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Bilinear and Bilateral Summation Formulae of Certain Polynomials in the Form of Operator Representations

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Abstract : Based on the technique used by Khan and Shukla [1] certain bilinear and bilateral summation formulae for various polynomials, whose Rodrigues or Rodrigues type formulae are known, have been obtained in terms of operational representations using derivatives and difference operators.The results obtained are believed to be new.

Keywords : Bilinear and bilateral operator; Operational representations.
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1 Introduction

Recently in 2009, Khan and Shukla [1] evolved a new technique to give operator representations of certain polynomials. Using the same technique Khan and Nisar [2–4] obtained operator representations of certain polynomials. Using the said technique and Rodrigues type formulae of various polynomials certain bilinear and bilateral summation formulae of these polynomials have been obtained in the form of operator representations.

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2 The Definitions, Notations, and Results Used

In deriving the operational representations of various polynomials, Khan and Shukla [1] used the following results which we also need here.

$$D^{\mu}x^{\lambda}=\frac{\Gamma(1+\lambda)}{\Gamma(1+\lambda-\mu)}x^{\lambda-\mu},\quad D\equiv\frac{d}{dx},\tag{2.1}$$

where λ and μ, $\lambda\geq\mu$ are arbitrary real numbers.

In particular, use has been made of the following results:

$$D^{r}e^{-x}=(-1)^{r}e^{-x}\tag{2.2}$$

$$D^{r}x^{-\alpha}=(\alpha)_{r}(-1)^{r}x^{-\alpha-r},\ \alpha\text{ is not an integer}\tag{2.3}$$

$$D^{r}x^{-\alpha-n}=(\alpha+n)_{r}(-1)^{r}x^{-\alpha-n-r}\tag{2.4}$$

$$D^{n-r}x^{\alpha-1+n}=\frac{(\alpha)_{n}}{(\alpha)_{r}}x^{\alpha-1+r}\tag{2.5}$$

$$D^{n-r}\ x^{-\alpha}=\frac{(\alpha)_{n}(-1)^{n}}{(1-\alpha-n)_{r}}\ \ x^{-\alpha-n+r},\ \alpha\text{ is not an integer},\tag{2.6}$$

where n and r are denote positive integers and

$$(\alpha)_{n}=\alpha\ (\alpha+1)\cdots(\alpha+n-1);\quad(\alpha)_{0}=1.$$

The finite difference operator Δ is defined as

$$\Delta f(x)=f(x+1)-f(x)\tag{2.7}$$

The gamma functions $\Gamma(z)$ is defined as

$$\Gamma(z)=\int_{0}^{\infty}\ e^{-t}t^{z-1}dt,\ Re(z)>0.\tag{2.8}$$

We also need the Rodrigues type formulae of the following polynomials (see [5–7]).

Laguerre polynomials: It is denoted by the symbol $L_{n}^{(\alpha)}(x)$ and its Rodrigues formula is

$$D^{n}\left(e^{-x}x^{\alpha+n}\right)=n!x^{\alpha}e^{-x}L_{n}^{(\alpha)}(x).\tag{2.9}$$

Hermite polynomials: It is denoted by the symbol $H_{n}(x)$ and its Rodrigues formula is

$$D^{n}\left(e^{-x^{2}}\right)=(-1)^{n}e^{-x^{2}}H_{n}(x).\tag{2.10}$$

Jacobi polynomials: It is denoted by the symbol $P_{n}^{(\alpha,\beta)}(x)$ and its Rodrigues formula is

$$D^{n}\left((x-1)^{n+\alpha}(x+1)^{n+\beta}\right)=2^{n}\ n!(x-1)^{\alpha}(x+1)^{\beta}P_{n}^{(\alpha,\beta)}(x).\tag{2.11}$$

Ultraspherical polynomials: The special case $\beta = \alpha$ of Jacobi polynomials is called Ultraspherical polynomials and is denoted by $P_n^{(\alpha,\alpha)}(x)$ and its Rodrigues formula is

$$D^n \left((x^2-1)^{n+\alpha}\right) = 2^n \; n! \; (x^2-1)^\alpha P_n^{(\alpha,\alpha)}(x). \tag{2.12}$$

Gagenbaur polynomials: The Gagenbaur polynomials $C_n^\nu(x)$ is generalization of Laguerre polynomials and its Rodrigues formula is

$$D^n \left((1-x^2)^{n+\nu-\frac{1}{2}}\right) = \frac{2^n \; n! \; (\nu+\frac{1}{2})_n}{(-1)^n (2\nu)_n (1-x^2)^{\frac{1}{2}-\nu}} C_n^\nu(x). \tag{2.13}$$

