



# A New Implicit Hybrid Algorithm for an Equilibrium Problem and a Countable Family of Relatively Nonexpansive Mappings in Banach Spaces<sup>1</sup>

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**Abstract :** In this paper, we introduce a new implicit shrinking algorithm for finding a common element of the set of solutions of an equilibrium problem and the set of common fixed points of a countable family of relatively nonexpansive mappings in the framework of Banach spaces. Our results are refinement as well as generalization of several well-known results in the current literature. As a consequence, we give some applications for solving variational inequality problems and convex minimization problems in Banach spaces.

**Keywords :** Relatively nonexpansive mapping; Implicit hybrid algorithm; Asymptotic fixed point; Equilibrium problems; Shrinking projection method.

**2010 Mathematics Subject Classification :** 47H05; 47J25.

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## 1 Introduction and Preliminaries

Over the past few decades, iterative algorithms play a key role in solving nonlinear equation in various fields of investigation. Therefore, algorithmic construction for the approximation of fixed points of various mappings is a problem of interest in various setting of spaces. Numerous implicit and explicit algorithms have been developed for the approximation of fixed point results. Most of the

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<sup>1</sup>This research was supported by the Higher Education Commission of Pakistan.

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problems in applied sciences such as monotone inclusion problems, saddle point problems, variational inequality problems, minimization problems, Nash equilibria in noncooperative games, vector equilibrium problems as well as certain fixed point problems reduce in terms of finding solution of an equilibrium problem which is defined as follows: Let  $C$  be a nonempty closed and convex subset of a real Banach space  $E$  and let  $f : C \times C \rightarrow \mathbb{R}$  (the set of reals) be a bifunction. The equilibrium problem for  $f$  is to find its equilibrium points, i.e. the set  $EP(f) = \{x \in C : f(x, y) \geq 0, \text{ for all } y \in C\}$ . For solving the equilibrium problem, let us assume that the bifunction  $f$  satisfies the following conditions:

- (A1)  $f(x, x) = 0$  for all  $x \in C$ ;
- (A2)  $f$  is monotone, i.e.  $f(x, y) + f(y, x) \leq 0$  for all  $x, y \in C$ ;
- (A3)  $\limsup_{t \downarrow 0} f(tz + (1-t)x, y) \leq f(x, y)$  for all  $x, y, z \in C$ ,
- (A4)  $f(x, \cdot)$  is convex and lower semicontinuous for all  $x \in C$ ; see [1, 2].

The Lyapunov functional  $\varphi : E \times E \rightarrow \mathbb{R}$  is defined by

$$\varphi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2 \text{ for all } x, y \in E.$$

It is obvious from the definition of  $\varphi$  that

- (1)  $\varphi(x, y) \geq 0$  for all  $x, y \in E$  and  $\varphi(x, y) = 0$  if and only if  $x = y$ ;
- (2)  $(\|x\| - \|y\|)^2 \leq \varphi(x, y) \leq (\|x\| + \|y\|)^2$  for all  $x, y \in E$ .

In a real Hilbert space, we have,  $\varphi(x, y) = \|x - y\|^2$  for all  $x, y \in E$ . For details, see [3, 4].

Let  $C$  be a nonempty closed and convex subset of a Banach space  $E$  and let  $T : C \rightarrow C$  be a nonlinear mapping. We denote  $F(T)$  the set of fixed points of  $T$ . A point  $x \in C$  is said to be *asymptotic fixed point* of  $T$  [5] if there exists a sequence  $\{x_n\} \subset C$  which converges weakly to  $x$  and  $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$ . The set of asymptotic fixed points of  $T$  is denoted by  $\widehat{F}(T)$ . Recall that a mapping  $T : C \rightarrow C$  is *nonexpansive* if  $\|Tx - Ty\| \leq \|x - y\|$  for all  $x, y \in C$  and *relatively nonexpansive* if

- (1)  $F(T)$  is nonempty;
- (2)  $\varphi(u, Tx) \leq \varphi(u, x)$  for all  $u \in F(T)$  and  $x \in C$ ;
- (3)  $\widehat{F}(T) = F(T)$ .

