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Idempotents and Similar Operators

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Abstract : If T is an idempotent operator on a Hilbert space H such that the spectrum of T contains two different points, then T has the representation $I \oplus \begin{bmatrix} 0 & A \\ 0 & I \end{bmatrix}$, where A is a unique, positive, and similar operator. Furthermore, if T on $\Im = K \oplus H$ has the form $\begin{bmatrix} \alpha I & A \\ 0 & \beta I \end{bmatrix}$, where α and β are scalars, then a characterization of the norm of T in terms of the norm of A is proved. Examples of such operators are also given. In particular, genetic operators are defined. Finally, a necessary and a sufficient condition for similar operators is provided.

Keywords : Idempotent; Spectrum; Essential spectrum; Similar operators.
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1 Introduction

Throughout this article, H and K will denote Hilbert spaces, and $\Im = K \oplus H$. The identity operator is denoted by I. We write $\mathrm{Sp}(T)$ and $r(T)$ for the spectrum and the spectral radius of an operator T respectively.

Let T_1 and T_2 be two operators in $B(H)$ and $B(K)$ respectively. Then, T_1 is similar to T_2 if there exists an invertible operator S from H onto K such that $T_2 S = S T_1$.

An operator T is called a contraction if the norm of T is less than one, [1]. Also, an operator on a Hilbert space is similar to a contraction if and only if its spectrum is contained in the interior of the unit disc; see [1].

Let H_1 be a subset of H. Let T be a continuous map from H_1 to H, then T is called a t-contraction if there exists a constant $t \geq 0$ such that for any nonempty bounded subset H_0 of H we have $\alpha(T(H_0)) \leq t\alpha(H_0)$, where $\alpha(T(H_0))$ is the measure of noncompactness of $T(H_0)$ and $\alpha(T) = \inf\{t > 0 : T \text{ is a } t\text{-contraction}\}$, see [2].

We note that if Φ is a compact operator and Φ_0 is a contraction such that $\alpha(\varphi_0) < 1$, then an operator $\psi = \Phi + \varphi_0$ is a t-contraction with $||\psi|| < 1$. W also note that an operator A is a $||A||$-contraction and $\alpha(A) \leq ||A||$.

2 Idempotent Operators

In this section, we prove some results on idempotent operators where the spectrum contains two distinct points.

Theorem 2.1. *Let T be an idempotent operator on H with $Sp(T) = \{\alpha, \beta\}$, where α and β are distinct scalars. Then T is similar to an operator of the form $I \oplus \begin{bmatrix} 0 & A \\ 0 & I \end{bmatrix}$, where A is a unique, positive (strict), and similar operator.*

Proof. Let T be an idempotent operator with $Sp(T) = \{\alpha, \beta\}$, where α and β are distinct scalars. Then $(\frac{T-\alpha I}{\beta-\alpha})$ is idempotent. Let $N(T)$ and $N(T^*)$ denote the null spaces of T and T^* respectively. If K_0 is the orthogonal complement of $N(T) \cap N(T^*)$ in H and T is idempotent, then T can be written as $0 \oplus S$ on H, where $S = \begin{bmatrix} 0 & A_1 \\ 0 & A_2 \end{bmatrix}$ on K_0, see [3]. If $R(S^*)$ denotes the range of S^* then $K_0 = N(S) \oplus R(S^*)$.

We also note that if S is idempotent (which is the case here), then $A_2 = I$ and S has a better representation of the form $I \oplus \begin{bmatrix} 0 & D \\ 0 & I \end{bmatrix}$ on $N(B) \oplus [N(S) \oplus \overline{(R(S^*)} \ominus N(B)]$ where $\ominus$ denotes the orthogonal complement of $N(B)$. The $R(D)$ is dense (D is one to one) and $A = U^*D$, where U is the unitary operator. This means the operator D can be decomposed so that $D = UA$. In this case, a simple matrix multiplication yields:

$$\begin{bmatrix} 0 & U^*D \\ 0 & I \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} 0 & A \\ 0 & I \end{bmatrix}.$$

Thus, by the definition of similar operators, the operator S is similar to $I \oplus \begin{bmatrix} 0 & A \\ 0 & I \end{bmatrix}$, and the representation of T is valid.