Bessel polynomials: It is denoted by the symbol $Y_n^\alpha(x)$ and and its Rodrigues formula is

$$D^n \left(x^{2n+\alpha} e^{\frac{-2}{x}}\right) = 2^n \; x^\alpha e^{\frac{-2}{x}} \; Y_n^\alpha(x). \tag{2.14}$$

Charlier polynomials: It is denoted by the symbol $C_n^a(x)$ and its Rodrigues type formula is

$$\Delta^n \left[\frac{a^x}{\Gamma(x-n+1)}\right] = \frac{(-1)^n \; a^x}{\Gamma(x+1)} C_n^a(x), \tag{2.15}$$

where $\Delta f(x) = f(x+1) - f(x)$.

Meixner polynomials: It is denoted by the symbol $M_n(x;\beta,c)$ and its Rodrigues type formula is

$$\Delta^n \left[\frac{c^x \; \Gamma(x+\beta)}{\Gamma(x-n+1)}\right] = \frac{\Gamma(x+\beta) c^{x+n}}{\Gamma(x+1)} M_n(x;\beta,c). \tag{2.16}$$

Hahn polynomials: It is denoted by the symbol $p_n(x;\alpha,\beta,\gamma)$ and its Rodrigues type formula is

$$\Delta^n \left[\frac{\Gamma(x+\alpha) \; \Gamma(x+\beta)}{\Gamma(x-n+1) \; \Gamma(x-n+r)}\right] = \frac{n! \; \Gamma(x+\alpha) \; \Gamma(x+\beta)}{\Gamma(x+1)\Gamma(x+\gamma)} p_n(x;\alpha,\beta,\gamma). \tag{2.17}$$

3 Operatioanal Representations

If $D_x \equiv \frac{\partial}{\partial x}$ *and* $D_y \equiv \frac{\partial}{\partial \; y}$, Khan and Shukla [1] wrote the binomial expansion for $(D_x + D_y)^n$ as

$$(D_x + D_y)^n \equiv \sum_{r=0}^{n} {}^nC_r D_x^{n-r} D_y^r \tag{3.1}$$

where ${}^nC_r = \frac{n!}{r!(n-r)!}$. By writing the finite series on the right of (3.1), Khan and Shukla [1] wrote (3.1) also as

$$(D_x + D_y)^n \equiv \sum_{r=0}^{n} {}^nC_r D_x^r D_y^{n-r}. \tag{3.2}$$

If F(x,y) is a function of x and y, they obtained the following from (3.1) and (3.2),

$$(D_x + D_y)^n F(x,y) \equiv \sum_{r=0}^{n} \frac{(-n)_r(-1)^r}{r!} D_x^{n-r} D_y^r \ F(x,y), \tag{3.3}$$

$$(D_x + D_y)^n F(x,y) \equiv \sum_{r=0}^{n} \frac{(-n)_r(-1)^r}{r!} D_x^r D_y^{n-r} \ F(x,y). \tag{3.4}$$

In particular, if $F(x,y) = f(x)g(y)$. Khan and Shukla [1] wrote (3.3) and (3.4) in the form

$$(D_x + D_y)^n f(x)g(y) \equiv \sum_{r=0}^{n} \frac{(-n)_r(-1)^r}{r!} D_x^{n-r} \ f(x) D_y^r \ g(y), \tag{3.5}$$

$$(D_x + D_y)^n f(x)g(y) \equiv \sum_{r=0}^{n} \frac{(-n)_r(-1)^r}{r!} D_x^r f(x) D_y^{n-r} g(y). \tag{3.6}$$

In a manner similar to above, we also have

$$(\Delta_x + \Delta_y)^n = \sum_{r=0}^{n} \frac{(-n)_r(-1)^r}{r!} \Delta_x^{n-r} \ \Delta_y^r \tag{3.7}$$

which can also written as

$$(\Delta_x + \Delta_y)^n = \sum_{r=0}^{n} \frac{(-n)_r(-1)^r}{r!} \Delta_x^r \ \Delta_y^{n-r} \tag{3.8}$$

and

$$(\Delta_x + D_y)^n = \sum_{r=0}^{n} \frac{(-n)_r(-1)^r}{r!} \Delta_x^{n-r} \ D_y^r \tag{3.9}$$

which can also written as

$$(D_x + \Delta_y)^n = \sum_{r=0}^{n} \frac{(-n)_r(-1)^r}{r!} \Delta_x^r \ D_y^{n-r}. \tag{3.10}$$

