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Some Lacunary Difference Sequence Spaces defined by Musielak-Orlicz Functions

Bipan Hazarika

Department of Mathematics, Rajiv Gandhi University,
Rono Hills, Doimukh, Itanagar-791 112, Arunachal Pradesh, India
e-mail : bh_rgu@yahoo.co.in

Abstract : In this article we are introduced the lacunary sequence spaces defined by Musielak-Orlicz functions and study their algebraic and topological properties. Also we obtain some relations related to these spaces.

Keywords : Difference sequence space; Lacunary sequence; Musielak-Orlicz function.
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1 Introduction

Throughout the article $w, c, c_0, \ell_\infty, \ell_1$ denote the spaces of all, convergent, null, bounded and absolutely summable sequences of complex numbers, respectively. The zero sequence is denoted by θ. Also $\mathbf{N}$ and $\mathbf{R}$ denote the set of all positive integers and set of real numbers respectively.

The difference sequence space was initially introduced by Kizmaz [1] and it was generalized by Et and Colak [2] defined in the following way:

$$Z(\Delta^m) = \{(x_k) \in w : \Delta^m x_k \in Z\},$$

for $Z = c, c_0, \ell_\infty$, where $m \in \mathbf{N}$; $\Delta^m x_k = \Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}$ and $\Delta^0 x_k = x_k$, for all $k \in \mathbf{N}$. The generalized difference operator is equivalent to the following binomial representation:

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$$\Delta^m x_k = \sum_{\nu=0}^{m} \binom{m}{\nu} (-1)^\nu x_{k+\nu}.$$

A *lacunary sequence* is an increasing integer sequence $\xi = (k_r)$, $r = 1, 2, 3, ...$ where $k_0 = 0$ with $h_r = k_r - k_{r-1} \to \infty$, as $r \to \infty$. We denote $I_r = (k_{r-1}, k_r]$ and $\eta_r = \frac{k_r}{k_{r-1}}$, for $r = 1, 2, 3, ...$

The lacunary strongly convergent sequence space N_ξ was defined by Freedman et al. [3] in the following way:

$$N_\xi = \left\{ (x_k) : \lim_{r\to\infty} h_r^{-1} \sum_{k\in I_r} |x_k - L| = 0, \text{ for some } L \right\}.$$

The space N_ξ is a *BK*- space with respect to the norm

$$||(x_k)||_\xi = \sup_r h_r^{-1} \sum_{k\in I_r} |x_k|.$$

N_ξ^0 denotes the subset of these sequences in N_ξ for which $L = 0, (N_\xi^0, ||.||_\xi)$ is also a *BK*- space. There is a relation beteen N_ξ and $|\sigma_1|$ of strongly Cesàro summable sequences (see Freedman et al. [3]). The space $|\sigma_1|$ is defined by

$$|\sigma_1| = \left\{ (x_k) \in w : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |x_k - L| = 0, \text{ for some } L \right\}.$$

For $\xi = (2^r)$, we have a relation between the spaces $|\sigma_1|$ and N_ξ, i.e. $| \ \sigma_1| = N_\xi$.

An *Orlicz function* is a function $M : [0, \infty) \to [0, \infty)$, which is continuous, non-decreasing and convex with $M(0) = 0, M(x) > 0$, for $x > 0$ and $M(x) \to \infty$, as $x \to \infty$. An Orlicz function M is said to satisfy Δ_2 - *condition* for small x or at 0 if for each $k > 0$ there exist $R_k > 0$ and $x_k > 0$ such that $M(kx) \le R_k M(x)$, for all $x \in (0, x_k]$. Moreover, an Orlicz function M is said to satisfy the Δ_2-condition if and only if

$$\lim_{x\to\infty} \sup \frac{M(2x)}{M(x)} < \infty.$$

Two Orlicz functions M_1 and M_2 are said to be *equivalent* if there are positive constants α, β and x_0 such that

$$M_1(\alpha x) \le M_2)(x) \le M_1(\beta x),$$

for all x with $0 \le x < x_0$.