For uniqueness, let $T = 0 \oplus I \oplus \begin{bmatrix} 0 & A \\ 0 & I \end{bmatrix}$ on $[N(T) \cap N(T^*)] \oplus [N(T-I) \cap N(T-I)^*]$. In fact, the dimensions of these spaces are uniquely determined by T. For two positive operators A_0 and A, let $\begin{bmatrix} 0 & A \\ 0 & I \end{bmatrix}$ and $\begin{bmatrix} 0 & A_0 \\ 0 & I \end{bmatrix}$ be similar. If

U is the unitary operator, then $U\begin{bmatrix} 0 & A \\ 0 & I \end{bmatrix} = \begin{bmatrix} 0 & A_0 \\ 0 & I \end{bmatrix} U$, and this gives us the following equations:

$$A_0U_4 = U_1A + U_2 \text{ and } U_3 = 0 \text{ where } U = \begin{bmatrix} U_1 & U_2 \\ U_3 & U_4 \end{bmatrix}.$$

Furthermore,

$$U^*\begin{bmatrix} 0 & A_0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} 0 & A_0 \\ 0 & I \end{bmatrix} U^* \Rightarrow U_2 = 0 \text{ and } A_0U_4 = U_1A.$$

Also, $A_0^2 = (U_4^*A_0)(A_0U_4) = (U_4^*A_0)(U_1A) = A^2 \Rightarrow A_0$ and A are similar operators. □

Theorem 2.2. *Let* $T = \begin{bmatrix} \alpha I & A \\ 0 & \beta I \end{bmatrix}$ *on* $\Im = K \oplus H$, *where* A *is the same as in Theorem* 1 *above, and* α *and* β *are scalars. Then* $||T|| = \left|\left|\begin{bmatrix} \alpha & ||A|| \\ 0 & \beta \end{bmatrix}\right|\right|$.

Proof. Let $\Im = K \oplus H$ and v be a unit vector in $\Im$. If v_1 and v_2 are the unit vectors in K and H respectively, then $v = pv_1 + qv_2$, where p and q are scalars with $|p|^2 + |q|^2 = 1$. Then

$$\begin{aligned} Tv &= (\alpha pv_1 + qAv_2) \oplus (\beta qv_2) \\ &\Rightarrow ||Tv||^2 = ||\alpha pv_1 + qAv_2||^2 \\ &= |\alpha|^2|p|^2 + |\beta|^2|q|^2 + |q|^2||Av_2||^2 + 2Re(\overline{\alpha pq}(Av, v_1)) \\ &\Rightarrow \left|\left|\begin{bmatrix} \alpha & (Av_2, v_1) \\ 0 & \beta \end{bmatrix} v\right|\right| \leq ||Tv||^2. \end{aligned}$$

Let

$$\eta = \sup\left\{\left|\left|\begin{bmatrix} \alpha & (Av_2, v_1) \\ 0 & \beta \end{bmatrix}\right|\right| : v_1 \in K, v_2 \in H, \text{ and } ||v_1|| = 1 = ||v_2||^2\right\}.$$

Then $\eta \leq ||T||$. Let the norm of tha matrix $\begin{bmatrix} \alpha & \gamma \\ 0 & \beta \end{bmatrix}$ for the first fixed α and β be a monotonically increasing function of $|\gamma|$. Then $||A|| = \sup\{|(Av_2, v_1)| : v_1 \in K, v_2 \in H, \text{ and } ||v_1|| = 1 = ||v_2||\}$ and $\left|\left|\begin{bmatrix} \alpha & ||A|| \\ 0 & \beta \end{bmatrix}\right|\right| \leq ||T||$. Let r be a real number such that $\overline{\alpha pq}||A||(\cos r + i \sin r) \geq 0$. Then, from above, we have $||T|| \leq \left|\left|\begin{bmatrix} \alpha & ||A|| \\ 0 & \beta \end{bmatrix}\right|\right|$, which proves the theorem. □

Corollary 2.3. $||Tv|| = ||T|| \Leftrightarrow ||Av_2|| = ||A||$.