Recently, numerous attempts have been made in order to guarantee the strong convergence through algorithmic construction for the approximation of fixed points. In 2004, Matsushita and Takahashi [6] introduced the following algorithm for a single relatively nonexpansive mapping  $T$  in a Banach space  $E$ : For an initial point  $x_0 \in C$ , define a sequence  $\{x_n\}$  by:

$$x_{n+1} = P_C J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT x_n), \quad n \geq 0, \quad (1.1)$$

where  $T$  is relatively nonexpansive mapping,  $J$  is the duality mapping on  $E$  and  $P_C$  is the generalized projection from  $E$  onto  $C$  and  $\{\alpha_n\}$  is a sequence in  $[0, 1]$ . They proved that the sequence  $\{x_n\}$  generated by (1.1) converges weakly to some fixed point of  $T$  under some suitable conditions on  $\{\alpha_n\}$ .

In 2008, Takahashi and Zembayashi [7] introduced the shrinking projection method for an equilibrium problem in a Banach space  $E$  as follows:

$$\begin{cases} x_0 = x \in C = C_0 \\ y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT x_n), \\ u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n}(y - u_n, Ju_n - Jy_n) \geq 0 \ \forall y \in C, \\ C_{n+1} = \{z \in C_n : \varphi(z, u_n) \leq \varphi(z, x_n)\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \geq 0, \end{cases} \quad (1.2)$$

where  $T$ ,  $J$  and  $P_C$  are the mappings as used in (1.1). They proved that the sequence  $\{x_n\}$  generated by (1.2) converges strongly to  $P_{F(T) \cap EP(f)}x_0$  under some appropriate conditions.

Recent developments in fixed point theory reflect that the algorithmic construction for the approximation of fixed point problems are vigorously proposed and analyzed for various classes of mappings in different spaces. Since, most of the problems from various disciplines of science are nonlinear in nature, therefore implicit algorithms have an advantage over explicit one in view of their accuracy for such nonlinear problems in the framework of Hilbert spaces and Banach spaces. The arity of the algorithm and the family of mappings also play an important role for the best approximation of nonlinear problems. The pioneering work of Xu and Ori [8] deals with the weak convergence of an implicit iterative algorithm for a finite family of nonexpansive mappings. Recently, hybrid algorithms are vigorously used for the development of approximate fixed point results. Furthermore, finding a common element of the set of solutions of an equilibrium problem and the set of fixed points in Hilbert spaces and Banach spaces is a problem of interest and, is therefore, studied by many authors; see also [1, 2, 9, 10] and references therein.

Inspired and motivated by these facts, we introduce an implicit shrinking projection algorithm for finding a common element of the set of solutions of an equilibrium problem and the set of common fixed points of a countable family of relatively nonexpansive mappings in a Banach space. As an application, we apply our result to solve a variational inequality problem and a convex minimization problem in Banach spaces.

Let  $E$  be a real Banach space with its dual  $E^*$ . For  $x^* \in E^*$ , its value at  $x \in E$  is denoted by  $\langle x, x^* \rangle$ . Denote  $S_E = \{x \in E : \|x\| = 1\}$ .

A Banach space  $E$  is said to be *strictly convex* if for  $x, y \in S_E$  with  $x \neq y$  implies  $\|x + y\| < 2$ , *uniformly convex* if for any two sequences  $\{x_n\}$  and  $\{y_n\}$  in  $S_E$  satisfying  $\lim_{n \rightarrow \infty} \|\frac{x_n + y_n}{2}\| = 1$  implies  $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$  and *reflexive* if  $T : E \rightarrow E^{**}$  is bijective, where  $E^{**}$  is a dual of  $E^*$ . Furthermore, define