Lindenstrauss and Tzafriri [4] used the idea of the Orlicz function to construct the sequence space:

$$\ell_M = \left\{ (x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}.$$

The space ℓ_M becomes a Banach space, with respect to the norm

$$|| \ x \ || = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}$$

which is called an Orlicz sequence space. The space ℓ_M is closely related to the space ℓ_p which is an Orlicz sequence space with $M(t) = |t|^p$, for $1 \leq p < \infty$.

Later on, Orlicz sequence spaces were investigated by Parashar and Choudhary [5], Maddox [6], Tripathy et al. [7–10] and many others.

2 Definitions and Notations

A sequence $\mathbf{M} = (M_k)$ of Orlicz functions is called a *Musielak-Orlicz function* (for details see [11, 12]). Also a Musielak-Orlicz function $\phi = (\phi_k)$ is called a *complementary function* of a Musielak-Orlicz function $\mathbf{M}$ if

$$\phi_k(t) = \sup\{| \ t|s - M_k(s) : s \geq 0\}, \text{ for } k = 1, 2, 3, ...$$

For a given Musielak-Orlicz function $\mathbf{M}$, the Musielak-Orlicz sequence space $l_{\mathbf{M}}$ and its subspace $h_{\mathbf{M}}$ are defined as follows:

$$l_{\mathbf{M}} = \{x = (x_k) \in w : I_{\mathbf{M}}(cx) < \infty, \text{ for some } c > 0\};$$

$$h_{\mathbf{M}} = \{x = (x_k) \in w : I_{\mathbf{M}}(cx) < \infty, \text{ for all } c > 0\},$$

where $I_{\mathbf{M}}$ is a convex modular defined by

$$I_{\mathbf{M}} = \sum_{k=1}^{\infty} M_k(x_k), x = (x_k) \in l_{\mathbf{M}}.$$

We consider $l_{\mathbf{M}}$ equipped with the Luxemburg norm

$$||x|| = \inf \left\{ k > 0 : I_{\mathbf{M}}\left(\frac{x}{k}\right) \leq 1 \right\}$$

or equipped with the Orlicz norm

$$||x||^0 = \inf \left\{ \frac{1}{k}(1 + I_{\mathbf{M}}(kx)) : k > 0 \right\}.$$

The main aim of this article is to introduce the following sequence spaces and examine some properties of the resulting sequence spaces. Let $p = (p_k)$ denote the sequences of positive real numbers, for all $k \in \mathbf{N}$. Let $\mathbf{M} = (M_k)$ be a Musielak-Orlicz function and $v = (v_k)$ be any sequence of non-zero complex numbers. Let X be a seminormed space over the field of complex numbers with the semi norm

q and $w(X)$ denotes the space of all sequences $x = (x_k)$, where $x_k \in X$. Then we define the following sequence spaces:

$$[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_1 = \Big\{(x_k) \in w(X) : \lim_{r\to\infty} h_r^{-1} \sum_{k\in I_r} \Big[M_k\Big(q\Big(\frac{v_k\Delta^m x_k - L}{\rho}\Big)\Big)\Big]^{p_k} \to 0, \text{ for some } \rho > 0 \text{ and } L \in \mathbf{C}\Big\};$$

$$[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_0 = \Big\{(x_k) \in w(X) : \lim_{r\to\infty} h_r^{-1} \sum_{k\in I_r} \Big[M_k\Big(q\Big(\frac{v_k\Delta^m x_k}{\rho}\Big)\Big)\Big]^{p_k} \to 0, \text{ for some } \rho > 0\Big\};$$

$$[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_\infty = \Big\{(x_k) \in w(X) : \sup_r h_r^{-1} \sum_{k\in I_r} \Big[M_k\Big(q\Big(\frac{v_k\Delta^m x_k}{\rho}\Big)\Big)\Big]^{p_k} < \infty, \text{ for some } \rho > 0\Big\}.$$

Definition 2.1. A sequence space E is said to be *solid* (or *normal*) if $(\alpha_k x_k) \in E$, whenever $(x_k) \in E$ and for all sequence (α_k) of scalars with $|\alpha_k| \le 1$, for all $k \in \mathbf{N}$.