In particular, if $F(x,y) = f(x)g(y)$, then (3.7) and (3.8) in the form

$$(\Delta_x + \Delta_y)^n \ f(x)g(y) = \sum_{r=0}^{n} \frac{(-n)_r(-1)^r}{r!} \Delta_x^{n-r} \ f(x) \Delta_y^r \ g(y) \tag{3.11}$$

$$(\Delta_x + \Delta_y)^n \ f(x)g(y) = \sum_{r=0}^{n} \frac{(-n)_r(-1)^r}{r!} \Delta_x^r \ f(x) \Delta_y^{n-r} \ g(y) \tag{3.12}$$

$$(\Delta_x + D_y)^n \ f(x)g(y) = \sum_{r=0}^{n} \frac{(-n)_r(-1)^r}{r!} \Delta_x^{n-r} f(x) \ D_y^r g(y) \tag{3.13}$$

$$(D_x + \Delta_y)^n \ f(x)g(y) = \sum_{r=0}^{n} \frac{(-n)_r(-1)^r}{r!} \Delta_x^r f(x) \ D_y^{n-r} g(y), \tag{3.14}$$

Now by taking special values of $f(x)$ and $g(y)$ in (3.5), (3.11), (3.13) and (3.14), we obtain the following partial differential operator representations of bilinear and bilateral summation formulae for various polynomials whose Rodrigues or Rodrigues type formula are known.

4 Bilinear Operator Representations

$$(D_x+D_y)^n\left\{x^{\alpha+n}y^{\beta+n}e^{-x-y}\right\}=n!x^{\alpha}y^{\beta+n}e^{-x-y}\sum_{r=0}^{n}L_{n-r}^{(\alpha+r)}(x)L_r^{(\beta+n-r)}(y)\left(\frac{x}{y}\right)^r, \tag{4.1}$$

$$(D_x+D_y)^n\left\{e^{-x^2-y^2}\right\}=(-1)^n e^{-x^2-y^2}\sum_{r=0}^{n}\frac{n!}{r!(n-r)!}H_{n-r}(x)H_r(y), \tag{4.2}$$

$$\begin{aligned}&(D_x+D_y)^n\left\{(x-1)^{n+\alpha}(x+1)^{n+\beta}(y-1)^{n+\gamma}(y+1)^{n+\delta}\right\}\\&\qquad=2^n\ n!(x-1)^{\alpha}(x+1)^{\beta}(y-1)^{n+\gamma}(y+1)^{n+\delta}\\&\qquad\quad\times\sum_{r=0}^{n}P_{n-r}^{(\alpha+r,\beta+r)}(x)P_r^{(n+\gamma-r,n+\delta-r)}(y)\left\{\frac{x^2-1}{y^2-1}\right\}^r,\end{aligned} \tag{4.3}$$

$$\begin{aligned}&(D_x+D_y)^n\left\{(x^2-1)^n(y^2-1)^n\right\}\\&\qquad=2^n\ n!(y^2-1)^n\sum_{r=0}^{n}P_{n-r}^{(r,r)}(x)P_r^{(n-r,n-r)}(y)\left\{\frac{x^2-1}{y^2-1}\right\}^r,\end{aligned} \tag{4.4}$$

$$\begin{aligned}&(D_x+D_y)^n\left\{(1-x^2)^{n+\nu-\frac12}(1-y^2)^{n+\mu-\frac12}\right\}\\&=\frac{(-1)^n\ n!(\nu+\frac12)_n(1-x^2)^{\nu-\frac12}}{(2\nu)_n(1-y^2)^{\frac12-n-\mu}}\sum_{r=0}^{n}\frac{(\nu)_r(1-\mu-\frac12-n)_r}{(2\nu+n)_r(\frac12-\mu-n)_r(1-\mu-n)_r(2\mu+2n)_r}\\&\qquad\times\ C_{n-r}^{\nu+r}(x)C_r^{n-r+\mu}(y)\left\{\frac{x^2-1}{1-y^2}\right\}^r,\end{aligned} \tag{4.5}$$

$$\begin{aligned}&(D_x+D_y)^n\left\{x^{2n+\alpha}e^{-\frac{2}{x}}y^{2n+\beta}e^{-\frac{2}{y}}\right\}\\&\qquad=2^n\ x^{\alpha}y^{2n+\beta}e^{-2\ (\frac1x+\frac1y)}\sum_{r=0}^{n}\frac{(-n)_r}{r!}Y_{n-r}^{\alpha+2r}(x)Y_r^{2n+\beta-2r}(y)\left\{\frac{-x^2}{y^2}\right\}^r,\end{aligned} \tag{4.6}$$