$h : S_E \times S_E \times \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$  by

$$h(x, y, t) = \frac{\|x + ty\| - \|x\|}{t}$$

for  $x, y \in S_E$  and  $t \in \mathbb{R} \setminus \{0\}$ . The norm of  $E$  is said to be *Gâteaux differentiable* if  $\lim_{t \rightarrow 0} h(x, y, t)$  exists for each  $x, y \in S_E$ . The Banach space  $E$  is said to be *smooth* if its norm is *Gâteaux differentiable*. The norm of  $E$  is said to be *Fréchet differentiable*, if for each  $x \in E$ ,  $\lim_{t \rightarrow 0} h(x, y, t)$  is attained uniformly for  $y \in S_E$ . The Banach space  $E$  is said to be *uniformly smooth* if its norm is *Fréchet differentiable*.

The normalized duality mapping  $J : E \rightarrow E^*$  is defined by

$$Jx = \left\{ x^* \in E^* : \|x\|^2 = \langle x, x^* \rangle = \|x^*\|^2 \right\}$$

for all  $x \in E$ . It is remarked that the normalized duality mapping  $J$  is nonempty, closed and convex in a Banach space and is single valued in a real reflexive and smooth Banach space. Furthermore,  $J^{-1} : E^* \rightarrow E$ , the inverse of the normalized duality mapping  $J$ , is also a duality mapping in uniformly convex and uniformly smooth Banach space. Both  $J$  and  $J^{-1}$  are uniformly norm-to-norm continuous on each bounded subset of  $E$  or  $E^*$ , respectively. For more details, see [11, 12].

The following well known results are needed in the sequel for the development of our main result.

**Lemma 1.1** (Matsushita and Takahashi [13]). *Let  $C$  be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space  $E$  and let  $T : C \rightarrow C$  be a relatively nonexpansive mapping. Then  $F(T)$  is closed and convex.*

**Lemma 1.2** (Kamimura and Takahashi [14]). *Let  $E$  be a uniformly convex and smooth Banach space and let  $\{x_n\}, \{y_n\}$  be two sequences in  $E$  such that either  $\{x_n\}$  or  $\{y_n\}$  is bounded. If  $\lim_{n \rightarrow \infty} \varphi(x_n, y_n) = 0$ , then  $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$ .*

Let  $C$  be a nonempty, closed and convex subset of a Hilbert space  $H$  and  $P_C : H \rightarrow C$  defined by

$$\|x - P_C x\| = \inf\{\|x - y\| : \text{for all } y \in C\},$$

is known as metric (nearest point) projection of  $H$  onto  $C$ . This fact characterizes Hilbert space and consequently not available in more general Banach space. In this sequel, Alber [3] introduced a generalized projection operator in Banach space as follows: Let  $E$  be a reflexive, strictly convex and smooth Banach space and let  $C$  be a nonempty, closed and convex subset of  $E$ . Then the generalized projection  $P_C : E \rightarrow C$  is a mapping that assigns to any point  $x \in E$ , the point  $x_0$  which is the solution to the minimization problem  $\varphi(x_0, x) = \min_{y \in C} \varphi(y, x)$ . The existence and uniqueness of the generalized projection operator follows from the properties of  $\varphi$ . In a real Hilbert space, the generalized projection coincides with the metric projection operator.

The following two lemmas are due to Alber [3] concerning the generalized projection operator.

**Lemma 1.3** (Alber [3]). *Let  $C$  be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space  $E$ , let  $x \in E$  and let  $x_0 \in C$ . Then,  $P_C x = x_0$  if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0, \text{ for all } y \in C.$$

**Lemma 1.4** (Alber [3]). *Let  $E$  be a reflexive, strictly convex and smooth Banach space, let  $C$  be a nonempty closed convex subset of  $E$  and let  $x \in E$ . Then*

$$\varphi(y, P_C x) + \varphi(P_C x, x) \leq \varphi(y, x), \text{ for all } y \in C.$$