Definition 2.2. A sequence space E is said to be *symmetric* if $(x_k) \in E$ implies $(x_{\pi(k)}) \in E$, where π is a permutation of $\mathbf{N}$.

Definition 2.3. A sequence space E is said to be *convergence free* if $(y_k) \in E$ whenever $(x_k) \in E$ and $x_k = 0$ implies $y_k = 0$.

Let $K = \{k_1 < k_2 < \cdots\} \subset \mathbf{N}$ and E be a sequence space. A *K-step space* of E is a sequence space $\lambda_K^E = \{(x_{k_n}) \in w : (k_n) \in E\}$. A canonical preimage of a sequence $(x_{k_n}) \in \lambda_K^E$ is a sequence $\{y_n\} \in w$ defined by

$$y_n = \begin{cases} x_n, & \text{if } n \in K; \\ 0, & \text{otherwise.} \end{cases}$$

A canonical preimage of a step space λ_K^E is a set of canonical preimages of all elements in λ_K^E, i.e. y is in canonical preimage of λ_K^E if and only if y is canonical preimage of some $x \in \lambda_K^E$.

Definition 2.4. A sequence space E is said to be *monotone* if it contains the canonical preimages of its step spaces.

The following results will be used for establishing some results of this article.

Lemma 2.5 (Kamthan and Gupta [13, p. 53]). *A sequence space E is solid implies E is monotone.*

Lemma 2.6 (Freedman et al. [3, Lemma 2.1]). *In order to $|\sigma_1| \subseteq N_\xi$ it is necessary and sufficient that* $\lim_r \inf \eta_r > 1$.

Lemma 2.7 (Freedman et al. [3, Lemma 2.2]). *In order to $N_\xi \subseteq |\sigma_1|$ it is necessary and sufficient that* $\lim_r \sup \eta_r < \infty$.

Lemma 2.8 (Et and Nuray [14, Theorem 2.2]). *If X is a Banach space normed by $||.||$, then $\Delta^m(X)$ is also a Banach space normed by*

$$||x||_\Delta = \sum_{i=1}^{m} |x_i| + f(\Delta^m x).$$

3 Main Results

Theorem 3.1. *Let $p = (p_k)$ in ℓ_∞ of strictly positive real numbers and $\xi = (k_r)$ be a lacunary sequence. Then $[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_1, [N_\xi, \mathbf{M}, \Delta^m, p, q, v]_0$ and $[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_\infty$ are linear spaces.*

Proof. The proof of the theorem is easy, so omitted. □

Theorem 3.2. *Let $\mathbf{M} = (M_k)$ be a Musielak-Orlicz function and $p = (p_k)$ in ℓ_∞ of strictly positive real numbers and $\xi = (k_r)$ be a lacunary sequence. Then $[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_0$ is a paranormed space (not totally paranormed) with the paranorm*

$$g_\Delta(x) = \sum_{i=1}^{m} |x_i| + \inf\Big\{\rho^{\frac{p_k}{H}} : \sup_r h_r^{-1} \sum_{k \in I_r} \Big[M_k\Big(q\Big(\frac{v_k \Delta^m x_k}{\rho}\Big)\Big)\Big] \leq 1, \text{ for some } \rho > 0 \text{ and } r = 1, 2, 3, ...\Big\},$$

where $H = \max\{1, \sup p_k\}$.

Proof. Clearly $g_\Delta(x) = g_\Delta(-x)$. Since $M_k(0) = 0$, for all $k \in \mathbf{N}$, we get $g_\Delta(\bar{\theta}) = 0$, for $x = \bar{\theta}$. Let $x = (x_k)$ and $y = (y_k)$ be two elements in $[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_0$ and let us choose $\rho_1 > 0$ and $\rho_2 > 0$ be such that

$$\sup_r h_r^{-1} \sum_{k \in I_r} \Big[M_k\Big(q\Big(\frac{v_k \Delta^m x_k}{\rho_1}\Big)\Big)\Big] \leq 1, \; r = 1, 2, 3, ...$$

and

$$\sup_r h_r^{-1} \sum_{k \in I_r} \Big[M_k\Big(q\Big(\frac{v_k \Delta^m y_k}{\rho_2}\Big)\Big)\Big] \leq 1, \; r = 1, 2, 3, ...$$

