\lambda, z) = \beta^{\gamma}(z)\left[GI_{-\gamma}(\lambda\beta(z)/b) + HK_{-\gamma}(\lambda\beta(z)/b)\right],$$

where G and H are arbitrary constants which can be determined from the boundary conditions. $\beta(z) = 1 + bz$, $\gamma = (1 + \overline{n})/2$.

For the N-layered stratified earths model, each layered has limit depth except for the lower most layered has infinite depth as mention in the previous section. The difference of this sections is the varying conductivity profile which is denoted by the binomially varying as

$$\sigma_1(z) = a(1 + b_1 z)^{\overline{n}_1}, 0 \leq z \leq z_1$$

and for $2 \leq z \leq N-1$,

$$\sigma_k(z) = \sigma_{k-1}(z_{k-1})(1+b_k(z-z_{k-1}))^{\overline{n}_k}, \qquad z_{k-1} \leq z \leq z_k,$$
$$\sigma_N(z) = \sigma_{N-1}(z_{N-1})(1+b_N(z-z_{N-1}))^{\overline{n}_N}, \qquad z > z_{N-1}$$

where $a, b_k > 0$ and $\overline{n}_k$ is an integer. The magnetic fields in each layered are

$$\widetilde{H}_k(\lambda, z) = \frac{I}{2\pi\lambda} + \alpha_k^{p_k}(z-z_{k-1})\Big[G_k I_{-p_k}(\lambda\alpha_k(z-z_{k-1})/b_k) + H_k K_{-p_k}(\lambda\alpha_k(z-z_{k-1})/b_k)\Big], \quad 1 \leq k \leq s, \tag{13}$$

and

$$\widetilde{H}_k(\lambda, z) = \alpha_k^{p_k}(z-z_{k-1})\Big[G_k I_{-p_k}(\lambda\alpha_k(z-z_{k-1})/b_k) + H_k K_{-p_k}(\lambda\alpha_k(z-z_{k-1})/b_k)\Big], \quad s+1 \leq k \leq N \tag{14}$$

where G_k and H_k are arbitrary constants which can be determined from the boundary conditions, $\alpha_k(z) = 1 + b_k z, and\ p_k = (1+\overline{n}_k)/2$.

6 2-layered Earths Model

Although stratified models are often relevant and can usually be applied to real geoelectric structures, few treatments of continuous geoelectric structures have been presented. Although stratified models with a large number of layers can represent identically a continuum's response to surface measurements, numerical computation is usually quite take long time. A better way to handle certain continuous structures might be to solve the equations directly for the desired structure. In our work, we design the model as two layered earths structure.

6.1 A Homogeneous Earth with an Exponential Varying Overburden

In the case of a homogeneous earth with an exponential varying overburden, the equation (9) and (10) become

$$H_1(r,z) = \int_0^\infty \lambda\left[\frac{I}{2\pi\lambda}\right]\left\{1 - \frac{e^{\alpha^- z} - e^{\alpha^+ z}}{(1+\delta^-)e^{\alpha^- h} - (1+\delta^+)e^{\alpha^+ h}}\right\} J_1(\lambda r)d\lambda, 0 \leq z \leq h,$$

$$H_2(r,z) = \int_0^\infty \left[\frac{I}{2\pi}\right]\left\{\frac{\left(\delta^- e^{\alpha^- h} - \delta^+ e^{\alpha^+ h}\right)e^{-\lambda(z-h)}}{(1+\delta^-)e^{\alpha^- h} - (1+\delta^+)e^{\alpha^+ h}}\right\} J_1(\lambda r)d\lambda, z > h$$

where $\delta^- = \alpha^-/\lambda$ and $\delta^+ = \alpha^+/\lambda$.

6.2 A Homogeneous Earth with a Linear Varying Overburden

In the case of a homogeneous earth with a linearly varying overburden, the equation (11) and (12) become

$$H_1(r,z) = \int_0^\infty \left[\frac{I}{2\pi}\right] \left\{1 - \frac{\alpha(z)\left[I_1(\lambda\alpha(z)/b) - \delta K_1(\lambda\alpha(z)/b)\right]}{\alpha(h)\left(I^* + \delta K^*\right)}\right\} J_1(\lambda r)d\lambda, 0 \le z \le h,$$

$$H_2(r,z) = \int_0^\infty \left[\frac{I}{2\pi}\right] \left\{\frac{\left[I_0(\lambda\alpha(h)/b) + \delta K_0(\lambda\alpha(h)/b)\right]}{\left(I^* + \delta K^*\right)}\right\} \overleftrightarrow{n} e^{-\lambda(z-h)} J_1(\lambda r)d\lambda, z > h.$$

6.3 A Homogeneous Earth with a Binomially Varying Overburden

In the case of a homogeneous earth with a binomially varying overburden, the equation (13) and (14) become

$$H_1(r,z) = \int_0^\infty \left[\frac{I}{2\pi}\right] \left\{1 - \frac{\alpha^{-p}(h)\alpha^p(z)\left[I_{-p}(\lambda\alpha(z)/b) - \delta K_{-p}(\lambda\alpha(z)/b)\right]}{\left(I^* + \delta K^*\right)}\right\} J_1(\lambda r)d\lambda,$$

$0 \le z \le h$,

$$H_2(r,z) = \int_0^\infty \left[\frac{I}{2\pi}\right] \left\{\frac{\left[I_q(\lambda\alpha(h)/b) + \delta K_q(\lambda\alpha(h)/b)\right]}{\left(I^* + \delta K^*\right)}\right\} e^{-\lambda(z-h)} J_1(\lambda r)d\lambda, z > h$$

where $\alpha(z) = 1 + bz$, $p = (1+n)/2$ and $q = (1-n)/2$.