**Lemma 1.5** (Blum and Oettli [1]). *Let  $C$  be a closed convex subset of a smooth, strictly convex and reflexive Banach space  $E$ , let  $f : C \times C \rightarrow \mathbb{R}$  be a bifunction satisfying (A1)-(A4), let  $r > 0$  and  $x \in E$ . Then there exists  $z \in C$  such that*

$$f(z, y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \text{ for all } y \in C.$$

**Lemma 1.6** (Takahashi and Zembayashi [15]). *Let  $C$  be a closed convex subset of a uniformly smooth, strictly convex, and reflexive Banach space  $E$ . Let  $f : C \times C \rightarrow \mathbb{R}$  be a bifunction satisfying (A1)-(A4). For  $r > 0$  and  $x \in E$ , define a mapping  $T_r : E \rightarrow C$  by*

$$T_r(x) = \left\{ z \in C : f(z, y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \text{ for all } y \in C \right\}$$

for all  $x \in C$ . Then, the following holds:

- (1)  $EP(f)$  is closed and convex;
- (2)  $T_r$  is single valued;
- (3)  $T_r$  is firmly nonexpansive-type mapping, i.e.,

$$\langle T_r x - T_r y, JT_r x - JT_r y \rangle \leq \langle T_r x - T_r y, Jx - Jy \rangle, \text{ for all } x, y \in E,$$

- (4)  $F(T_r) = EP(f)$ .

## 2 Main Result

In this section, we prove a strong convergence theorem by using a shrinking projection method based on an implicit hybrid algorithm for a countable family of relatively nonexpansive mappings in a Banach space.

Our main result is as under:

**Theorem 2.1.** *Let  $C$  be a nonempty, closed and convex subset of a uniformly smooth and uniformly convex Banach space  $E$ . Let  $f : C^2 \rightarrow \mathbb{R}$  be a bifunction satisfying (A1)-(A4) and let  $S_i : C \rightarrow C$ ,  $i \geq 1$ , be a countable family of relatively nonexpansive mappings such that  $F := \bigcap_{i=1}^{\infty} F(S_i) \cap EP(f) \neq \emptyset$ . Let  $\{x_n\}$  be a sequence generated by:*

$$\begin{cases} x_0 \in C_0 = C \\ y_{n,i} = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JS_i y_{n,i}), \quad i \geq 1, \\ u_{n,i} \in C \text{ such that } f(u_{n,i}, y) + \frac{1}{r_n} \langle y - u_{n,i}, Ju_{n,i} - Jy_{n,i} \rangle \geq 0, \forall y \in C, \\ C_{n+1} = \{z \in C_n : \sup_{i \geq 1} \varphi(z, u_{n,i}) \leq \varphi(z, x_n)\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n \geq 0, \end{cases}$$

where  $\{\alpha_n\} \subset (0, 1)$  and  $\{r_n\} \subset (0, \infty)$  satisfying  $\limsup_{n \rightarrow \infty} \alpha_n < 1$  and  $\liminf_{n \rightarrow \infty} r_n > 0$ . Then  $\{x_n\}$  converges strongly to  $P_F x_0$ , where  $P_F$  is a generalized projection of  $E$  onto  $F$ .

*Proof.* By Lemma 1.1 and Lemma 1.6 (4), we know that  $F$  is a closed and convex. Next we show that  $C_n$  is closed and convex. Clearly,  $C_0 = C$  is closed and convex. Suppose that  $C_k$  is closed and convex for  $k \in \mathbb{N}$ . For each  $z \in C_k$ , we observe that

$$\begin{aligned} C_{k+1} &= \{z \in C_k : \sup_{i \geq 1} \varphi(z, u_{k,i}) \leq \varphi(z, x_k)\} \\ &= \bigcap_{i \geq 1} \{z \in C_k : \varphi(z, u_{k,i}) \leq \varphi(z, x_k)\} \\ &= \bigcap_{i \geq 1} \{z \in C_k : 2\langle z, Jx_k - Ju_{k,i} \rangle + \|u_{k,i}\|^2 - \|x_k\|^2 \leq 0\}. \end{aligned}$$

This implies that  $C_{k+1}$  is closed and convex. By induction, we get that  $C_n$  is closed and convex for all  $n \geq 0$ .