$$(\Delta_x+\Delta_y)^n\left[\frac{a^x}{\Gamma(x-n+1)}\frac{a^y}{\Gamma(y-n+1)}\right] \tag{4.7}$$
$$=(-1)^n a^{x+y}\sum_{r=0}^{n}\frac{n!}{r!(n-r)!}\frac{1}{\Gamma(x-r+1)\Gamma(y+r+1)}C_{n-r}^{(a)}(x-r)C_r^{(a)}(y+r),$$

$$(\Delta_x+\Delta_y)^n\left[\frac{c^x\Gamma(x+\beta)}{\Gamma(x-n+1)}\frac{d^y\Gamma(y+r)}{\Gamma(y-n+1)}\right] \tag{4.8}$$
$$=c^{x+y}d^y\sum_{r=0}^{n}\frac{n!}{r!(n-r)!}\frac{\Gamma(x-r+\beta)\Gamma(y+r+\gamma)}{\Gamma(x-r+1)\Gamma(y+r+1)}M_{n-r}(x-r;\beta,c)M_r(y+r;\gamma,d),$$

$$\begin{aligned}(\Delta_x+\Delta_y)^n&\left[\frac{\Gamma(x+\alpha)\Gamma(x+\beta)}{\Gamma(x-n+1)\Gamma(x-n+\gamma)}\frac{\Gamma(y+\delta)\Gamma(y+\phi)}{\Gamma(y-n+1)\Gamma(y-n+\psi)}\right]\\&=n!\sum_{r=0}^{n}\frac{\Gamma(x-r+\alpha)\Gamma(x-r+\beta)}{\Gamma(x-r+1)\Gamma(x-r+\gamma)}\frac{\Gamma(y+r+\phi)\Gamma(y+r+\delta)}{\Gamma(y+r+1)\Gamma(y+r+\psi)}\\&\quad\times\ p_{n-r}(x-r;\alpha,\beta,\gamma)p_r(y+r;\delta,\phi,\psi).\end{aligned} \tag{4.9}$$

5 Bilateral Operational Representations

$$(D_x+D_y)^n\left\{x^{\alpha+n}e^{-x}e^{-y^2}\right\}=n!x^{\alpha}e^{-x-y^2}\sum_{r=0}^{n}\frac{1}{r!}L_{n-r}^{(\alpha+r)}(x)H_r(y)(-x)^r, \tag{5.1}$$

$$\begin{aligned}&(D_x+D_y)^n\left\{(x-1)^{n+\alpha}(x+1)^{n+\beta}e^{-y^2}\right\}\\&\quad=2^n\ n!(x-1)^{\alpha}(x+1)^{\beta}e^{-y^2}\sum_{r=0}^{n}\frac{1}{r!}P_{n-r}^{(\alpha+r,\beta+r)}(x)H_r(y)\left\{\frac{1-x^2}{2}\right\}^r,\end{aligned} \tag{5.2}$$

$$(D_x+D_y)^n\left\{e^{-x}x^{n+\alpha}(x-1)^{n+\alpha}(y-1)^{n+\alpha}(y+1)^{n+\beta}\right\} \tag{5.3}$$
$$=n!x^{\alpha}e^{-x}(y-1)^{n+\alpha}(y+1)^{n+\beta}\sum_{r=0}^{n}L_{n-r}^{(\alpha+r)}(x)P_r^{(n+\alpha-r,n+\beta-r)}(y)\left(\frac{2x}{y^2-1}\right)^r,$$

$$\begin{aligned}&(D_x+D_y)^n\left\{(x^2-1)^n e^{-y}y^{n+\alpha}\right\}\\&\qquad=n!2^n y^{\alpha+n}e^{-y}\sum_{r=0}^{n}P_{n-r}^{(r,r)}(x)L_r^{(\alpha+n-r)}(y)\left(\frac{x^2-1}{2y}\right)^r,\end{aligned} \tag{5.4}$$

$$(D_x+D_y)^n\left\{(x^2-1)^n e^{-y^2}\right\} = n!2^n\ e^{-y^2}\sum_{r=0}^{n}\frac{1}{r!}P_{n-r}^{(r,r)}(x)H_r(y)\left(\frac{1-x^2}{2y}\right)^r, \tag{5.5}$$

$$\begin{aligned}&(D_x+D_y)^n\left\{(x^2-1)^n(y-1)^{n+\alpha}(y+1)^{n+\beta}\right\}\\ &= n!2^n\ (y-1)^{n+\alpha}(y+1)^{n+\beta}\sum_{r=0}^{n}P_{n-r}^{(r,r)}(x)P_r^{(n+\alpha-r,n+\beta-r)}(y)\left(\frac{x^2-1}{y^2-1}\right)^r,\end{aligned} \tag{5.6}$$

