Thai Journal of Mathematics (2003) 1: 1-19

Lattices of M-solid Generalized Varieties and M-solid Pseudovarieties

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Abstract: An identity is satisfied as a hyperidentity in a variety V of algebras if after any replacement of the operation symbols occurring in this identity by terms of the appropriate arity, one gets again identities of V. In a similar way one defines the weaker concept of an M-hyperidentity. If every identity in a variety is satisfied as an M-hyperidentity, the variety is said to be M-solid. Using the ultimately M-hypersatisfaction of an identity filter by a class of algebras, these concepts can be transferred to M-solid generalized varieties and M-solid pseudovarieties. We study the corresponding Galois connections and the complete lattices of all M-solid generalized varieties of a given type and of all M-solid pseudovarieties of this type. We investigate mappings between these lattices and apply the results to M-solid pseudovarieties of semigroups.

Keywords: Pseudovariety, M-solid pseudovariety, generalized variety.
2000 Mathematics Subject Classification: 08B05, 08C05, 08C99

1 Preliminaries

A generalized variety of type τ is a class of algebras of type τ that is closed under formation of homomorphic images, subalgebras, finite direct products, and arbitrary direct powers. For a class K of algebras of type τ, $\mathbf{H}(K), \mathbf{S}(K), \mathbf{P}(K), \mathbf{P_f}(K), \mathbf{Pow}(K)$ denote the classes of all homomorphic images, subalgebras, direct products, finite direct products, and direct powers, respectively. Since varieties are classes of algebras of type τ which are closed under homomorphic images, subalgebras and arbitrary direct products, every variety is a generalized variety. Equivalently, varieties are classes of algebras of the same type τ satisfying every equation from a set Σ of equations as identities. To characterize generalized varieties by equations we need the concept of an identity filter. A *filter* in a lattice $\mathcal{L}$ is the universe F of a sublattices of $\mathcal{L}$ with the property that from $X \in F$, $Y \in L$

and $Y \supseteq X$ there follows $Y \in F$. A filter F is called a *principal filter* if there exists an $X \in L$ such that F is generated by X: $F = \langle\{X\}\rangle$. In this case we will skip one pair of brackets and simply write $\langle X\rangle$.

Definition 1.1 Let $(f_i)_{i\in I}$ be a sequence of operation symbols of type τ where f_i is n_i - ary and let X be a set of variables. By $W_\tau(X)$ we denote the set of all terms of type τ. An identity filter Δ of type τ is a filter in the power set lattice $\mathcal{P}(W_\tau(X)^2)$. By $IF(\tau)$ we denote the class of all identity filters of type τ.

Definition 1.2 We say that an algebra $\mathcal{A} := (A; (f_i^A)_{i\in I})$ of type τ *ultimately satisfies* an identity filter $\Delta \in IF(\tau)$ if there exists a set $\Sigma \in \Delta$ such that $\mathcal{A}$ satisfies each equation from Σ as identity, i.e.

$$\mathcal{A} \underset{u.s.}{\models} \Delta :\Leftrightarrow \exists\Sigma \in \Delta\ (\mathcal{A} \models \Sigma).$$

Then one can prove:

Proposition 1.3 ([8]) *Let Δ be a filter of identities, then the class of all algebras which ultimately satisfy Δ is a generalized variety, and conversely, for any generalized variety W, there exists an identity filter Δ such that W consists precisely of all algebras which ultimately satisfy Δ.*

We will now give another characterization of generalized varieties using the Galois connection which is induced by the relation

$$\underset{u.s.}{\models}\ \subseteq Alg(\tau) \times IF(\tau).$$

For a class $K \subseteq Alg(\tau)$ and for a set $\mathcal{F} \subseteq IF(\tau)$ we define

$$GMod\,\mathcal{F} := \{\mathcal{A} \in Alg(\tau) \mid \forall\Delta \in \mathcal{F}\ (\mathcal{A} \underset{u.s.}{\models} \Delta)\}$$

and

$$Filt\ K := \{\Delta \in IF(\tau) \mid \forall\mathcal{A} \in K\ (\mathcal{A} \underset{u.s.}{\models} \Delta)\}.$$

Clearly, the pair $(GMod, Filt)$ is a Galois connection between $Alg(\tau)$ and $IF(\tau)$.

As usual for a Galois connection, we have two closure operators $GModFilt$ and $FiltGMod$ and their sets of fix points, i.e. the sets

$$\{\mathcal{F} \subseteq IF(\tau) \mid FiltGMod\,\mathcal{F} = \mathcal{F}\} \text{ and } \{K \subseteq Alg(\tau) \mid GModFilt\,K = K\}$$

form complete sublattices of the power set lattice of $IF(\tau)$ and of the power set lattice of $Alg(\tau)$, respectively. Usually, instead of $GMod\ \{\Delta\}$ we write simply $GMod\ \Delta$, for $\Delta \in IF(\tau)$.

Lemma 1.4 *For every subset $\mathcal{F} \subseteq IF(\tau)$, the class $GMod\ \mathcal{F}$ is a generalized variety.*

Proof. Let $\mathcal{F} \subseteq IF(\tau)$. By definition we have

$$\begin{aligned} GMod\ \mathcal{F} &:= \{\mathcal{A} \in Alg(\tau) \mid \forall \Delta \in \mathcal{F}\ (\mathcal{A} \underset{u.s.}{\models} \Delta)\} \\ &:= \{\mathcal{A} \in Alg(\tau) \mid \forall \Delta \in \mathcal{F}\ \exists \Sigma_\Delta \in \Delta(\mathcal{A} \models \Sigma_\Delta)\}. \end{aligned}$$

We form $\Delta' = \bigcap\limits_{\Delta \in \mathcal{F}} \Delta$. We want to show $GMod\ \mathcal{F} = GMod\ \Delta'$. Let $\mathcal{A} \in GMod\ \mathcal{F}$. By definition, for all $\Delta \in \mathcal{F}$ there exists $\Sigma_\Delta \in \Delta$ such that $\mathcal{A} \models \Sigma_\Delta$. Then we have $\mathcal{A} \models \bigcup\limits_{\Delta \in \mathcal{F}} \Sigma_\Delta$. Since $\Sigma_\Delta \subseteq \bigcup\limits_{\Delta \in \mathcal{F}} \Sigma_\Delta$ and $\Sigma_\Delta \in \Delta$, we have $\bigcup\limits_{\Delta \in \mathcal{F}} \Sigma_\Delta \in \Delta$ for all $\Delta \in \mathcal{F}$. This means that $\bigcup\limits_{\Delta \in \mathcal{F}} \Sigma_\Delta \in \Delta'$. Now we have $\mathcal{A} \underset{u.s.}{\models} \Delta'$, i.e $\mathcal{A} \in GMod\ \Delta'$. For the opposite inclusion, let $\mathcal{A} \in GMod\ \Delta'$. By definition, there exists $\Sigma \in \Delta'$ such that $\mathcal{A} \models \Sigma$. Since $\Delta' \subseteq \Delta$ for all $\Delta \in \mathcal{F}$ we get $\mathcal{A} \underset{u.s.}{\models} \Delta$, i.e. $\mathcal{A} \in GMod\ \mathcal{F}$. Now we have $GMod\ \mathcal{F} = GMod\ \Delta'$. By Proposition 1.3, $GMod\ \mathcal{F}$ is a generalized variety. □

Proposition 1.5 *A class K of algebras of type τ is a generalized variety if and only if $K = GModFilt\ K$.*

Proof. Let K be a generalized variety. Then by Proposition 1.3 there exists an identity filter Δ such that $K = GMod\ \Delta$ and then $GModFilt\ K = GModFiltGMod\ \Delta = GMod\ \Delta = K$ using a property of Galois connections. The converse follows from Lemma 1.4. □

By Proposition 1.5 and the property of a Galois connection, the class of all generalized varieties of the same type τ forms a complete lattice with the intersection as meet operation.