For simplicity, we divide the remaining proof into the following six steps.

**Step 1.**  $F \subset C_n$  for all  $n \geq 0$ .

**Step 2.**  $\lim_{n \rightarrow \infty} \varphi(x_n, x_0)$  exists.

**Step 3.**  $\{x_n\}$  is a Cauchy sequence.

**Step 4.**  $x_n \rightarrow q \in \bigcap_{i \geq 1} F(S_i)$ .

**Step 5.**  $x_n \rightarrow q \in EP(f)$ .

**Step 6.**  $q = P_F x_0$ .

**Proof of step 1.**  $F \subset C_0 = C$  is obvious. Suppose that  $F \subset C_k$  for  $k \in \mathbb{N}$ . For any  $p \in F$ , we first estimate that

$$\begin{aligned} \varphi(p, y_{k,i}) &= \varphi(p, J^{-1}(\alpha_k Jx_k + (1 - \alpha_k)JS_i y_{k,i})) \\ &= \|p\|^2 - 2\langle p, \alpha_k Jx_k + (1 - \alpha_k)JS_i y_{k,i} \rangle + \|\alpha_k Jx_k + (1 - \alpha_k)JS_i y_{k,i}\|^2 \\ &\leq \|p\|^2 - 2\alpha_k \langle p, Jx_k \rangle - 2(1 - \alpha_k) \langle p, JS_i y_{k,i} \rangle + \alpha_k \|x_k\|^2 \\ &\quad + (1 - \alpha_k) \|S_i y_{k,i}\|^2 \\ &= \alpha_k \left( \|p\|^2 - 2\langle p, Jx_k \rangle + \|x_k\|^2 \right) \\ &\quad + (1 - \alpha_k) \left( \|p\|^2 - 2\langle p, JS_i y_{k,i} \rangle + \|S_i y_{k,i}\|^2 \right) \\ &= \alpha_k \varphi(p, x_k) + (1 - \alpha_k) \varphi(p, S_i y_{k,i}) \\ &\leq \alpha_k \varphi(p, x_k) + (1 - \alpha_k) \varphi(p, y_{k,i}). \end{aligned} \tag{2.1}$$

Since  $\alpha_k > 0$ , therefore (2.1) reduces to

$$\varphi(p, y_{k,i}) \leq \varphi(p, x_k). \tag{2.2}$$

Note that  $u_{k,i} = T_{r_k} y_{k,i}$ ,  $i \geq 1$ . Since  $T_{r_k}$  is relatively nonexpansive, so we have

$$\varphi(p, u_{k,i}) = \varphi(p, T_{r_k} y_{k,i}) \leq \varphi(p, y_{k,i}) \leq \varphi(p, x_k).$$

This shows that  $p \in C_{k+1}$ ; consequently,  $F \subset C_{k+1}$ . By a simple induction, we also get that  $F \subset C_n$  for all  $n \geq 0$ . Moreover,  $P_{C_{n+1}}x_0$  is well defined.

**Proof of step 2.** Since  $x_n = P_{C_n}x_0$  and  $x_{n+1} = P_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$  for all  $n \geq 0$ , we obtain

$$\varphi(x_n, x_0) \leq \varphi(x_{n+1}, x_0).$$

This shows that  $\{\varphi(x_n, x_0)\}$  is non-decreasing. On the other hand, it follows from  $x_n = P_{C_n}x_0$  and Lemma 1.4 that

$$\varphi(x_n, x_0) = \varphi(P_{C_n}x_0, x_0) \leq \varphi(p, x_0) - \varphi(p, P_{C_n}x_0) \leq \varphi(p, x_0),$$

for each  $p \in F$ . Therefore,  $\varphi(x_n, x_0)$  is bounded and hence  $\lim_{n \rightarrow \infty} \varphi(x_n, x_0)$  exists.