$$\begin{aligned}&(D_x+D_y)^n\left\{(x^2-1)^n y^{2n+\beta}e^{\frac{-2}{y}}\right\}\\ &\qquad = n!2^n\ y^{2n+\beta}e^{\frac{-2}{y}}\sum_{r=0}^{n}\frac{1}{r!}P_{n-r}^{(r,r)}(x)Y_r^{2n+\beta-2r}(y)\left(\frac{x^2-1}{y}\right)^r,\end{aligned} \tag{5.7}$$

$$\begin{aligned}&(D_x+D_y)^n\left\{(1-x^2)^{n+\nu-\frac{1}{2}}e^{-y^2}\right\}\\ &= \frac{(-1)^n\ n!(\nu+\frac{1}{2})_n(1-x^2)^{\nu-\frac{1}{2}}e^{-y}}{(2\nu)n}\sum_{r=0}^{n}\frac{(\nu)_r}{(2\nu+n)}C_{n-r}^{\nu+r}(x)H_r(y)\left\{2x^2-1\right\}^r,\end{aligned} \tag{5.8}$$

$$\begin{aligned}&(D_x+D_y)^n\left\{(1-x^2)^{n+\nu-\frac{1}{2}}e^{-y}y^{n+\alpha}\right\}\\ &\qquad = \frac{(-1)^n\ n!(\nu+\frac{1}{2})_n(1-x^2)^{\nu-\frac{1}{2}}e^{-y}y^{\alpha+n}}{(2\nu)_n}\\ &\qquad\quad \times\sum_{r=0}^{n}\frac{(\nu)_r}{(2\nu+n)_r}C_{n-r}^{\nu+r}(x)L_r^{(\alpha+n-r)}(y)\left\{\frac{2(1-x^2)}{y}\right\}^r,\end{aligned} \tag{5.9}$$

$$\begin{aligned}&(D_x+D_y)^n\left\{(1-x^2)^{n+\nu-\frac{1}{2}}(y-1)^{n+\alpha}(y+1)^{n+\beta}\right\}\\ &\qquad = \frac{(-1)^n n!2^n(\nu+\frac{1}{2})_n(1-x^2)^{\nu-\frac{1}{2}}(y-1)^{n+\alpha}(y+1)^{n+\beta}}{(2\nu)_n}\\ &\qquad\quad \times\sum_{r=0}^{n}\frac{(\nu)_r}{(2\nu+n)_r}C_{n-r}^{\nu+r}(x)P_r^{(n+\alpha-r,n+\beta-r)}(y)\left\{\frac{4(1-x^2)}{y^2-1}\right\}^r,\end{aligned} \tag{5.10}$$

$$\begin{aligned}&(D_x+D_y)^n\left\{(1-x^2)^{n+\nu-\frac{1}{2}}y^{2n+\beta}e^{\frac{-2}{y}}\right\}\\ &\qquad = \frac{(-1)^n\ n!2^n(\nu+\frac{1}{2})_n(2\nu)_n(1-x^2)^{\nu-\frac{1}{2}}}{y}^{2n+\beta}\ e^{\frac{-2}{y}}\\ &\qquad\quad \times\sum_{r=0}^{n}\frac{(\nu)_r}{r!(2\nu+n)_r}C_{n-r}^{\nu+r}(x)Y_r^{2n+\beta-2r}(y)\left\{\frac{4(1-x^2)}{y^2}\right\}^r,\end{aligned} \tag{5.11}$$

$$(D_x+D_y)^n\left\{(1-x^2)^{n+\nu-\frac{1}{2}}(y^2-1)^n\right\}$$
$$=(-1)^n\ n!2^n\ (\nu+\frac{1}{2})_n(1-x^2)^{\nu-\frac{1}{2}}(y^2-1)^n$$
$$\times\sum_{r=0}^{n}\frac{1}{(\nu+\frac{1}{2})_r(2\nu+n)_r}C_{n-r}^{\nu+r}(x)P_r^{(n-r,n-r)}(y)\left\{\frac{1-x^2}{y^2-1}\right\}^r,\quad (5.12)$$

$$(D_x+D_y)^n\left\{x^{\alpha+n}e^{-x}y^{2n+\beta}e^{\frac{-2}{y}}\right\}$$
$$=n!x^{\alpha}e^{-x}y^{2n+\beta}e^{\frac{-2}{y}}\sum_{r=0}^{n}\frac{2^r}{r!}L_{n-r}^{\alpha+r}(x)Y_r^{2n+\beta-2r}(y)\left\{\frac{x}{y^2}\right\}^r,\quad (5.13)$$