7 Numerical Experiments

In our forward model examples, we compute the magnetic field due to direct current source on the ground surface of the models in section 6.1, 6.2 and 6.3. The models are applied the current 1-Ampere, injected by the probe length of 1 meter perpendicular to the ground surface. The results are performed as the graphs in Figure 1, 2 and 3. The graphs are shown the behavior of the magnetic field against source-receiver spacing (r) at different depth. The curves of each model at the same depth are not too much different, but they are quite different as the depth is varied. The magnetic field intensities drop very fast as we increase the source-receiver spacing to 10 meters.

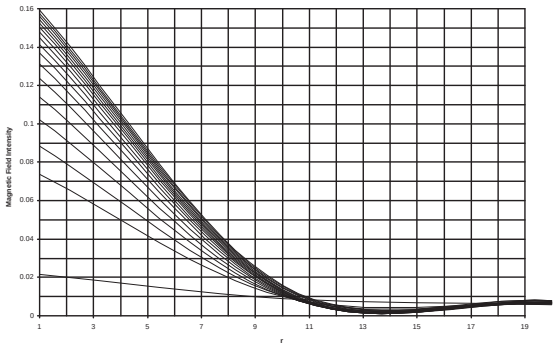

Figure 1: The behavior of magnetic field (from model in section 6.1) against r at different depth $z = 0, 0.2, 0.4, ..., 3.2, 3.4m$.

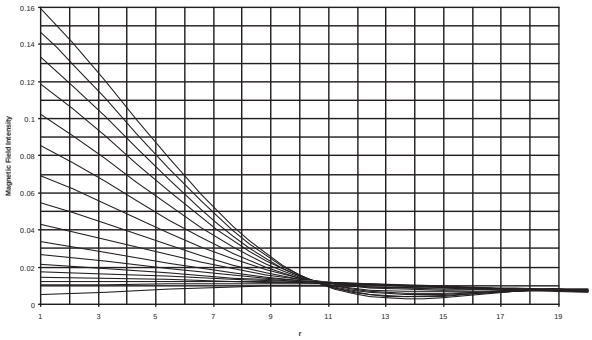

Figure 2: The behavior of magnetic field(from model in section 6.2) against r at different depth $z = 0, 0.2, 0.4, \ldots, 3.2, 3.4m$.

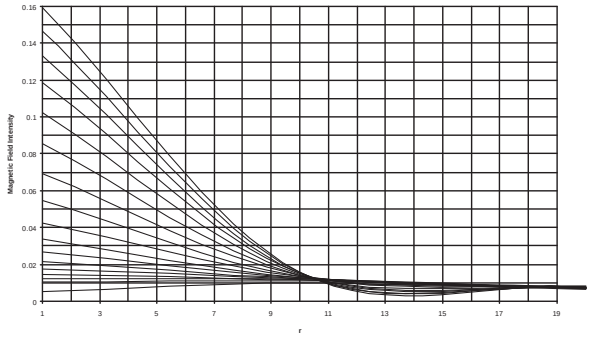

Figure 3: The behavior of magnetic field(from model in section 6.3) against r at different depth $z = 0, 0.2, 0.4, \ldots, 3.2, 3.4m$.

8 Inversion Example

In our inverse model example, we simulate reflection of radiation data from our forward model by injection the current 1-Ampere to the ground. The conductivity distribution below the ground surface is assumed to be continuous and depends only on depth. In our example, the model is given by

$$\sigma_1(z) = e^{-0.1960475832z},\ 0 \leq z \leq 1,$$
$$\sigma_2(z) = \sigma_1(1),\ z > 1.$$

The forward model to simulate the set of real data generates the magnetic field. The theoretical values are perturbed by superimposing a Gaussian relative error to the 3 per cent level. The associated errors can be regarded as realizations of normal random variables with zero means and variances $s_i^2; i = 1, 2, \ldots, m$. Table 1 shows the result from our procedure. We start the model with initial guess $b = 1$ and $n = -1$. The result from our procedure converge to $b = 0.1960475832$ which is the true value after using 6 iterations only.

i	0	1	2	3	4	5
b	1.000000000	0.4966546754	0.201211235	0.192002436	0.195923534	0.196487811
Δ_{H_i}	0.3413×10^{-3}	1.5661×10^{-7}	4.2316×10^{-10}	5.2547×10^{-13}	$8.8795\ \times10^{-15}$	3.6478×10^{-17}

Table 1 : Successive iterates using initial estimates for $n = -1, b = 1$ in an exponentially decreasing ground profile with true values being $n = -1$ and $b = 0.1960475832$

9 Conclusions

In this paper, we conduct the method to explore the parameter of the conductivity of ground. The method used the integral transform technique to produce the magnetic field which can be computed easily. The magnetic field is plotted against source-receiver spacing (r) at different depths. The inversion process is used to find out the parameters of the conductivity of ground. The conjugate gradient method is used to construct iterative procedure. The method performs good results and shows the robustness of the procedure.

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