Now we want to mention another characterization of generalized varieties using the concept of a directed family of sets given by C.J. Ash. [1]. A family Γ of sets or of classes is said to be *directed* if for all A, $B \in \Gamma$ there exists $C \in \Gamma$ with $A \subseteq C$ and $B \subseteq C$.

Theorem 1.6 ([1]) *For a class K of algebras of type τ the following conditions are equivalent:*

(i) K is a generalized variety.

(ii) K is the union of some directed family of varieties of type τ.

Later on we need also the concept of an ideal which is dual to the concept of a filter. The universe I of a sublattice of a lattice $\mathcal{L}$ is called an *ideal of* $\mathcal{L}$ if the following condition is satisfied: If $X \in I$, $Y \in \mathcal{L}$ and $Y \subseteq X$ then $Y \in I$.

A pseudovariety K is a class of finite algebras which is closed under homomorphic images, subalgebras and finite direct products. Clearly, if V is a variety then its finite part V^{fin} is a pseudovariety. But the converse is wrong [2]. Pseudovarieties can be defined by identity filters as follows:

Proposition 1.7 *([8]) Let Δ be a filter of identities. Then the class of all finite algebras which ultimately satisfy Δ is a pseudovariety and conversely, for any pseudovariety K there exists an identity filter Δ such that K consists precisely of all finite algebras which ultimately satisfy Δ.*

We get one more characterization of pseudovarieties restricting the relation $\underset{u.s.}{\models}$ to the class $Alg_{fin}(\tau)$ of all finite algebras from $Alg(\tau)$. For this restricted relation and the Galois connection we use the same denotations $\underset{u.s.}{\models} \subseteq Alg_{fin}(\tau) \times IF(\tau)$, $PMod\,\mathcal{F}$, $Filt\,K$ for every $K \subseteq Alg_{fin}(\tau)$, $\mathcal{F} \subseteq IF(\tau)$. Then K is a pseudovarieties iff $K = PModFilt\,K$. For more details see [4].

The importance of generalized varieties for pseudovarieties becomes clear by the following proposition by C.J. Ash. For a generalized variety W we denote by W^{fin} the set of all finite algebras in W. We remark that W^{fin} is the greatest pseudovariety contained in W.

Proposition 1.8 ([1]) *For every pseudovariety K there exists a generalized variety W such that $K = W^{fin}$.*

It is known that every variety is a generalized variety, since it is the union of the directed family of itself. Now we consider the lattice $\mathcal{L}(\tau)$ of all varieties of type τ and the lattice $\mathcal{L}^G(\tau)$ of all generalized varieties of type τ. Then we obtain that $\mathcal{L}(\tau)$ is a sublattice of $\mathcal{L}^G(\tau)$. To prove this we need a well-known Lemma which is a consequence of some part of Ash's paper [1].

Lemma 1.9 ([1]) *Let K be a class of algebras of type τ. Then K is a variety if and only if there exists a principal identity filter Δ such that $K = GMod\ \Delta$.*

Also the following proposition should be well-known, but nevertheless we will give a proof.

Proposition 1.10 $(\mathcal{L}(\tau); \overset{var}{\bigwedge}, \overset{var}{\bigvee})$ *is a sublattice of* $(\mathcal{L}^G(\tau); \overset{gen}{\bigwedge}, \overset{gen}{\bigvee})$.

Proof. It is clear that $\mathcal{L}(\tau) \subseteq \mathcal{L}^G(\tau)$. Let $V_1, V_2 \in \mathcal{L}(\tau)$. We want to show $V_1 \overset{var}{\bigwedge} V_2 = V_1 \overset{gen}{\bigwedge} V_2$ and $V_1 \overset{var}{\bigvee} V_2 = V_1 \overset{gen}{\bigvee} V_2$. It is clear that $V_1 \overset{var}{\bigwedge} V_2 = V_1 \overset{gen}{\bigwedge} V_2$ and $V_1 \overset{var}{\bigvee} V_2 \supseteq V_1 \overset{gen}{\bigvee} V_2$. It is left to show that $V_1 \overset{var}{\bigvee} V_2 \subseteq V_1 \overset{gen}{\bigvee} V_2$. Let $\Delta = \{\Sigma \subseteq W_\tau(X)^2 | Id\mathcal{A}_1 \cap Id\mathcal{A}_2 \subseteq \Sigma$ for some $\mathcal{A}_1 \in V_1$ and $\mathcal{A}_2 \in V_2\}$.

At first we show that Δ is a principal filter. Let $\Sigma \in \Delta$, $\Sigma' \subseteq W_\tau(X)^2$ and $\Sigma \subseteq \Sigma'$. Since $\Sigma \in \Delta$, there exists $\mathcal{A}_1 \in V_1$ and $\mathcal{A}_2 \in V_2$ such that $Id\mathcal{A}_1 \cap Id\mathcal{A}_2 \subseteq \Sigma \subseteq \Sigma'$. It follows that $\Sigma' \in \Delta$. Let $\{\Sigma_i | i \in I\} \subseteq \Delta$. For all $i \in I$, there exist $\mathcal{A}_{1i} \in V_1$ and $\mathcal{A}_{2i} \in V_2$ such that $Id\mathcal{A}_{1i} \cap Id\mathcal{A}_{2i} \subseteq \Sigma_i$, then $\bigcap_{i \in I}(Id\mathcal{A}_{1i} \cap Id\mathcal{A}_{2i}) \subseteq \bigcap_{i \in I} \Sigma_i$ implies that $(\bigcap_{i \in I} Id\mathcal{A}_{1i}) \cap (\bigcap_{i \in I} Id\mathcal{A}_{2i}) \subseteq \bigcap_{i \in I} \Sigma_i$. Since $\bigcap_{i \in I} Id\mathcal{A}_{ji} \supseteq IdV_j = Id(\mathcal{F}_{V_j}(X))$ for all $j = 1, 2$ where $\mathcal{F}_{V_j}(X)$ is the free algebra generated by X with respect to V_j and since $\mathcal{F}_{V_j}(X) \in V_j$ for all $j \in \{1, 2\}$, we get $\bigcap_{i \in I} \Sigma_i \in \Delta$. Now we have that Δ is closed under arbitrary intersections. Altogether, Δ is principal.

The next step is to show $V_1 \overset{gen}{\bigvee} V_2 = GMod\ \Delta$. Clearly, $V_1 \cup V_2 \subseteq GMod\ \Delta$. Let W be an arbitrary generalized variety containing $V_1 \cup V_2$. Let $W = GMod\ \Delta'$ for some $\Delta' \in IF(\tau)$. Let $\mathcal{A} \in GMod\ \Delta$. Then there exists $\Sigma \in \Delta$ such that $\mathcal{A} \models \Sigma$ implies that $\mathcal{A} \models Id\mathcal{B}_1 \cap Id\mathcal{B}_2 \subseteq \Sigma$ for some $\mathcal{B}_1 \in V_1$, $\mathcal{B}_2 \in V_2$. Since $\mathcal{B}_1$, $\mathcal{B}_2 \in W$, $\mathcal{B}_1 \models \Sigma_1'$ and $\mathcal{B}_2 \models \Sigma_2'$ for some Σ_1', $\Sigma_2' \in \Delta'$, then $\Sigma_1' \subseteq Id\mathcal{B}_1$ and $\Sigma_2' \subseteq Id\mathcal{B}_2$ implies that $Id\mathcal{B}_1 \cap Id\mathcal{B}_2 \subseteq \Sigma_1' \cap \Sigma_2' \in \Delta'$, then $\Sigma \in \Delta'$. Therefore, $\mathcal{A} \in W$. Now we have $V_1 \overset{gen}{\bigvee} V_2 = GMod\Delta$. Since Δ is principal and by Lemma 1.9, $V_1 \overset{gen}{\bigvee} V_2$ is a variety. Then $V_1 \overset{var}{\bigvee} V_2 \subseteq V_1 \overset{gen}{\bigvee} V_2$. Altogether, $V_1 \overset{var}{\bigvee} V_2 = V_1 \overset{gen}{\bigvee} V_2$ □