$$(D_x+D_y)^n\left\{e^{-x^2}y^{2n+\beta}e^{\frac{-2}{y}}\right\}$$
$$=(-1)^n\ e^{-x^2}y^{2n+\beta}e^{\frac{-2}{y}}\sum_{r=0}^{n}{}^nC_r\ 2^rH_{n-r}(x)Y_r^{2n+\beta-2r}(y)\left\{\frac{-1}{y^2}\right\}^r,\quad (5.14)$$

$$(\Delta_x+D_y)^n\left\{\frac{a^x}{\Gamma(x-n+1)}y^{n+\alpha}e^{-y}\right\}$$
$$=\frac{(-1)^n\ a^xy^{\alpha+n}e^{-y}}{\Gamma(x+1)}\sum_{r=0}^{n}(-n)_r(-x)_r\ C_{n-r}^{(a)}(x-r)L_r^{(\alpha+n-r)}(y)\left(\frac{-1}{y}\right)^r,\quad (5.15)$$

$$(\Delta_x+D_y)^n\left\{\frac{a^x}{\Gamma(x-n+1)}(y-1)^{n+\alpha}(y=1)^{n+\beta}\right\}$$
$$=\frac{(-1)^n\ a^x(y-1)^{n+\alpha}(y+1)^{n+\beta}}{\Gamma(x+1)}$$
$$\times\sum_{r=0}^{n}(-n)_r(-x)_r\ C_{n-r}^{(a)}(x-r)P_r^{(n+\alpha+n-r,n+\beta-r)}(y)\left(\frac{2}{1-y^2}\right)^r,\quad (5.16)$$

$$(\Delta_x+D_y)^n\left\{\frac{a^x}{\Gamma(x-n+1)}e^{-y^2}\right\}$$
$$=\frac{a^xe^{-y^2}}{\Gamma(x+1)}\sum_{r=0}^{n}(-n)_r(-x)_r\ C_{n-r}^{(a)}(x-r)H_r(y),\quad (5.17)$$

$$(\Delta_x+D_y)^n\left\{\frac{a^x}{\Gamma(x-n+1)}(y^2-1)^n\right\}$$
$$=\frac{(-1)^na^x(y^2-1)^n}{\Gamma(x+1)}\sum_{r=0}^{n}(-n)_r(-x)_r\ C_{n-r}^{(a)}(x-r)P_r^{(n-r,n-r)}(y),\quad (5.18)$$

$$(\Delta_x + D_y)^n \left\{ \frac{a^x}{\Gamma(x-n+1)} (1-y^2)^{n+\nu-\frac{1}{2}} \right\}$$
$$= \frac{(-1)^n}{\Gamma(x+1)(1-y^2)^{\frac{1}{2}-n-\nu}} \sum_{r=0}^{n} \frac{(-n)_r(-x)_r(1-\nu-\frac{1}{2}-n)_r}{r!(\frac{1}{2}-\nu-n)_r(1-\nu-n)_r}$$
$$\times\ C_{n-r}^{(a)}(x-r) C_r^{n-r+\nu}(y) \left\{ \frac{1}{4(y^2-1)} \right\}^r, \tag{5.19}$$

$$(\Delta_x + D_y)^n \left\{ \frac{a^x}{\Gamma(x-n+1)} y^{2n+\beta} e^{-2y} \right\}$$
$$= \frac{(-1)^n y^{2n+\beta} e^{-2y}}{\Gamma(x+1)} \sum_{r=0}^{n} \frac{(-n)_r(-x)_r}{r!}\ C_{n-r}^{(a)}(x-r) Y_r^{2n+\beta-r}(y) \left\{ \frac{-2}{y^2} \right\}^r, \tag{5.20}$$

$$(\Delta_x + D_y)^n \left\{ \frac{c^x\ \Gamma(x+\beta)}{\Gamma(x-n+1)} y^{n+\alpha} e^{-y} \right\}$$
$$= c^{x+n} y^{\alpha+n} e^{-y} \sum_{r=0}^{n} \frac{(-n)_r \Gamma(x-r+\beta)}{\Gamma(x-r+1)}$$
$$\times\ M_{n-r}(x-r;\beta,c) L_r^{\alpha+n-r}(y) \left\{ \frac{-1}{y} \right\}^r, \tag{5.21}$$