Proof. Let $v \in \Im$ and $||Tv|| = ||T||$. Let $q = 0$. Then from Theorem 1, we have $||Tv|| = ||T|| = ||\alpha q v_2|| = |\alpha \geq ||T^*(v_1 \oplus 0)||| = \sqrt{(|\alpha|^2 + ||A^* v_1||^2)} \Rightarrow A = 0$. Next, let $q \neq 0$. Then from Theorem 2.1, it follows that $||Av_2|| = ||A||$. The other implication is a straight forward application of Theorem 1 and the fact that if v_1 is a unit vector in H, then $v_1 = \frac{Av_2}{||Av_2||}$ with $||A|| = (Av_2, v_1)$. □

Corollary 2.4. $||T - \alpha I||^2 = |\alpha - \beta|^2 + ||A||^2$.

Proof. The corollary is an easy consequence of Theorem 2.1. □

Example 2.5. *It is known that the boundary of the spectrum of an operator is contained in the approximate spctrum of the operator. Since* $W(T)$*(*$W(T)$ *is the numerical range of* T*) is convex, we have* $coSp(T) \subset \overline{W(T)}$*. For more on numerical ranges, refer to [4]. The equality does not hold in general. For example, let* $H = C^2$*, and let* $T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$*. Then,* $Sp(T) = \{0\}$*, although* T *is not idempotent. On the other hand,*

$$W(T) = \{zw^* : z, w, \in C, |z|^2 + |w|^2 = 1\} = \left\{\lambda \in C : |\lambda| \leq \frac{1}{2}\right\}.$$

Remark 2.6. *If* T *is a normal operator, then* $coSp(T) = \overline{W(T)}$*.*

Example 2.7. *Let* $T = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \in B(C)^2$ *be a hermitian operator such that*

$$||(z, w)|| = \max\left\{|z|, |w|, \frac{|z - w|}{\sqrt{2}}\right\}.$$

Then $Sp(T)$ *contains two distinct points* 0 *and* 1*. Also, by Theorem 2.2, it follows that* $W(T)$ *is either the closed line segment with two points* α *and* β *or the disc with foci at* α *and* β*, where the major and minor axes are given by* $||T - \alpha I||$ *and* $\sqrt{||T - \alpha I||^2 - |\alpha - \beta|^2}$*, respectively. In fact,* $W(T) = co\{0, 1, \frac{1+i}{2}\}$*.*

Example 2.8. *Let* $T = x_1 \otimes x_2$ *with rank of* $T = 1$*. Then* $W(T)$ *is the closed disc with foci* (x_1, x_2) *and zero. Major and monor axes are given by* $||x_1||||x_2||$ *and* $\sqrt{||x_1||^2||x_2||^2 - |(x_1 - x_2)|^2}$*, respectively. The numerical radius* $w(T)$ *is given by* $\frac{1}{2}(||x_1||||x_2|| + |(x_1, x_2)|)$*. Compare this with [5] and [6].*

Remark 2.9. *If* $T = \begin{bmatrix} \alpha & (Av_2, v_1) \\ 0 & \beta \end{bmatrix}$*, then* $W(T)$ *is the closed disc with foci at* α *and* β*.*

The major and minor axes are given by $\sqrt{(Av_2, v_1)^2 + |\alpha - \beta|^2}$ and $|(Av_2, v_1)|$ respectively. The numerical range $W(T)$ is closed by Corollary 2.4. In fact, $W(T)$ is closed if and only if $||A|| = |(Av_2, v_1)|$.

Example 2.10. *This example has its roots in the biological sciences. We note that genetic operators are idempotent. Let* $\Gamma = \{x = (x_0, \ldots, x_{n-1}) : h(x) = 1, x_j \geq 0\}$ *be the set of all possible populations, where* $h(x) = \sum_{j=0}^{n-1} x_j, x \in \Re^n$. *Let* P *be a population distribution in* Γ *from which the first parent is drawn and* $q \in \Gamma$ *for the second parent. Let* $L(p,q) \in \Gamma$ *be the population distribution resulting from crossing other random parents from* p *and* q. *Let* e_i *be the vector with a* 1 *at position* i *and zeroes elsewhere. Then* e_i *represents a population comprised entirely of copies of* i. *In this case, the other component (*k*th component) of* $L(e_i, e_j)$ *equals the probability that crossing* i *and* j *will produce* k. *Therefore,*

$$\begin{aligned} L(p,q)_k &= \sum p_i q_j L(e_i, e_j)_k \\ &\Rightarrow L(p,q) = \sum_k \left(\sum_{i,j} p_i q_j L(e_i, e_j)_k \right) e_k \\ &= \sum_{i,j} p_i q_j L(e_i, e_j). \end{aligned}$$