2 M-solid Generalized Varieties and a Second Galois Connection

To define M-solid generalized varieties we need the concept of a hypersubstitution. Hypersubstitutions of type τ are mappings from the set of all operation symbols of type τ into the set of all terms of type τ which preserve the arities. This means, to each n_i-ary operation symbol of type τ we assign an n_i-ary term from $W_\tau(X)$. Hypersubstitutions can be extended to mappings $\hat{\sigma} : W_\tau(X) \to W_\tau(X)$ which are defined on the set $W_\tau(X)$ of all terms of type τ by the following inductive definition:

(i) $\hat{\sigma}[x] := x$ for every variable $x \in X$.

(ii) $\hat{\sigma}[f_i(t_1, \ldots, t_{n_i})] := \sigma(f_i)(\hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$ for composite terms $f_i(t_1, \ldots, t_{n_i})$.

Let $Hyp(\tau)$ be the set of all hypersubstitutions of type τ. On the set $Hyp(\tau)$ we may define a binary operation $\circ_h$ by $\sigma_1 \circ_h \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$. Let σ_{id} be the hypersubstitution which maps each operation symbol f_i to the term $f_i(x_1, ..., x_{n_i})$. Then $(Hyp(\tau); \circ_h, \sigma_{id})$ is a monoid. Hypersubstitutions can be applied to equations $s \approx t$ of type τ. This gives new equations of the form $\hat{\sigma}[s] \approx \hat{\sigma}[t]$. For arbitrary sets $\Sigma \subseteq W_\tau(X)^2, \Sigma \neq \emptyset$ and for a submonoid $M \subseteq Hyp(\tau)$ we define

$$\chi_M^E[\Sigma] := \bigcup_{s \approx t \in \Sigma} \bigcup_{\sigma \in M} \hat{\sigma}[s] \approx \hat{\sigma}[t].$$

It is not difficult to prove that $\chi_M^E : \mathcal{P}(W_\tau(X)^2) \to \mathcal{P}(W_\tau(X)^2)$ is a closure operator which is called completely additive because of its definition as union of singleton sets.

To apply hypersubstitutions to algebras we consider so-called derived algebras. If $\mathcal{A} = (A; (f_i^A)_{i \in I})$ is an algebra of type τ, then $\sigma(\mathcal{A}) = (A; (\sigma(f_i)^A)_{i \in I})$ is called *algebra derived from* $\mathcal{A}$ *by* σ. Here $\sigma(f_i)^{\mathcal{A}}$ is the term operation induced on the algebra $\mathcal{A}$ by the term $\sigma(f_i)$.
For the class $K \subseteq Alg(\tau)$ and for a submonoid $M \subseteq Hyp(\tau)$ we define

$$\chi_M^A[K] := \bigcup_{\sigma \in M} \bigcup_{\mathcal{A} \in K} \sigma(\mathcal{A}).$$

χ_M^A is also a completely additive closure operator. Between both operators χ_M^A and χ_M^E there is a close interconnection given by

$$\mathcal{A} \models \chi_M^E[s \approx t] \Leftrightarrow \chi_M^A[\mathcal{A}] \models s \approx t.$$

Because of this property we call (χ^A_M, χ^E_M) a conjugate pair of operators. For more background on conjugate pairs of additive closure operators see [6] and [7]. A variety V is called $M-solid$ if $\chi^A_M[V] = V$. We may define M-solid generalized varieties in a natural way.

Definition 2.1 A generalized variety W of algebras of type τ is called $M - solid$ if $\chi^A_M[W] = W$.

Definition 2.2 Let $\Delta \in IF(\tau)$ be an identity filter of type τ and let $\mathcal{A} \in Alg(\tau)$ be an algebra of type τ. Then we say that $\mathcal{A}$ ultimately M-hypersatisfies Δ if for every hypersubstitution σ in M there exists a set Σ_σ of equations in Δ such that $\mathcal{A}$ satisfies $\sigma(\Sigma_\sigma)$:

$$\mathcal{A} \underset{u.Mh.s.}{\models} \Delta :\Leftrightarrow \forall\sigma \in M\ \exists\Sigma_\sigma \in \Delta\ (\mathcal{A} \models \sigma(\Sigma_\sigma)).$$

The concept of ultimately hypersatisfaction (for $M = Hyp(\tau)$) was defined in [8]. By $H_M GMod\,\Delta$ for an identity filter Δ we denote the class of all algebras of type τ which ultimately M-hypersatisfy Δ. The characterization of solid generalized varieties given in [8] can be easily generalized to $M - solid$ generalized varieties.

Theorem 2.3 *For a class W of algebras of type τ the following conditions are equivalent:*

(i) W is an $M - solid$ generalized variety.

(ii) W is a generalized variety and for each identity filter $\Delta \in IF(\tau)$, W ultimately M-hypersatisfies Δ whenever W ultimately satisfies Δ, i.e.

$$W \underset{u.Mh.s.}{\models} \Delta \Leftrightarrow W \underset{u.s.}{\models} \Delta.$$

(iii) There exists an identity filter $\Delta \in IF(\tau)$ such that W consists precisely of all algebras which ultimately M-hypersatisfy Δ, in this case we write, $W = H_M GMod\,\Delta$.

Proof. $(i) \Rightarrow (ii)$. Assume that W is an M-solid generalized variety, i.e. $W = \chi^{\mathcal{A}}_M[W]$. Let $\Delta \in IF(\tau)$. We have

$$\begin{aligned} W \underset{u.s.}{\models} \Delta \ &\Leftrightarrow\ \chi^{\mathcal{A}}_M[W] \underset{u.s.}{\models} \Delta \\ &\Leftrightarrow\ \forall\sigma \in M\ \forall\mathcal{A} \in W\ \exists\Sigma \in \Delta(\sigma(\mathcal{A}) \models \Sigma) \\ &\Leftrightarrow\ \forall\sigma \in M\ \forall\mathcal{A} \in W\ \exists\Sigma \in \Delta(\mathcal{A} \models \sigma(\Sigma)) \\ &\Leftrightarrow\ W \underset{u.Mh.s.}{\models} \Delta. \end{aligned}$$

$(ii) \Rightarrow (iii)$. Assume that W is a generalized variety and for each identity filter $\Delta \in IF(\tau)$, W ultimately M-hypersatisfies Δ whenever W ultimately satisfies Δ. Since by assumption W is a generalized variety, there exists $\Delta \in IF(\tau)$ such that $W = GMod\ \Delta$, i.e. $W \underset{u.s.}{\models} \Delta$. Then by assumption $W \underset{u.Mh.s.}{\models} \Delta$ and then $W \subseteq H_M GMod\ \Delta$. Again by our assumption from $W \underset{u.Mh.s.}{\models} \Delta$ we get $W \underset{u.s.}{\models} \Delta$ and this means $H_M GMod\ \Delta \subseteq GMod\ \Delta = W$. Altogether we have the equality $W = H_M GMod\ \Delta$.