$$(\Delta_x + D_y)^n \left\{ \frac{c^x\ \Gamma(x+\beta)}{\Gamma(x-n+1)} e^{-y^2} \right\}$$
$$= c^{x+n} e^{-y^2} \sum_{r=0}^{n} \frac{(-n)_r \Gamma(x-r+\beta)}{r!\Gamma(x-r+1)} M_{n-r}(x-r;\beta,c) H_r(y), \tag{5.22}$$

$$(\Delta_x + D_y)^n \left\{ \frac{c^x\ \Gamma(x+\beta)}{\Gamma(x-n+1)} (y-1)^{n+\gamma}(y+1)^{n+\delta} \right\}$$
$$= c^{x+n}(y-1)^{n+\gamma}(y+1)^{n+\delta} \sum_{r=0}^{n} \frac{(-n)_r \Gamma(x-r+\beta)}{\Gamma(x-r+1)}$$
$$\times\ M_{n-r}(x-r;\beta,c) P_r^{(n+\gamma-r,n+\delta-r)}(y) \left\{ \frac{2}{1-y^2} \right\}^r, \tag{5.23}$$

$$(\Delta_x + D_y)^n \left\{ \frac{c^x\ \Gamma(x+\beta)}{\Gamma(x-n+1)} (y^2-1)^n \right\}$$
$$= c^{x+n}(y^2-1)^n \sum_{r=0}^{n} \frac{(-n)_r \Gamma(x-r+\beta)}{\Gamma(x-r+1)}$$
$$\times\ M_{n-r}(x-r;\beta,c) P_r^{(n-r,n-r)}(y) \left\{ \frac{2}{1-y^2} \right\}^r, \tag{5.24}$$

$$
\begin{aligned}
&(\Delta_x + D_y)^n \left\{ \frac{c^x\ \Gamma(x+\beta)}{\Gamma(x-n+1)} (1-y^2)^{n+\nu-\frac{1}{2}} \right\} \\
&\qquad = c^{x+n}(1-y^2)^{n+\nu-\frac{1}{2}} \sum_{r=0}^{n} \frac{(-n)_r \Gamma(x-r+\beta)}{\Gamma(x-r+1)} \frac{(1-\nu-\frac{1}{2}-n)_r}{(\frac{1}{2}-\nu-n)_r(1-\nu-n)_r} \\
&\qquad \times\ M_{n-r}(x-r;\beta,c) C_r^{n-r+\nu}(y) \left\{ \frac{1}{4(1-y^2)} \right\}^r, \qquad (5.25)
\end{aligned}
$$

$$
\begin{aligned}
&(\Delta_x + D_y)^n \left\{ \frac{c^x\ \Gamma(x+\beta)}{\Gamma(x-n+1)} y^{2n+\beta} e^{-\frac{2}{y}} \right\} \\
&\qquad = c^{x+n} y^{2n+\beta} e^{-\frac{2}{y}} \sum_{r=0}^{n} \frac{(-n)_r \Gamma(x-r+\beta)}{r!\Gamma(x-r+1)} \\
&\qquad \times\ M_{n-r}(x-r;\beta,c) Y_r^{2n+\beta-2r}(y) \left\{ \frac{-2}{y^2} \right\}^r, \qquad (5.26)
\end{aligned}
$$

$$
\begin{aligned}
&(\Delta_x + \Delta_y)^n \left\{ \frac{c^x\ \Gamma(x+\beta)}{\Gamma(x-n+1)} y^{2n+\beta} \frac{a^y}{\Gamma(y-n+1)} \right\} \\
&\qquad = c^{x+n} a^y \sum_{r=0}^{n} \frac{(-n)_r \Gamma(x-r+\beta)}{r!\Gamma(x-r+1)\Gamma(y+r+1)} \\
&\qquad \times\ M_{n-r}(x-r;\beta,c) C_r^{a}(y+r), \qquad (5.27)
\end{aligned}
$$

$$
\begin{aligned}
&(\Delta_x + D_y)^n \left\{ \frac{\Gamma(x+\alpha)\ \Gamma(x+\beta)}{\Gamma(x-n+1)\Gamma(x-n+\gamma)} y^{n+\alpha} e^{-y} \right\} \\
&\qquad = n! y^{\alpha+n} e^{-y} \sum_{r=0}^{n} \frac{\Gamma(x-r+\alpha)\Gamma(x-r+\beta)}{\Gamma(x-r+1)\Gamma(x-r+\gamma)} \\
&\qquad \times\ p_{n-r}(x-r;\alpha,\beta,\gamma) L_r^{(\alpha+n-r)}(y) \left\{ \frac{1}{y} \right\}^r, \qquad (5.28)
\end{aligned}
$$