The sum $L(p,q)$ *is completely determined by vectors* $L(e_i, e_j)$*, and this can be extended to* $\Re^n$. *We define the following operator:*

$$\Psi : \Re^n \to \Re^n \text{ by } \Psi(x_1, \ldots, x_n) = \sum_{i,j} x_i x_j L(e_i, e_j).$$

Then, Ψ *defines an idempotent operator. These operators are nonlinear. For more information on this topic, refer to [7].*

Remark 2.11. *It is easy to see that* $||\Psi|| = 1$. *If* A *is a linear operator, then the compositions* $A \circ \Psi$ *and* $\Psi \circ A$ *are idempotent. This operator represents the effect of applying crossover to a population vector.*

3 Similar Operators

Definition 3.1. The essential spectrum of an operator A, denoted by $Sp_e(A)$ is the set of scalars λ such that at least one of the following conditions holds:

(a) Range of $(\lambda I - A) = R((\lambda I - A))$ is not closed.

(b) $\bigcup_{n=1}^{\infty} N((\Lambda I - A))^n$ is infinite dimensional.

The essential spectral radius is defined by $r_e(A) = \sup\{|\lambda| : \lambda \in Sp_e(A)\}$.

Theorem 3.2. *The essential spectra of two similar operators on a Hilbert space* H *are the same.*

Proof. Let P and Q be two similar operators on H. Suppose that S is an invertivle operator on H such that $Q = S^{-1}PS$. Then the essential spectra of P and Q are equal since their spectra are the same. Furthermore, $S^{-1}\overline{[(\lambda I - P)|} = \overline{(\lambda I - Q)}$ and $R[(\lambda I - P)] = S^{-1}[R(\lambda I - P)]$. Also, for each positive interger n, $N(\lambda I - Q)^n = S^{-1}[N(\lambda I - P)^n]$. ∎

Theorem 3.3. *An operator T on H is similar to a contraction if and only if $r_e(T) < 1$.*

Proof. Let S be a bounded and invertible operator on H. Then by Theorem 3.2, we have the following inequality: $r_e(T) = r_e(S^{-1}TS) \leq \alpha(S^{-1}TS)$, where $\alpha(S^{-1}TS)$ is a measure of noncompactness of $(S^{-1}TS)$. If S is linear and bijective, then $r_e(T) \leq \inf\{\alpha(S^{-1}TS)\}$. Now, let $\delta > 0$. Then, for a finite dimensional operator ψ, we have from Theorem 3.2, $r(T+\psi) \leq r_e(T) + \delta/m, m = 2, 3, 4, \ldots$. Hence, $||S^{-1}(T+\psi)S|| \leq r_e(T) + \delta$. Furthermore, α is a seminorm and therefore the following is true:

$$\alpha(S^{-1}TS) = \alpha(S^{-1}TS + S^{-1}\psi S) = \alpha(S^{-1}(T+\psi)S) \leq ||S^{-1}(T+\psi)S||,$$

which implies that $\alpha(S^{-1}TS) \leq r_e(T) + \delta$. Since δ is arbitrarily chosen, the theorem follows from Theorem 3.2. □

Remark 3.4. *We denote by X' the set of invertible elements of a unital C^* algebra X. Let x be an element of X such that the spectral radius $r(x)$, of x is strictly less than one. Then, the series $\sum_{n=0}^{\infty} ||x^n||^2$ is convergent and belongs to X. Also, if $x_0 = \sum_{n=0}^{\infty} ||x^n||^* x^n$, then $x_0 \geq 1$. Let y be an element of X such that $y = \sqrt{x_0}$ and $y \geq 1$, then $y \in X'$. Hence,*

$$||yxy^{-1}|| = \left|\left| y^{-1} \sum_{n=1}^{\infty} (x^*)^n y^{-1} \right|\right| = r(1 - y^{-2}) < 1 \Rightarrow ||yxy^{-1}|| < 1.$$

Let M be a closed two sided ideal of X. If x is in X and X/M is a C^*-algebra, then $r(x+M) = \inf_{y \in M} r(x+y), y \in X'$.

Remark 3.5. *If $\kappa(H)$ denotes the set of compact operators on H and if $T \in \kappa(H)$, then $||\kappa(H) + T|| = \alpha(T)$.*

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