$(iii) \Rightarrow (i)$. Assume there exists $\Delta \in IF(\tau)$ such that $W = H_M GMod\ \Delta$. We have to show that W is an M-solid generalized variety. Let $\mathcal{A} \in W = H_M GMod\ \Delta$. By definition for all $\sigma \in M$, there exists $\Sigma \in \Delta$ such that $\mathcal{A} \models \sigma(\Sigma)$. Then subalgebras, homomorphic images and arbitrary powers of $\mathcal{A}$ also satisfy $\sigma(\Sigma)$. It follows that W is closed under these formations. Let $\mathcal{A}_1,\ \mathcal{A}_2 \in W$. By definition for all $\sigma \in M$ there exist $\Sigma_1,\ \Sigma_2 \in \Delta$ such that $\mathcal{A}_1 \models \sigma(\Sigma_1)$ and $\mathcal{A}_2 \models \sigma(\Sigma_2)$. Then we have $\mathcal{A}_1 \times \mathcal{A}_2 \models \sigma(\Sigma_1) \cap \sigma(\Sigma_2) = \sigma(\Sigma_1 \cap \Sigma_2)$. Since $\Sigma_1 \cap \Sigma_2 \in \Delta$, we get $\mathcal{A}_1 \times \mathcal{A}_2 \underset{u.Mh.s.}{\models} \Delta$, i.e. $\mathcal{A}_1 \times \mathcal{A}_2 \in W$. This can be generalized for more than two algebras and W is closed under finite direct products. Therefore W is a generalized variety.
To prove that W is closed under the formation of derived algebras, let $\sigma \in M$ and $\mathcal{A} \in W = H_M GMod\ \Delta$. We want to show that $\sigma(\mathcal{A}) \in W$. With $\delta \in M$, we have $\sigma \circ_h \delta \in M$. Since $\mathcal{A} \in H_M GMod\ \Delta$, there exists $\Sigma \in \Delta$ such that $\mathcal{A} \models (\sigma \circ_h \delta)(\Sigma) = \hat{\sigma}(\hat{\delta}(\Sigma))$. By the conjugate pair property, we have $\sigma(\mathcal{A}) \models \delta(\Sigma)$. Then $\sigma(\mathcal{A}) \underset{u.Mh.s.}{\models} \Delta$, i.e. $\sigma(\mathcal{A}) \in H_M GMod \Delta = W$. Altogether, W is an M-solid generalized variety. □

Now we will give a second characterization of M-solid generalized varieties using the Galois connection which is induced by the relation $\underset{u.Mh.s.}{\models}$.

For a class $K \subseteq Alg(\tau)$ and for a set $\mathcal{F} \subseteq IF(\tau)$ we define

$$\begin{aligned} H_M GMod\ \mathcal{F} &:= \{\mathcal{A} \in Alg(\tau) \mid \forall \Delta \in \mathcal{F}\ (\mathcal{A} \underset{u.Mh.s.}{\models} \Delta)\} \\ H_M Filt\ K &:= \{\Delta \in IF(\tau) \mid \forall \mathcal{A} \in K\ (\mathcal{A} \underset{u.Mh.s.}{\models} \Delta)\}. \end{aligned}$$

Then the pair $(H_M GMod, H_M Filt)$ is the Galois connection induced by $\underset{u.Mh.s.}{\models}$. It follows that the products $H_M GMod H_M Filt$ and $H_M Filt H_M GMod$

are closure operators and that the sets

$$\{K \subseteq Alg(\tau) | H_M GMod H_M Filt\ K = K\}$$

and

$$\{\mathcal{F} \subseteq IF(\tau) | H_M Filt H_M GMod\ \mathcal{F} = \mathcal{F}\}$$

of their fixed points form complete lattices which are dually isomorphic.

Corollary 2.4 *Let $\mathcal{F} \subseteq IF(\tau)$. Then every class of algebras of the form $H_M GMod\ \mathcal{F}$ is an M-solid generalized variety and conversely, for every M-solid generalized variety W there is a set $\mathcal{F} \subseteq IF(\tau)$ of filters such that $W = H_M GMod\ \mathcal{F}$.*

Proof. Let $\mathcal{F} \subseteq IF(\tau)$. Then

$$\begin{aligned} H_M GMod\ \mathcal{F} &:= \{\mathcal{A} \in Alg(\tau) \mid \forall \Delta \in \mathcal{F}\ (\mathcal{A} \underset{u.Mh.s.}{\models} \Delta)\} \\ &:= \{\mathcal{A} \in Alg(\tau) \mid \forall \Delta \in \mathcal{F}\ \forall \sigma \in M\ \exists \Sigma^{\sigma}_{\Delta} \in \Delta\ (\mathcal{A} \models \sigma(\Sigma^{\sigma}_{\Delta}))\}. \end{aligned}$$

We form $\Delta' := \bigcap\limits_{\Delta \in \mathcal{F}} \Delta$. We want to show that $H_M GMod\,\mathcal{F} = H_M GMod\ \Delta'$. Let $\mathcal{A} \in H_M GMod\ \mathcal{F}$. Then by definition, for all $\Delta \in \mathcal{F}$ and for all $\sigma \in M$ there exists $\Sigma^{\sigma}_{\Delta} \in \Delta$ such that $\mathcal{A} \models \sigma(\Sigma^{\sigma}_{\Delta})$. It follows that $\mathcal{A} \models \sigma(\bigcup\limits_{\Delta \in \mathcal{F}} \Sigma^{\sigma}_{\Delta})$. Since $\Sigma^{\sigma}_{\Delta} \subseteq \bigcup\limits_{\Delta \in \mathcal{F}} \Sigma^{\sigma}_{\Delta}$ and $\Sigma^{\sigma}_{\Delta} \in \Delta$, one has $\bigcup\limits_{\Delta \in \mathcal{F}} \Sigma^{\sigma}_{\Delta} \in \Delta$ for every $\Delta \in \mathcal{F}$. It follows that $\bigcup\limits_{\Delta \in \mathcal{F}} \Sigma^{\sigma}_{\Delta} \in \Delta'$. Then we have $\mathcal{A} \underset{u.Mh.s.}{\models} \Delta'$. It means that $\mathcal{A} \in H_M GMod\ \Delta'$, i.e. $H_M GMod\ \mathcal{F} \subseteq H_M GMod\ \Delta'$. For the opposite inclusion, suppose that $\mathcal{A} \in H_M GMod\ \Delta'$. By definition, for all $\sigma \in M$ there exists $\Sigma_{\sigma} \in \Delta'$ such that $\mathcal{A} \models \sigma(\Sigma_{\sigma})$. Since $\Delta' \subseteq \Delta$ for all $\Delta \in \mathcal{F}$, we have $\Sigma_{\sigma} \in \Delta$. Now we obtain $\mathcal{A} \underset{u.Mh.s.}{\models} \Delta$, then $\mathcal{A} \in H_M Mod\ \mathcal{F}$, i.e. $H_M GMod\ \mathcal{F} \supseteq H_M GMod\ \Delta'$. Altogether, $H_M GMod\ \mathcal{F} = H_M GMod\ \Delta'$. By Theorem 2.3, we get that $H_M GMod\ \mathcal{F}$ is an M-solid generalized variety. The converse is clear by Theorem 2.3 if we set $\mathcal{F} = \{\Delta\}$ for some $\Delta \in IF(\tau)$. □

Corollary 2.5 *Let W be a class of algebras of type τ. Then the following conditions are equivalent.*

(i) W is an M-solid generalized variety.

(ii) $Filt\ W = H_M Filt\ W$.

(iii) $H_M GModH_M Filt\ W = W$.