Let $\rho = \rho_1 + \rho_2$, then we have

$$\begin{aligned}
\sup_r h_r^{-1} \sum_{k\in I_r} \Big[M_k\Big(q\Big(\frac{v_k\Delta^m(x_k+y_k)}{\rho}\Big)\Big)\Big] \\
&\leq \Big(\frac{\rho_1}{\rho_1+\rho_2}\Big) \sup_r h_r^{-1} \sum_{k\in I_r} \Big[M_k\Big(q\Big(\frac{v_k\Delta^m x_k}{\rho_1}\Big)\Big)\Big] \\
&\quad + \Big(\frac{\rho_2}{\rho_1+\rho_2}\Big) \sup_r h_r^{-1} \sum_{k\in I_r} \Big[M_k\Big(q\Big(\frac{v_k\Delta^m y_k}{\rho_2}\Big)\Big)\Big] \\
&\leq 1.
\end{aligned}$$

Since $\rho > 0$, we have

$$\begin{aligned}
g_\Delta(x+y) &= \sum_{i=1}^m |x_i+y_i| + \inf\Big\{\rho^{\frac{p_k}{H}} : \sup_r h_r^{-1} \sum_{k\in I_r} \Big[M_k\Big(q\Big(\frac{v_k\Delta^m(x_k+y_k)}{\rho}\Big)\Big)\Big] \leq 1, \\
&\qquad r = 1,2,3,...\Big\} \\
&\leq \sum_{i=1}^m |x_i| + \inf\Big\{\rho_1^{\frac{p_k}{H}} : \sup_r h_r^{-1} \sum_{k\in I_r} \Big[M_k\Big(q\Big(\frac{v_k\Delta^m x_k}{\rho_1}\Big)\Big)\Big] \leq 1, \\
&\qquad \text{for some } \rho_1 > 0 \text{ and } r = 1,2,3,...\Big\} \\
&\quad + \sum_{i=1}^m |y_i| + \inf\Big\{\rho_2^{\frac{p_k}{H}} : \sup_r h_r^{-1} \sum_{k\in I_r} \Big[M_k\Big(q\Big(\frac{v_k\Delta^m y_k}{\rho_2}\Big)\Big)\Big] \leq 1, \\
&\qquad \text{for some } \rho_2 > 0 \text{ and } r = 1,2,3,...\Big\} \\
&= g_\Delta(x) + g_\Delta(y),
\end{aligned}$$

i.e. $g_\Delta(x+y) \leq g_\Delta(x) + g_\Delta(y)$.

Finally, let λ be a given non-zero scalar in **C**. Then the continuity of the product follows from the following expression.

$$\begin{aligned}
g_\Delta(\lambda x) &= \sum_{i=1}^m |\lambda x_i| + \inf\Big\{\rho^{\frac{p_k}{H}} : \sup_r h_r^{-1} \sum_{k\in I_r} \Big[M_k\Big(q\Big(\frac{v_k\Delta^m(\lambda x_k)}{\rho}\Big)\Big)\Big] \leq 1, \\
&\qquad \text{for some } \rho > 0 \text{ and } r = 1,2,3,...\Big\} \\
&= \lambda \sum_{i=1}^m |x_i| + \inf\Big\{(|\lambda|\eta)^{\frac{p_k}{H}} : \sup_r h_r^{-1} \sum_{k\in I_r} \Big[M_k\Big(q\Big(\frac{v_k\Delta^m x_k}{\eta}\Big)\Big)\Big] \leq 1, \\
&\qquad \text{for some } \rho > 0 \text{ and } r = 1,2,3,...\Big\}
\end{aligned}$$

where $\eta = \frac{\rho}{|\lambda|} > 0$. This completes the proof of the theorem. □

The proof of the following theorem is easy, so omitted.

Theorem 3.3. *Let* $\mathbf{M} = (M_k)$ *and* $\phi = (\phi_k)$ *be two Musielak-Orlicz functions and* $p = (p_k) \in \ell_\infty$ *of strictly positive real numbers. Then*

(i) $[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_Z \subseteq [N_\xi, \phi.\mathbf{M}, \Delta^m, p, q, v]_Z$

(ii) $[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_Z \cap [N_\xi, \phi, \Delta^m, p, q, v]_Z \subseteq [N_\xi, \phi + \mathbf{M}, \Delta^m, p, q, v]_Z$,

where $Z = 0, 1, \infty$.