**Proof of step 3.** Since  $x_m = P_{C_m}x_0 \in C_m \subset C_n$  for  $m > n$ , so by Lemma 1.4, we have

$$\begin{aligned} \varphi(x_m, x_n) &= \varphi(x_m, P_{C_n}x_0) \leq \varphi(x_m, x_0) - \varphi(P_{C_n}x_0, x_0) \\ &= \varphi(x_m, x_0) - \varphi(x_n, x_0). \end{aligned}$$

Letting  $m, n \rightarrow \infty$ , we have  $\varphi(x_m, x_n) \rightarrow 0$ . By Lemma 1.2, we have  $\|x_m - x_n\| \rightarrow 0$ . Hence  $\{x_n\}$  is Cauchy. Therefore, there exists a point  $q \in C$  such that  $x_n \rightarrow q$  as  $n \rightarrow \infty$ . In particular, we also have

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \quad (2.3)$$

**Proof of step 4.** As  $x_{n+1} \in C_n$ , so  $\varphi(x_{n+1}, u_{n,i}) \leq \varphi(x_{n+1}, x_n)$ . Tending  $n \rightarrow \infty$ , we have  $\lim_{n \rightarrow \infty} \varphi(x_{n+1}, u_{n,i}) = 0$  for all  $i \geq 1$ . Again by Lemma 1.2, we have

$$\lim_{n \rightarrow \infty} \|x_{n+1} - u_{n,i}\| = 0, \quad i \geq 1. \quad (2.4)$$

This implies  $u_{n,i} \rightarrow q$  as  $n \rightarrow \infty$ . Furthermore, (2.3) and (2.4) yield that

$$\lim_{n \rightarrow \infty} \|x_n - u_{n,i}\| = 0, \quad i \geq 1. \quad (2.5)$$

Since  $J$  is uniformly norm-to-norm continuous on bounded sets, so

$$\lim_{n \rightarrow \infty} \|Jx_n - Ju_{n,i}\| = 0. \quad (2.6)$$

From (2.2), we know that  $\varphi(p, y_{n,i}) \leq \varphi(p, x_n)$  for all  $i \geq 1$ . So by Lemma 1.4, we have

$$\begin{aligned} \varphi(u_{n,i}, y_{n,i}) &= \varphi(T_{r_n}y_{n,i}, y_{n,i}) \\ &\leq \varphi(p, y_{n,i}) - \varphi(p, T_{r_n}y_{n,i}) \\ &\leq \varphi(p, x_n) - \varphi(p, T_{r_n}y_{n,i}) \\ &= \varphi(p, x_n) - \varphi(p, u_{n,i}), \quad i \geq 1. \end{aligned} \quad (2.7)$$

From (2.5), (2.6) and (2.7), we have  $\lim_{n \rightarrow \infty} \varphi(u_{n,i}, y_{n,i}) = 0$  for all  $i \geq 1$ ; consequently, Lemma 1.2 asserts that

$$\lim_{n \rightarrow \infty} \|u_{n,i} - y_{n,i}\| = 0, \quad i \geq 1. \quad (2.8)$$

From (2.5) and (2.8), we also have

$$\lim_{n \rightarrow \infty} \|x_n - y_{n,i}\| = 0, \quad i \geq 1. \quad (2.9)$$

Thus,  $y_{n,i} \rightarrow q$  as  $n \rightarrow \infty$ . On the other hand, we observe that

$$\|JS_i y_{n,i} - Jy_{n,i}\| = \frac{\alpha_n}{1 - \alpha_n} \|Jx_n - Jy_{n,i}\|, \quad i \geq 1. \quad (2.10)$$

Since  $\limsup_{n \rightarrow \infty} \alpha_n < 1$ , it follows from (2.9) and (2.10) that

$$\lim_{n \rightarrow \infty} \|JS_i y_{n,i} - Jy_{n,i}\| = 0, \quad i \geq 1.$$

This also implies that

$$\lim_{n \rightarrow \infty} \|S_i y_{n,i} - y_{n,i}\| = 0, \quad i \geq 1.$$

Therefore,  $q \in \bigcap_{i=1}^{\infty} \widehat{F}(S_i) = \bigcap_{i=1}^{\infty} F(S_i)$ .