Proof. $(i) \Rightarrow (ii)$. Assume that W is an M-solid generalized variety. Then we have:

$$\Delta \in Filt\ W \Leftrightarrow \forall \mathcal{A} \in W\ (\mathcal{A} \underset{u.s.}{\models} \Delta) \Leftrightarrow \forall \mathcal{A} \in W\ (\mathcal{A} \underset{u.Mh.s.}{\models} \Delta)$$

$$\Leftrightarrow \Delta \in H_M Filt\ W$$

by Theorem 2.3.

$(ii) \Rightarrow (iii)$. Assume that $Filt\ W = H_M Filt\ W$. Then we have: $H_M GMod(H_M Filt\ W) = H_M GMod(Filt\ W) \subseteq GModFilt\ W = W$, and it is clear that $H_M GMod(H_M Filt\ W) \supseteq W$.

$(iii) \Rightarrow (i)$. Obvious by Corollary 2.4. □

Now we know that the set of all M-solid generalized varieties of type τ forms a complete lattice, denoted by $\mathcal{S}_M^G(\tau)$, which is contained in the complete lattice of all generalized varieties of type τ. Our next aim is to show that the lattice of all M-solid generalized varieties forms even a complete sublattice of the set of all generalized varieties of the same type. For the proof we will apply the characterization of complete sublattices via so-called Galois closed subrelations.

Definition 2.6 Let R and R' be relations between sets A and B. Let (μ, ι) and (μ', ι') be the Galois connections between A and B induced by R and R', respectively. The relation R' is called a *Galois − closed subrelation* of R if,

(i) $R' \subseteq R$, and

(ii) $\forall T \subseteq A, \forall S \subseteq B\ (\mu'(T) = S \wedge \iota'(S) = T) \Rightarrow (\mu(T) = S \wedge \iota(S) = T)$.

From this definition we can prove the following characterization of complete sublattices of a complete lattice.

Theorem 2.7 *Let $R \subseteq A \times B$ be a relation between sets A and B, with induced Galois connection (μ, ι). Let $\mathcal{H}_{\iota\mu}$ be the corresponding lattice of closed subsets of A .*

(i) If $R' \subseteq A \times B$ is a Galois-closed subrelation of R, then the class $\mathcal{U}_{R'} := \mathcal{H}_{\iota'\mu'}$ is a complete sublattice of $\mathcal{H}_{\iota\mu}$.

(ii) If $\mathcal{U}$ is a complete sublattice of $\mathcal{H}_{\iota\mu}$, then the relation

$$\mathcal{R}_{\mathcal{U}} := \cup\{T \times \mu(T) | T \in \mathcal{U}\}$$

is a Galois-closed subrelation of R.

(iii) For any Galois-closed subrelation R' of R and any complete sublattice $\mathcal{U}$ of $\mathcal{H}_{\iota\mu}$ we have $\mathcal{U}_{R_{\mathcal{U}}} = \mathcal{U}$ and *$R_{\mathcal{U}_{R'}} = R'$.*

For the proof and more background see [7]and [3].

Now we apply Theorem 2.7 to our situation and set $A := Alg(\tau)$, $B := IF(\tau)$,$R := \underset{u.s.}{\models}$, $R' := \underset{u.Mh.s.}{\models}$, $(\mu, \iota) := (Filt, GMod)$, $(\mu', \iota') := (H_M Filt, H_M GMod)$, $\mathcal{H}_{\iota\mu} := \mathcal{L}^G(\tau)$, $\mathcal{H}_{\iota'\mu'} := \mathcal{S}^G_M(\tau)$.

Proposition 2.8 *For every monoid M of hypersubstitutions the relation $\underset{u.Mh.s.}{\models} \subseteq Alg(\tau) \times IF(\tau)$ is a Galois-closed subrelation of the relation $\underset{u.s.}{\models} \subseteq Alg(\tau) \times IF(\tau)$.*

Proof. It is clear that $\underset{u.Mh.s.}{\models} \subseteq \underset{u.s.}{\models}$.
Let $\mathcal{F} \subseteq IF(\tau)$ be a set of identity filters of type τ and let $K \subseteq Alg(\tau)$ be a class of finite algebras of type τ such that $H_M Filt\ K = \mathcal{F}$ and $H_M GMod\ \mathcal{F} = K$. The last equation means by Corollary 2.4 that K is an M-solid generalized variety and then by Corollary 2.5 we have $Filt\ K = H_M Filt\ K = \mathcal{F}$ and thus $Filt\ K = \mathcal{F}$. It is left to show that $GMod\ \mathcal{F} = K$. We have $GMod\mathcal{F} = GModH_M Filt\ K = GModFilt\ K$ if we use again that $K = H_M GMod\mathcal{F}$ is an M-solid generalized variety and therefore $H_M Filt\ K = Filt\ K$. Since every M-solid generalized variety is a generalized variety, we have $GMod\ \mathcal{F} = GModFilt\ K = K$. This shows that $\underset{u.Mh.s.}{\models}$ is a Galois closed subrelation of $\underset{u.s.}{\models}$. □

Now we apply Theorem 2.7 and obtain:

Theorem 2.9 *For every monoid $M \subseteq Hyp(\tau)$ of hypersubstitutions the lattice $\mathcal{S}^G_M(\tau)$ is a complete sublattice of the lattice $\mathcal{L}^G(\tau)$.*

An M-solid pseudovariety K is a pseudovariety with $\chi^{\mathcal{A}}_M[K] = K$. Let $Alg_{fin}(\tau)$ be the class of all finite algebras of type τ. Then we consider the restricted relation $\underset{u.Mh.s.}{\models} \subseteq Alg_{fin}(\tau) \times IF(\tau)$ which was defined first in [8]. The characterization given in [8] can be generalized easily to M-solid pseudovarieties. M-solid pseudovarieties can be defined by identity filters as follows:

Theorem 2.10 *For a class K of finite algebras the following conditions are equivalent:*

(i) K is an $M-$ solid pseudovariety.

(ii) K is a pseudovariety and for each identity filter $\Delta \in IF(\tau)$, K ultimately M-hypersatisfies Δ whenever K ultimately satisfies Δ, i.e.

$$K \underset{u.Mh.s.}{\models} \Delta \Leftrightarrow K \underset{u.s.}{\models} \Delta.$$

(iii) There exists an identity filter $\Delta \in IF(\tau)$ such that K consists precisely of all finite algebras which ultimately M-hypersatisfy Δ, in this case we write, $K = H_M PMod\, \Delta$.

The Galois connection $(H_M GMod, H_M Filt)$ can be also restricted to a Galois connection $(H_M PMod, H_M Filt)$ which is induced be the restricted relation. Then K is an M-solid pseudovariety if and only if

$$K = H_M PMod H_M Filt\ K.$$

In [4] was also proved that the lattice $\mathcal{S}_M^{PS}(\tau)$ of all M-solid pseudovarieties of type τ is a complete sublattice of the lattice $\mathcal{L}^{PS}(\tau)$ of all pseudovarieties of type τ.

3 Mappings of the Lattice of all M-solid Generalized Varieties

Let I be an ideal in the lattice $\mathcal{L}(\tau)$ of all varieties of type τ. From the properties of an ideal it follows that I is a directed family of varieties of type τ. Therefore $\cup I$ is a generalized variety. Let $\mathcal{L}^I(\tau)$ be the lattice of all ideals of $\mathcal{L}(\tau)$. Then we will find the interconnection between the lattice $\mathcal{L}^I(\tau)$ and the lattice $\mathcal{L}^G(\tau)$ of all generalized varieties of type τ.