Theorem 3.4. *The inclusion* $[N_\xi, \mathbf{M}, \Delta^{m-1}, q]_Z \subseteq [N_\xi, \mathbf{M}, \Delta^m, q]_Z$ *holds, for* $m \geq 1$. *In general* $[N_\xi, \mathbf{M}, \Delta^i, q]_Z \subseteq [N_\xi, \mathbf{M}, \Delta^m, q]_Z$, *for* $i = 0, 1, 2, ..., m-1$ *and the inclusions are strict, where* $Z = 0, 1, \infty$.

Proof. Let $(x_k) \in [N_\xi, \mathbf{M}, \Delta^{m-1}, q]_0$. Then there exists $\rho > 0$ such that

$$\lim_{r\to\infty} h_r^{-1} \sum_{k\in I_r} \Big[M_k\Big(q\Big(\frac{\Delta^{m-1}x_k}{\rho}\Big)\Big)\Big] \to 0.$$

Since $\mathbf{M}$ is nondecreasing and convex, we have

$$\begin{aligned}
h_r^{-1} & \sum_{k\in I_r} \Big[M_k\Big(q\Big(\frac{\Delta^m x_k}{2\rho}\Big)\Big)\Big] \\
&= h_r^{-1} \sum_{k\in I_r} \Big[M_k\Big(q\Big(\frac{\Delta^{m-1}x_k - \Delta^{m-1}x_{k+1}}{2\rho}\Big)\Big)\Big] \\
&\leq h_r^{-1} \Bigg\{\sum_{k\in I_r} \Big[M_k\Big(q\Big(\frac{\Delta^{m-1}x_k}{2\rho}\Big)\Big)\Big] + \sum_{k\in I_r} \Big[M_k\Big(q\Big(\frac{\Delta^{m-1}x_{k+1}}{2\rho}\Big)\Big)\Big]\Bigg\} \\
&\leq h_r^{-1} \sum_{k\in I_r} \frac{1}{2}\Big[M_k\Big(q\Big(\frac{\Delta^{m-1}x_k}{\rho}\Big)\Big)\Big] + h_r^{-1} \sum_{k\in I_r} \frac{1}{2}\Big[M_k\Big(q\Big(\frac{\Delta^{m-1}x_{k+1}}{\rho}\Big)\Big)\Big] \\
&< h_r^{-1} \sum_{k\in I_r} \Big[M_k\Big(q\Big(\frac{\Delta^{m-1}x_k}{\rho}\Big)\Big)\Big] + h_r^{-1} \sum_{k\in I_r} \Big[M_k\Big(q\Big(\frac{\Delta^{m-1}x_{k+1}}{\rho}\Big)\Big)\Big].
\end{aligned}$$

Taking limit $r \to \infty$, we have

$$h_r^{-1} \sum_{k\in I_r} \Big[M_k\Big(q\Big(\frac{\Delta^m x_k}{\rho}\Big)\Big)\Big] \to 0,$$

i.e. $(x_k) \in [N_\xi,\ \mathbf{M},\ \Delta^m,\ q]_0$. The rest of the cases can be proved in the similar way. By using induction, we have $[N_\xi, \mathbf{M}, \Delta^i, q]_Z \subseteq [N_\xi, \mathbf{M}, \Delta^m, q]_Z$, for $i = 0, 1, 2, ..., m-1$. ☐

The above inclusion is strict follows from the following example.

Example 3.5. *Let* $M_k(x) = x^2$, *for all* $x \in [0, \infty)$, $\xi = (2^r)$, *for all* $k \in \mathbf{N}$ *and* $q(x) = |x|$. *Consider a sequence* (x_k) *defined by*

$$(x_k) = (k^{m-1}, k^{m-1}, k^{m-1}, ...).$$

Then $\Delta^m x_k = 0$, *but* $\Delta^{m-1}x_k = (-1)^{m-1}(m-1)!$, *for all* $n \in \mathbf{N}$. *Thus* $(x_k) \in [N_\xi, \mathbf{M}, \Delta^m, q]_0$, *but* $(x_k) \notin [N_\xi, \mathbf{M}, \Delta^{m-1}, q]_0$.