**Proof of step 5.** Since  $\liminf_{n \rightarrow \infty} r_n > 0$ , it follows from (2.8) that

$$\lim_{n \rightarrow \infty} \frac{\|Ju_{n,i} - Jy_{n,i}\|}{r_n} = 0, \quad i \geq 1. \quad (2.11)$$

From  $u_{n,i} = T_{r_n} y_{n,i}$  for all  $n \geq 0$  and  $i \geq 1$ , we have

$$f(u_{n,i}, y) + \frac{1}{r_n} \langle y - u_{n,i}, Ju_{n,i} - Jy_{n,i} \rangle \geq 0, \quad \text{for all } y \in C.$$

From (A2), we have

$$\begin{aligned} \frac{1}{r_n} \langle y - u_{n,i}, Ju_{n,i} - Jy_{n,i} \rangle &\geq -f(u_{n,i}, y) \\ &\geq f(y, u_{n,i}), \quad \text{for all } y \in C. \end{aligned}$$

From  $u_{n,i} \rightarrow q$  and (A4), we obtain  $f(y, q) \leq 0$  for all  $y \in C$ . Let  $y_t = ty + (1-t)q$  for  $0 < t < 1$  and  $y \in C$ . Then  $y_t \in C$  and hence  $f(y_t, q) \leq 0$ . From (A1) and (A4), we have  $0 = f(y_t, y_t) \leq tf(y_t, y) + (1-t)f(y_t, q) \leq tf(y_t, y)$ . Thus,  $f(y_t, y) \geq 0$ . From (A3), we have  $f(q, y) \geq 0$  for all  $y \in C$ . Therefore,  $q \in EP(f)$  and hence  $q \in F$ .

**Proof of step 6.** From  $x_n = P_{C_n} x_0$  we have

$$\langle x_n - p, Jx_0 - Jx_n \rangle \geq 0, \quad \text{for all } p \in F.$$

Taking limit in the above inequality, we have  $\langle q - p, Jx_0 - Jq \rangle \geq 0$ , for all  $p \in F$ . So by Lemma 1.3, we conclude that  $q = P_F x_0$ . This completes the proof.  $\square$



### 3 Applications

This section deals with the application of equilibrium problems as it play a central role in numerous disciplines including economics, management science, operations research, and engineering. We discuss variational inequality problem and convex minimization problem in a Banach space.

#### 3.1 Variational Inequality Problem

Numerous algorithms have been developed for the computation of equilibrium points. Variational inequality theory, a powerful computational algorithm, is one of them which has numerous applications in various disciplines of sciences such as mathematical programming, game theory, mechanics and geometry. Now, we formally define classical variational inequality problem in connection with the equilibrium problem discussed in Theorem 2.1 as follows: Let  $A : C \rightarrow E^*$  be a nonlinear mapping, the variational inequality problem is to find a point  $x \in C$  such that  $\langle Ax, y - x \rangle \geq 0$  for all  $y \in C$ . The set of solutions of variational inequality problem is denoted as

$$VI(C, A) = \{x \in C : \langle Ax, y - x \rangle \geq 0, \text{ for all } y \in C\}.$$