Lemma 3.1 *Let C be a directed family of varieties of type τ and let $\langle C\rangle$ be the ideal generated by C. Then $\cup\langle C\rangle = \cup C$.*

Proof. Let C be a directed family of varieties of type τ. Let $D = \{W \in \mathcal{L}(\tau) | W \subseteq V_i$ for some $V_i \in C\}$. We want to show that D is an ideal. Let $W' \subseteq W \in D$. Then $W' \subseteq W \subseteq V_i$ for some $V_i \in C$. Then we obtain $W' \in D$. Let $W_1, W_2 \in D$. Then we have $W_1 \subseteq V_1$ and $W_2 \subseteq V_2$ for some $V_1, V_2 \in C$. Since $V_1, V_2 \in C$, there exists a variety V_3 such that $V_1, V_2 \subseteq V_3$, so $W_1 \vee W_2 \subseteq V_3$, then $W_1 \vee W_2 \in D$. Now we have that D is an ideal. The next step is to show $\langle C \rangle = D$. It is clear that $C \subseteq D$. Let J be an arbitrary ideal of varieties of type τ containing C. For showing $D \subseteq J$, let $W \in D$, then there exists $V_i \in C$ such that $W \subseteq V_i$. Since $C \subseteq J$ and J is an ideal we get $W \subseteq V_i \in J$, i.e. $D \subseteq J$. Now we have $\langle C \rangle = D$ and it is left to show $\cup C = \cup \langle C \rangle$. It is easy to see that $\cup C \subseteq \cup \langle C \rangle$. For the opposite inclusion, let $\mathcal{A} \in \cup \langle C \rangle$. Then there exists a variety $W \in \langle C \rangle$ such that $\mathcal{A} \in W$. Since $\langle C \rangle = \{W \in \mathcal{L}(\tau) | W \subseteq V_i$ for some $V_i \in C\}$, there exists a variety $V_i \in C$ such that $\mathcal{A} \in W \subseteq V_i$. Then $\mathcal{A} \in \cup C$, i.e. $\cup C \supseteq \cup \langle C \rangle$. Altogether, $\cup C = \cup \langle C \rangle$. □

Lemma 3.2 *Let I and J be ideals of varieties of type τ. Then* $\cup (I \vee J) = (\cup I) \vee (\cup J)$.

Proof. Clearly, $(\cup I) \vee (\cup J) \subseteq \cup (I \vee J)$. For the opposite inclusion, we want first to show that $(\cup I) \vee (\cup J) = \cup C$ where $C = \{V(\mathcal{A}_1) \vee V(\mathcal{A}_2) | \mathcal{A}_1 \in \cup I$ and $\mathcal{A}_2 \in \cup J\}$ where $V(\mathcal{A})$ is the variety generated by $\mathcal{A}$. We have to show that C is directed. Let $\mathcal{A}_1, \mathcal{B}_1 \in \cup I$, $\mathcal{A}_2, \mathcal{B}_2 \in \cup J$ and $V(\mathcal{A}_1) \vee V(\mathcal{A}_2), V(\mathcal{B}_1) \vee V(\mathcal{B}_2) \in C$. Since I and J are ideals, $V(\mathcal{A}_1), V(\mathcal{B}_1) \in I$, $V(\mathcal{A}_2), V(\mathcal{B}_2) \in J$, then $V(\mathcal{A}_1) \vee V(\mathcal{B}_1) = V(\mathcal{A}_1 \times \mathcal{B}_1) \in I$ and $V(\mathcal{A}_2) \vee V(\mathcal{B}_2) = V(\mathcal{A}_2 \times \mathcal{B}_2) \in J$. We obtain $V(\mathcal{A}_1) \vee V(\mathcal{B}_1), V(\mathcal{A}_2) \vee V(\mathcal{B}_2) \subseteq V(\mathcal{A}_1 \times \mathcal{B}_1) \vee V(\mathcal{A}_2 \times \mathcal{B}_2)$. Since $\mathcal{A}_1 \times \mathcal{B}_1 \in \cup I$ and $\mathcal{A}_2 \times \mathcal{B}_2 \in \cup J$, C is directed. By definition of C, we have $(\cup I) \cup (\cup J) \subseteq \cup C$. Let Z be an arbitrary generalized variety containing $(\cup I) \cup (\cup J)$. Then there exists a directed family of varieties, say C_Z, such that $Z = \cup C_Z$. To show $\cup C \subseteq Z$, let $\mathcal{A} \in \cup C$. Then $\mathcal{A} \in V(\mathcal{A}_1) \vee V(\mathcal{A}_2)$ for some $\mathcal{A}_1 \in \cup I$ and for some $\mathcal{A}_2 \in \cup J$. Since Z contains $(\cup I) \cup (\cup J)$, we have $\mathcal{A}_1$ and $\mathcal{A}_2 \in Z$. So $\mathcal{A}_1 \in W_1$, $\mathcal{A}_2 \in W_2$ for some $W_1, W_2 \in C_Z$. Since C_Z is directed, there exists $W \in C_Z$ such that $W_1, W_2 \subseteq W$. So $\mathcal{A} \in V(\mathcal{A}_1) \vee V(\mathcal{A}_2) \subseteq W$, then $\mathcal{A} \in \cup C_Z = Z$. Now we have $(\cup I) \vee (\cup J) = \cup C$. Recall that $I \vee J = \{W \subseteq W_I \vee W_J | W_I \in I$ and $W_J \in J\}$. We get $\cup (I \vee J) = \cup \{W_I \vee W_J | W_I \in I$ and $W_J \in J\}$. Let $\mathcal{A} \in \cup (I \vee J)$. Then $\mathcal{A} \in W_I \vee W_J$ for some $W_I \in I$, $W_J \in J$. Since W_I and W_J are varieties, $W_I = V(\mathcal{F}_{W_I}(X))$ and $W_J = V(\mathcal{F}_{W_J}(X))$. Because $\mathcal{F}_{W_I}(X) \in W_I$ and $\mathcal{F}_{W_J}(X) \in W_J$, we get $\mathcal{F}_{W_I}(X) \in \cup I$ and $\mathcal{F}_{W_J}(X) \in \cup J$. It follows that $W_I \vee W_J = V(\mathcal{F}_{W_I}(X)) \vee V(\mathcal{F}_{W_J}(X)) \in C$. Therefore, $\mathcal{A} \in (\cup I) \vee (\cup J)$, i.e.$(\cup I) \vee (\cup J) \supseteq \cup (I \vee J)$. □

Theorem 3.3 *The lattice $\mathcal{L}^I(\tau)$ of ideals of varieties of type τ is isomorphic to the lattice $\mathcal{L}^G(\tau)$ of generalized varieties of type τ.*

Proof. We define a mapping $\varphi : \mathcal{L}^I(\tau) \to \mathcal{L}^G(\tau)$ by $I \mapsto \cup I$ for every $I \in \mathcal{L}^I(\tau)$.
It is clear that φ is well-defined and $\varphi(I \wedge J) = \varphi(I) \wedge \varphi(J)$, for every $I, J \in \mathcal{L}^I(\tau)$. By Lemma 3.2, we have $\varphi(I \vee J) = \varphi(I) \vee \varphi(J)$. And by Lemma 3.1 φ is onto.
Next we want to show that φ is injective. Let $I, J \in \mathcal{L}^I(\tau)$ such that $\cup I = \cup J$. Let $W \in I$. Since W is a variety , $W = V(\mathcal{F}_W(X))$ and $\mathcal{F}_W(X) \in W \subseteq \cup I = \cup J$. Then there exists $V \in J$ such that $\mathcal{F}_W(X) \in V$, so $V(\mathcal{F}_W(X)) \subseteq V$ implies $V(\mathcal{F}_W(X)) = W \in J$, i.e. $I \subseteq J$. The opposite inclusion can be proved similarly. Therefore, φ is injective. Altogether, $\mathcal{L}^I(\tau) \cong \mathcal{L}^G(\tau)$. $\square$

Since all M-solid varieties of type τ form a complete lattice $\mathcal{S}_M(\tau)$, we can consider the lattice of all ideals of this lattice.