Theorem 3.6. *Let $\xi = (k_r)$ be a lacunary sequence and let $\mathbf{M} = (\mathbf{M_k})$ be a Musielak-Orlicz function. Then*

(i) $[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_0 \subseteq [N_\xi, \mathbf{M}, \Delta^m, p, q, v]_1 \subseteq [N_\xi, \mathbf{M}, \Delta^m, p, q, v]_\infty$, *and the inclusion is strict.*

(ii) *If* $|v_k| \leq 1$, *then* $[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_Z \subseteq [N_\xi, \mathbf{M}, \Delta^m, p, q,]_Z$, *for* $Z = 0, 1, \infty$.

Proof. (i) The inclusion $[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_0 \subseteq [N_\xi, \mathbf{M}, \Delta^m, p, q, v]_1$ is obvious. Let (x_k) be an element of $[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_1$. Then there exists $\rho > 0$ such that

$$\lim_{r\to\infty} h_r^{-1} \sum_{k\in I_r} \Big[M_k\Big(q\Big(\frac{v_k\Delta^m x_k - L}{\rho}\Big)\Big)\Big]^{p_k} \to 0.$$

Since M_k is non decreasing and convex for all $k \in \mathbf{N}$, we have

$$h_r^{-1} \sum_{k\in I_r} \Big[M_k\Big(q\Big(\frac{v_k\Delta^m x_k}{\rho}\Big)\Big)\Big]^{p_k} \leq D h_r^{-1} \sum_{k\in I_r} \Big[M_k\Big(q\Big(\frac{v_k\Delta^m x_k - L}{\rho}\Big)\Big)\Big]^{p_k} + D \max\Big[1, M_k\Big(q\Big(\frac{L}{\rho}\Big)\Big)\Big]^H,$$

where $G = \sup_k p_k, D = \max\{1, 2^{G-1}\}$.

Thus the sequence (x_k) belongs to $[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_\infty$.

The inclusions are strict follows from the following example.

Example 3.7. *Let*

$$p_k = \begin{cases} 4, & \text{if } k \text{ is even;} \\ 5, & \text{if } k \text{ is odd.} \end{cases}$$

Let $m \geq 0$ be given. Let $v_k = k$, $M_k(x) = x^2$, for all $k \in \mathbf{N}$ and $q(x) = |x|$. Let $\xi = (2^r)$ be a lacunary sequence. Consider a sequence (x_k) defined by

$$(x_k) = (k^m, k^m, k^m, ...).$$

Thus the sequence (x_k) belongs to $[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_1$, but (x_k) does not belong to $[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_0$.

The proof of the part (ii) is easy, so omitted. □

Theorem 3.8. *Let $\mathbf{M} = (M_k)$ and $\phi = (\phi_k)$ be two Musielak-Orlicz functions. If M_k and ϕ_k are equivalent for each $k \in \mathbf{N}$ and $\xi = (k_r)$ be a lacunary sequence. Then*

$$[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_Z = [N_\xi, \phi, \Delta^m, p, q, v]_Z,$$

where $Z = 0, 1, \infty$.

Proof. The proof of the theorem is easy, so omitted. □

Theorem 3.9. *Let $\mathbf{M} = (M_k)$ be any Musielak-Orlicz function and let q_1 and q_2 be two semi norms. Then* *(i)* $[N_\xi, \mathbf{M}, \Delta^m, p, q_1, v]_Z \cap [N_\xi, \mathbf{M}, \Delta^m, p, q_2, v]_Z \subseteq [N_\xi, \mathbf{M}, \Delta^m, p, q_1 + q_2, v]_Z$;

(ii) *if* q_1 *is stronger than* q_2, $[N_\xi, \mathbf{M}, \Delta^m, p, q_1, v]_Z \subseteq [N_\xi, \mathbf{M}, \Delta^m, p, q_2, v]_Z$, *where* $Z = 0, 1, \infty$.