**Theorem 3.1.** *Let  $C$  be a nonempty, closed and convex subset of a uniformly smooth and uniformly convex Banach space  $E$ . Let  $A : C \rightarrow E^*$  be a monotone and continuous mapping, and let  $S_i : C \rightarrow C$ ,  $i \geq 1$ , be a countable family of relatively nonexpansive mappings such that  $F := \bigcap_{i=1}^{\infty} F(S_i) \cap VI(C, A) \neq \emptyset$ . Let  $\{x_n\}$  be a sequence generated by*

$$\begin{cases} x_0 \in C_0 = C \\ y_{n,i} = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JS_i y_{n,i}), \quad i \geq 1, \\ u_{n,i} \in C \text{ such that } \langle Au_{n,i}, y - u_{n,i} \rangle + \frac{1}{r_n} \langle y - u_{n,i}, Ju_{n,i} - Jy_{n,i} \rangle \geq 0, \forall y \in C, \\ C_{n+1} = \{z \in C_n : \sup_{i \geq 1} \varphi(z, u_{n,i}) \leq \varphi(z, x_n)\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n \geq 0, \end{cases}$$

where  $\{\alpha_n\} \subset (0, 1)$  and  $\{r_n\} \subset (0, \infty)$  satisfying  $\limsup_{n \rightarrow \infty} \alpha_n < 1$  and  $\liminf_{n \rightarrow \infty} r_n > 0$ . Then,  $\{x_n\}$  converges strongly to  $P_F x_0$ , where  $P_F$  is a generalized projection of  $E$  onto  $F$ .

*Proof.* Define  $f(x, y) = \langle Ax, y - x \rangle$  for all  $x, y \in C$ . Then  $f$  satisfies the conditions (A1)-(A4). Therefore, by Theorem 2.1, we obtain the desired result.  $\square$

#### 3.2 Convex Minimization Problem

Mathematical optimization has applicable roots in various disciplines such as estimation and signal processing, communications and networks, electronic circuit design, data analysis and modeling, statistics (optimal design), and finance. Convex minimization problem (CMP), basically deals with the problems of minimizing

real valued convex function defined on the convex subset of the underlying space, i.e.  $\phi : C \rightarrow \mathbb{R}$  such that

$$CMP(\phi) = \{x \in C : \phi(x) \leq \phi(y), \text{ for all } y \in C\}.$$

**Theorem 3.2.** *Let  $C$  be a nonempty, closed and convex subset of a uniformly smooth and uniformly convex Banach space  $E$ . Let  $\phi : C \rightarrow \mathbb{R}$  be a proper, lower semicontinuous and convex function, and let  $S_i : C \rightarrow C$ ,  $i \geq 1$ , be a countable family of relatively nonexpansive mappings such that  $F := \bigcap_{i=1}^{\infty} F(S_i) \cap CMP(\phi) \neq \emptyset$ . Let  $\{x_n\}$  be a sequence generated by*

$$\begin{cases} x_0 \in C_0 = C \\ y_{n,i} = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JS_i y_{n,i}), \quad i \geq 1, \\ u_{n,i} \in C \text{ such that } \phi(y) + \frac{1}{r_n}\langle y - u_{n,i}, Ju_{n,i} - Jy_{n,i} \rangle \geq \phi(u_{n,i}), \forall y \in C, \\ C_{n+1} = \{z \in C_n : \sup_{i \geq 1} \varphi(z, u_{n,i}) \leq \varphi(z, x_n)\}, \\ x_{n+1} = PC_{n+1}x_0, \quad n \geq 0, \end{cases}$$

where  $\{\alpha_n\} \subset (0, 1)$  and  $\{r_n\} \subset (0, \infty)$  satisfying  $\limsup_{n \rightarrow \infty} \alpha_n < 1$  and  $\liminf_{n \rightarrow \infty} r_n > 0$ . Then,  $\{x_n\}$  converges strongly to  $P_F x_0$ , where  $P_F$  is a generalized projection of  $E$  onto  $F$ .

*Proof.* Define  $f(x, y) = \phi(y) - \phi(x)$  for all  $x, y \in C$ . Then  $f$  satisfies the conditions (A1)-(A4). Therefore, Theorem 2.1 can also be applied to such a convex minimization problem.  $\square$

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(Received 10 April 2011)

(Accepted 15 May 2011)