Lemma 3.4 *Let $(\mathcal{S}^I_M(\tau); \overset{im}{\bigwedge}, \overset{im}{\bigvee})$ be the lattice of all ideals of M-solid varieties of type τ and let $(\mathcal{S}^G_M(\tau); \overset{mg}{\bigwedge}, \overset{mg}{\bigvee})$ be the lattice of all M-solid generalized varieties of type τ. Then $\cup(I \overset{im}{\bigvee} J) = (\cup I) \overset{mg}{\bigvee} (\cup J)$ for all $I, J \in \mathcal{S}^I_M(\tau)$.*

Proof. Let $(\mathcal{L}(\tau); \overset{var}{\bigwedge}, \overset{var}{\bigvee})$ be the lattice of all varieties of type τ, $(\mathcal{S}_M(\tau); \overset{mv}{\bigwedge}, \overset{mv}{\bigvee})$ be the lattice of all M-solid varieties of type τ and $(\mathcal{L}^G(\tau); \overset{gen}{\bigwedge}, \overset{gen}{\bigvee})$ be the lattice of all generalized varieties of type τ. Let I, J be ideals of M-solid varieties of type τ. It is clear that $\cup(I \overset{im}{\bigvee} J) \supseteq (\cup I) \overset{mg}{\bigvee} (\cup J)$. Since, by Theorem 2.9, the lattice $\mathcal{S}^G_M(\tau)$ is a complete sublattice of the lattice $\mathcal{L}^G(\tau)$ and since in the proof of Lemma 3.2, we have $(\cup I) \overset{mg}{\bigvee} (\cup J) = (\cup I) \overset{gen}{\bigvee} (\cup J) = \cup C$ where $C = \{V(\mathcal{A}) \overset{var}{\bigvee} V(\mathcal{B}) \mid \mathcal{A} \in \cup I \text{ and } \mathcal{B} \in \cup J\}$. Because $I \overset{im}{\bigvee} J = \{W \in \mathcal{S}_M(\tau) \mid W \subseteq W_I \overset{mv}{\bigvee} W_J$ for some $W_I \in I$ and $W_J \in J\}$, we have $\cup(I \overset{im}{\bigvee} J) = \cup\{W_I \overset{mv}{\bigvee} W_J \mid W_I \in I \text{ and } W_J \in J\}$. Let $\mathcal{A} \in \cup(I \overset{im}{\bigvee} J)$. There exist $W_I \in I$ and $W_J \in J$ such that $\mathcal{A} \in W_I \overset{mv}{\bigvee} W_J = W_I \overset{var}{\bigvee} W_J$, since the lattice $\mathcal{S}_M(\tau)$ is a complete sublattice of the lattice $\mathcal{L}(\tau)$. Since $W_I = V(\mathcal{F}_{W_I}(X))$ and $W_J = V(\mathcal{F}_{W_J}(X))$, we have $\mathcal{A} \in V(\mathcal{F}_{W_I}(X)) \overset{var}{\bigvee} V(\mathcal{F}_{W_J}(X))$. Every relatively free algebra with respect to a variety is contained in the variety, so $V(\mathcal{F}_{W_I}(X)) \overset{var}{\bigvee} V(\mathcal{F}_{W_J}(X)) \in C$. It means that $\mathcal{A} \in \cup C = (\cup I) \overset{mg}{\bigvee} (\cup J)$, i.e. $\cup(I \overset{im}{\bigvee} J) \subseteq (\cup I) \overset{mg}{\bigvee} (\cup J)$. Altogether, $\cup(I \overset{im}{\bigvee} J) = (\cup I) \overset{mg}{\bigvee} (\cup J)$ □

Now we want to consider the mapping between the lattice $\mathcal{S}_M^I(\tau)$ of ideals of M-solid varieties of type τ and the lattice $\mathcal{S}_M^G(\tau)$ of M-solid generalized varieties of type τ. [Gra-V-P;97] proved that not every solid generalized variety of type τ is the union of a directed family of solid varieties of type τ. Therefore not every M-solid generalized variety is the union of an ideal of M-solid varieties. That means the restriction of the mapping used in the proof of Theorem 3.3 to ideals of M-solid varieties is not onto the lattice of all M-solid generalized varieties. As a consequence of this fact and Lemma 3.4, we have:

Corollary 3.5 *The lattice $\mathcal{S}_M^I(\tau)$ can be embedded into the lattice $\mathcal{S}_M^G(\tau)$.*

4 The Lattice of all M-solid Pseudovarieties

In this section we start with some basic knowledge of the concept of locally finite varieties. We will use locally finiteness of varieties to consider the mapping between the lattice of all M-solid generalized varieties and the lattice of all M-solid pseudovarieties. At the end of this section we apply the results to M-solid pseudovarieties of semigroups.

Definition 4.1 An algebra $\mathcal{A}$ of type τ is said to be *locally finite* if all its finitely generated subalgebras are finite. A class K of algebras of the same type τ is said to be *locally finite* if all its elements are locally finite.

It is well-known that a variety V is locally finite if and only if all finitely generated free algebras with respect to V are finite.

Lemma 4.2 *Let V_1 and V_2 be locally finite varieties of type τ. If $V_1^{fin} = V_2^{fin}$ then $V_1 = V_2$.*

Proof. Let V_i for every $i = 1, 2$ be locally finite varieties of type τ. Then V_i^{fin} contains all finitely generated free algebras with respect to V_i for every $i = 1, 2$. From $V_1^{fin} = V_2^{fin}$, we have $\{\mathcal{F}_{V_1}(n) | n \in N, n \geq 1\} = \{\mathcal{F}_{V_2}(n) | n \in N, n \geq 1\}$. Since a variety is uniquely determined by all finitely generated free algebras $\mathcal{F}_V(n)$ for $n \in N, n \geq 1$, we get $V_1 = V_2$. □

Theorem 4.3 *Let V be a locally finite variety of type τ, $\mathcal{L}(V)$ be the lattice of all subvarieties of V and $\mathcal{L}^{PS}(V^{fin})$ the lattice of all subpseudovarieties of V^{fin}. Then $\mathcal{L}(V)$ is isomorphic to $\mathcal{L}^{PS}(V^{fin})$.*

Proof. We define a mapping $\varphi : \mathcal{L}(V) \to \mathcal{L}^{PS}(V^{fin})$ by $W \mapsto W^{fin}$, for every $W \in \mathcal{L}(V)$. It is clear that φ is well-defined and $\varphi(W_1 \wedge W_2) = \varphi(W_1) \wedge \varphi(W_2)$, for every $W_1, W_2 \in \mathcal{L}(V)$. By Lemma 4.2, φ is one-to-one. Since $\mathcal{L}(\tau)$ is a sublattice of $\mathcal{L}^G(\tau)$ and by [Alm; 94] (Exercise 3.2.9), $(W_1 \vee W_2)^{fin} = W_1^{fin} \vee W_2^{fin}$, i.e. $\varphi(W_1 \vee W_2) = \varphi(W_1) \vee \varphi(W_2)$. The next step is to show that φ is onto. Let K be a subpseudovariety of V^{fin}. We form $W = V(K)$ - the variety generated by K. We want to show $W^{fin} = K$. Let $\mathcal{A} \in W^{fin}$. Then $\mathcal{A}$ is a homomorphic image of a free algebra $\mathcal{F}_W(n)$ for some $n \geq 1$. Since $\mathcal{F}_W(n)$ is isomorphic to a subdirect product of algebras in K and V is locally finite, $\mathcal{F}_W(n)$ is isomorphic to finite subdirect product of K. It follows that $\mathcal{F}_W(n) \in K$, i.e. $\mathcal{A} \in K$. That means $W^{fin} \subseteq K$. The opposite inclusion is obvious. Altogether, $\mathcal{L}(V)$ is isomorphic to $\mathcal{L}^{PS}(V^{fin})$. □