Proof. The proof of the theorem is easy, so omitted. □

We give the following two propositions without proof.

Proposition 3.10. *Let* $\xi = (k_r)$ *be a lacunary sequence. Then the followings hold:*

(i) *If* $\liminf_r \eta_r > 1$, *then for any Musielak-Orlicz function* $\mathbf{M} = (M_k)$, *for all* $k \in \mathbf{N}$,

$$[W, \mathbf{M}, \Delta^m, p, q, v]_0 \subseteq [N_\xi, \mathbf{M}, \Delta^m, p, q, v]_0,$$

where

$$[W, \mathbf{M}, \Delta^m, p, q, v]_0 = \Big\{(x_k) \in w(X) : \lim_{n\to\infty} \sum_{k=1}^{n} \Big[M_k\Big(q\Big(\frac{v_k \Delta^m x_k}{\rho}\Big)\Big)\Big]^{p_k} \to 0, \text{ for some } \rho > 0\Big\}.$$

(ii) *If* $\limsup_r \eta_r < \infty$, *then for any Musielak-Orlicz function* $\mathbf{M} = (M_k)$, *for all* $k \in \mathbf{N}$,

$$[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_0 \subseteq [W, \mathbf{M}, \Delta^m, p, q, v]_0.$$

Proposition 3.11. *Let* $\xi = (k_r)$ *be a lacunary sequence, with* $0 < \liminf_r \eta_r \leq \limsup_r \eta_r < \infty$, *then for any Musielak-Orlicz function* $\mathbf{M} = (M_k)$, *for all* $k \in \mathbf{N}$,

$$[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_0 = [W, \mathbf{M}, \Delta^m, p, q, v]_0.$$

Property 3.12. *The spaces* $[N_\xi, \mathbf{M}, p, q, v]_0$ *and* $[N_\xi, \mathbf{M}, p, q, v]_\infty$ *are solid as well as monotone. The spaces* $[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_Z$ *are not solid in general, for* $Z = 0, 1, \infty$.

Proof. Let $(x_k) \in [N_\xi, \mathbf{M}, p, q, v]_0$. Then there exists $\rho > 0$ such that

$$\lim_{r\to\infty} h_r^{-1} \sum_{k\in I_r} \Big[M_k\Big(q\Big(\frac{v_k x_k}{\rho}\Big)\Big)\Big]^{p_k} \to 0.$$

Let (α_k) be a sequence of scalars such that $|\alpha_k| \leq 1$, for all $k \in \mathbf{N}$. Since

$$|\alpha_k| \leq \max(1, |\alpha_k|^G) \leq 1, \text{ for all } k \in \mathbf{N}, \text{ where } G = \sup_k p_k < \infty.$$

Then for each r, we have

$$h_r^{-1} \sum_{k\in I_r} \Big[M_k\Big(q\Big(\frac{\alpha_k(v_k x_k)}{\rho}\Big)\Big)\Big]^{p_k} \leq h_r^{-1} \sum_{k\in I_r} \Big[M_k\Big(q\Big(\frac{v_k x_k}{\rho}\Big)\Big)\Big]^{p_k}. \tag{3.1}$$

Therefore $(\alpha_k x_k) \in [N_\xi, \mathbf{M}, p, q, v]_0$. Hence $[N_\xi, \mathbf{M}, p, q, v]_0$ is solid.

By the Lemma 2.5, it follows that the space $[N_\xi, \mathbf{M}, p, q, v]_0$ is monotone. Again by the inequality (3.1) and the Lemma 2.5, we can proved that the space $[N_\xi, \mathbf{M}, p, q, v]_\infty$ is solid as well as monotone. In order to prove that the spaces $[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_1$ and $[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_\infty$ are not solid in general, we consider the following example.