$$
\begin{aligned}
&(\Delta_x + D_y)^n \left\{ \frac{\Gamma(x+\alpha)\ \Gamma(x+\beta)}{\Gamma(x-n+1)\Gamma(x-n+\gamma)} e^{-y^2} \right\} \\
&\qquad = n! e^{-y^2} \sum_{r=0}^{n} \frac{\Gamma(x-r+\alpha)\Gamma(x-r+\beta)}{\Gamma(x-r+1)\Gamma(x-r+\gamma)} \\
&\qquad \times\ p_{n-r}(x-r;\alpha,\beta,\gamma) H_r(y), \qquad (5.29)
\end{aligned}
$$

$$(\Delta_x + D_y)^n \left\{ \frac{\Gamma(x+\alpha)\ \Gamma(x+\beta)}{\Gamma(x-n+1)\Gamma(x-n+\gamma)} (y-1)^{n+\delta}(y+1)^{n+\psi} \right\}$$
$$= n!(y-n)^{n+\delta}(y+1)^{n+\psi} \sum_{r=0}^{n} \frac{\Gamma(x-r+\alpha)\Gamma(x-r+\beta)}{\Gamma(x-r+1)\Gamma(x-r+\gamma)}$$
$$\times\ p_{n-r}(x-r;\alpha,\beta,\gamma) P_r^{(n+\delta-r,n+\psi-r)}(y) \left\{ \frac{2}{y^2-1} \right\}^r, \qquad (5.30)$$

$$(\Delta_x + D_y)^n \left\{ \frac{\Gamma(x+\alpha)\ \Gamma(x+\beta)}{\Gamma(x-n+1)\Gamma(x-n+\gamma)} (y^2-1)^{n} \right\}$$
$$= n!(y^2-1)^{n} \sum_{r=0}^{n} \frac{\Gamma(x-r+\alpha)\Gamma(x-r+\beta)}{\Gamma(x-r+1)\Gamma(x-r+\gamma)}$$
$$\times\ p_{n-r}(x-r;\alpha,\beta,\gamma) P_r^{(n-r,n-r)}(y) \left\{ \frac{2}{y^2-1} \right\}^r, \qquad (5.31)$$

$$(\Delta_x + D_y)^n \left\{ \frac{\Gamma(x+\alpha)\ \Gamma(x+\beta)}{\Gamma(x-n+1)\Gamma(x-n+\gamma)} (1-y^2)^{n+\nu-\frac{1}{2}} \right\}$$
$$= n!(1-y^2)^{n+\nu-\frac{1}{2}} \sum_{r=0}^{n} \frac{\Gamma(x-r+\alpha)\Gamma(x-r+\beta)(1-\nu-\frac{1}{2}-n)_r}{\Gamma(x-r+1)\Gamma(x-r+\gamma)(\frac{1}{2}-\nu-n)_r(1-\nu-n)_r}$$
$$\times\ p_{n-r}(x-r;\alpha,\beta,\gamma) C_r^{n-r+\nu}(y) \left\{ \frac{1}{4(y^2-1)} \right\}^r, \qquad (5.32)$$

$$(\Delta_x + D_y)^n \left\{ \frac{\Gamma(x+\alpha)\ \Gamma(x+\beta)}{\Gamma(x-n+1)\Gamma(x-n+\gamma)} y^{2n+\beta} e^{\frac{-2}{y}} \right\}$$
$$= n! y^{2n+\beta} e^{\frac{-2}{y}} \sum_{r=0}^{n} \frac{\Gamma(x-r+\alpha)\Gamma(x-r+\beta)}{\Gamma(x-r+1)\Gamma(x-r+\gamma)}$$
$$\times\ p_{n-r}(x-r;\alpha,\beta,\gamma) Y_r^{2n+\beta-2r}(y) \left\{ \frac{2}{y^2} \right\}^r, \quad (5.33)$$

$$(\Delta_x + \Delta_y)^n \left\{ \frac{\Gamma(x+\alpha)\ \Gamma(x+\beta)}{\Gamma(x-n+1)\Gamma(x-n+\gamma)} \frac{a^y}{\Gamma(y-n+1)} \right\}$$
$$= n! a^y \sum_{r=0}^{n} \frac{(-1)^r \Gamma(x-r+\alpha)\Gamma(x-r+\beta)}{r!\Gamma(x-r+1)\Gamma(x-r+\gamma)\Gamma(y+r+1)}$$
$$\times\ p_{n-r}(x-r;\alpha,\beta,\gamma) C_r^{a}(y+r). \qquad (5.34)$$

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