The proof shows that every subpseudovariety of a locally finite variety is the finite part of a subvariety. We know that the lattice $\mathcal{S}_M(\tau)$ of M-solid varieties of type τ is a complete sublattice of the lattice $\mathcal{L}(\tau)$ and [4] proved that the lattice $\mathcal{S}_M^{PS}(\tau)$ is a complete sublattice of the lattice $\mathcal{L}^{PS}(\tau)$. The restriction of the mapping used in the proof of Theorem 4.3 to the lattice $\mathcal{S}_M(V)$ of all M-solid subvarieties of V gives the following isomorphism:

Corollary 4.4 *Let V be a locally finite variety of type τ, $\mathcal{S}_M(V)$ be the lattice of all M-solid subvarieties of V and $\mathcal{S}_M^{PS}(V^{fin})$ be the lattice of all M-solid subpseudovarieties of V^{fin}. Then $\mathcal{S}_M(V)$ is isomorphic to $\mathcal{S}_M^{PS}(V^{fin})$.*

Proof. Let W be an M-solid subvariety of V. Then $W^{fin} \in \mathcal{S}_M^{PS}(V^{fin})$ since the finite part of W is closed under the operator $\chi_M^{\mathcal{A}}$. Conversely if $K \in \mathcal{S}_M^{PS}(V^{fin})$ then $V(K)$ is M-solid since K is closed under the operator $\chi_M^{\mathcal{A}}$. In a similar way as in the proof of Theorem 3.4 we show that $V(K)^{fin} = K$. □

Corollary 4.4 and its proof show that every M-solid subpseudovariety of a locally finite variety is the finite part of an M- solid variety.

Proposition 4.5 *Let V be a locally finite generalized variety of type τ. Then the lattice $\mathcal{L}^{PS}(V^{fin})$ of all subpseudovarieties of V^{fin} is isomorphic to a homomorphic image of the lattice $\mathcal{L}^G(V)$ of all generalized subvarieties of V.*

Proof. We define a mapping $\varphi : \mathcal{L}_M^G(V) \to \mathcal{L}^{PS}(V^{fin})$ by $\varphi(W) = W^{fin}$, for all $W \in \mathcal{L}^G(\tau)$. At first, we have to show that φ is a surjective lattice homomorphism.
It is clear that φ is well-defined and $\varphi(W_1 \wedge W_2) = \varphi(W_1) \wedge \varphi(W_2)$. By [2] (Exercise 3.2.9), $(W_1 \vee W_2)^{fin} = W_1^{fin} \vee W_2^{fin}$, i.e. $\varphi(W_1 \vee W_2) = \varphi(W_1) \vee \varphi(W_2)$. Now we have that φ is a lattice homomorphism. The next step is to show that φ is onto. Let $P \in \mathcal{L}^{PS}(V^{fin})$. We form $C = \{V(\mathcal{A}) | \mathcal{A} \in P\}$. Since if $\mathcal{A}_1, \mathcal{A}_2 \in P$, then $\mathcal{A}_1 \times \mathcal{A}_2 \in P$ and $V(\mathcal{A}_1) \vee V(\mathcal{A}_2) = V(\mathcal{A}_1 \times \mathcal{A}_2)$, it means that C is directed. It is clear that $P \subseteq (\cup C)^{fin}$. On the other hand, for every $\mathcal{A} \in (\cup C)^{fin}$, there exists an algebra $\mathcal{B} \in P$ such that $\mathcal{A} \in V(\mathcal{B}) \in C$. Since $(V(\mathcal{B}))^{fin} \subseteq P$, it follows that $\mathcal{A} \in P$. Now we have proved that φ is a surjective lattice homomorphism. By the homomorphic image theorem, the proposition is proved. □

Corollary 4.6 *Let V be a locally finite generalized variety of type τ. Then the lattice $\mathcal{S}_M^{PS}(V^{fin})$ is isomorphic to a homomorphic image of the lattice $\mathcal{S}_M^G(V)$ of all M-solid generalized subvarieties of V.*

Proof. We define a mapping $\varphi : \mathcal{S}_M^G(V) \to \mathcal{S}_M^{PS}(V^{fin})$ by $\varphi(W) = W^{fin}$, for all $W \in \mathcal{S}_M^G(\tau)$. By Theorem 2.9, the lattice $\mathcal{S}_M^G(V)$ is a complete sublattice of the lattice $\mathcal{L}^G(\tau)$ and by [4], the lattice $\mathcal{S}_M^{PS}(\tau)$ is a complete sublattice of the lattice $\mathcal{L}^{PS}(\tau)$. By Proposition 4.5, we obtain that φ is a lattice homomorphism. The next step is to show that φ is onto. Let $P \in \mathcal{S}_M^{PS}(V^{fin})$. We denote by $V_M(\mathcal{A})$ the M-solid variety generated by $\{\mathcal{A}\}$. We form $C = \{V_M(\mathcal{A}) \mid \mathcal{A} \in P\}$. It is clear that C is directed, i.e. $\cup C \in \mathcal{S}_M^G(V)$ and $P \subseteq (\cup C)^{fin}$. Since $V_M(\mathcal{A}) = \mathbf{HSP}(\chi_M^{\mathcal{A}}[\mathcal{A}])$, $\chi_M^{\mathcal{A}}[\mathcal{A}]$ is a class of finite algebras and $\chi_M^{\mathcal{A}}[\mathcal{A}] \subseteq P$ for all $\mathcal{A} \in P$, we have $(V_M(\mathcal{A}))^{fin} = \mathbf{HSP_f}(\chi_M^{\mathcal{A}}[\mathcal{A}]) \subseteq P$. It follows that $(\cup C)^{fin} \subseteq P$.Therefore, φ is onto. By the homomorphic image theorem, the corollary is proved. □

Finally we want to apply these results to M-solid pseudovarieties of semigroups. Since locally finiteness is an important condition for the results in this section, we have to characterize locally finite M-solid varieties of semigroups. This was done in [Den-P;03]. We need the definition of a proper hypersubstitution with respect to a variety V.

Definition 4.7 Let V be a variety of type τ. Then a hypersubstitution $\sigma \in Hyp(\tau)$ is called *proper* if for every $s \approx t \in IdV$ we have $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdV$.

Theorem 4.8 *([Den-P;03] Let $M \subseteq Hyp(\tau)$ be a monoid of hypersubstitutions of type $\tau = (2)$. Then $V = H_M Mod\{x_1(x_2x_3) \approx (x_1x_2)x_3\}$ is locally finite if and only if $\sigma_{x_1x_2x_1}$ or $\sigma_{x_2x_1x_2}$ is a proper hypersubstitution of V.*

Note that if $V = H_M Mod\{x_1(x_2x_3) \approx (x_1x_2)x_3\}$ is locally finite, then every M-solid variety of semigroups is locally finite, since $V = H_M Mod\{x_1(x_2x_3) \approx (x_1x_2)x_3\}$ is the greatest one.

Theorem 4.9 *If $\sigma_{x_1x_2x_1}$ or $\sigma_{x_2x_1x_2}$ is a proper hypersubstitution of $V = H_M Mod\{x_1(x_2x_3) \approx (x_1x_2)x_3\}$ then the lattice of all M-solid pseudovarieties of semigroups is isomorphic to the lattice of all M-solid varieties of semigroups.*

Proof. This directly follows from Corollary 4.4 and Theorem 4.8. □

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(Received 13 July 2003)

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