Example 3.13. *Let $M_k(x) = x^t$, for all $k \in \mathbf{N}$ and $t \geq 1$. Let $p_k = \frac{1}{k}$, $v_k = k$, for all $k \in \mathbf{N}$ and $q(x) = |\, x|$. Let $\xi = (2^r)$ be a lacunary sequence, for all $k \in \mathbf{N}$. Consider a sequence (x_k) defined by*

$$x_k = k^2, \text{ for all } k \in \mathbf{N}.$$

Then (x_k) belongs to $[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_1$ and $[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_\infty$, for $m = 1$. Let $(\alpha_k) = (-1)^k$, for all $k \in \mathbf{N}$. Then $(\alpha_k x_k)$ does not belong to the spaces $[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_1$ and $[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_\infty$. Hence the spaces $[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_1$ and $[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_\infty$ are not solid.

Therefore by the Lemma 2.5, it follows that the spaces $[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_1$ and $[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_\infty$ are not monotone.

Next to show that the space $[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_0$ is not solid in general. We consider the following example.

Example 3.14. *Under the restrictions on $\mathbf{M}, p, v, m, q$ and ξ as in Example 3.7. We consider a sequence (x_k) defined by*

$$x_k = 2, \text{ for all } k \in \mathbf{N}.$$

Let $(\alpha_k) = (-1)^k$, for all $k \in \mathbf{N}$. Then $(\alpha_k x_k)$ does not belong to $[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_0$. Hence the space $[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_0$ is not solid.

Therefore by the Lemma 2.5, it follows that the space $[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_0$ is not monotone. □

Property 3.15. *The space $[N_\xi, \mathbf{M}, p, q, v]_1$ is neither solid nor monotone.*

Proof. The space $[N_\xi, \mathbf{M}, p, q, v]_1$ is not monotone follows from the following example.

Example 3.16. *Let $p_k = 1 + \frac{1}{k^2}$ and $v_k = k$, for all $k \in \mathbf{N}$. Let $M_k(x) = x^t$, for all $k \in \mathbf{N}$ and $t \geq 1$ and $q(x) = |x|$. Let $\xi = (2^r)$ be a lacunary sequence for all $k \in \mathbf{N}$. Consider a sequence (x_k) defined by*

$$(x_k) = (2, 2, 2, ...), \text{ for all } k \in \mathbf{N}.$$

Consider the K^{th}-step space E_K for a sequence space E and defined a sequence (y_k) as follows:

$$y_k = \begin{cases} x_k, & \text{if } k \text{ is even;} \\ 0, & \text{otherwise.} \end{cases}$$

Then (y_k) does not belong to the K^{th}-step space E_K of the sequence space E. Hence the space $[N_\xi, \mathbf{M}, p, q, v]_1$ is not monotone.

Therefore by the Lemma 2.5, it follows that the space $[N_\xi, \mathbf{M}, p, q, v]_1$ is not solid. □

Property 3.17. *The spaces* $[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_Z$ *are not monotone in general, for* $Z = 0, 1, \infty$.

Proof. The proof of the result follows from the Examples 3.13 and 3.14, by considering the K^{th}- step space E_K for a sequence space E and defined a sequence (y_k) as follows:

$$y_k = \begin{cases} x_k, & \text{if k is even;} \\ 0, & \text{otherwise.} \end{cases}$$

Then the sequence (x_k) defined in the Example 3.13 belongs to $[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_Z$, but (y_k) does not belong to $[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_Z$, for $Z = 1, \infty$.

Similarly, (x_k) defined in the Example 3.14 belongs to $[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_0$, but (y_k) does not belong to $[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_0$. Hence the spaces $[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_Z$ are not monotone, for $Z = 0, 1, \infty$. □

Property 3.18. *The spaces* $[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_Z$ *are not symmetric in general, for* $Z = 0, 1, \infty$.

Proof. The proof of the result follows from the following example.

Example 3.19. *Let* $M_k(x) = x^2$, $p_k = k$ *and* $v_k = k^2$, *for all* $k \in \mathbf{N}$. *and* $q(x) = |x|$. *Let* $\xi = (2^r)$ *be a lacunary sequence for all* $k \in \mathbf{N}$. *Consider a sequence* (x_k) *defined by*

$$x_k = k^3, \text{ for all } k \in \mathbf{N}.$$

Then (x_k) *belongs to* $[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_0$, *for* $m = 1$. *Consider the sequence* (y_k) *which is the rearrangement of the sequence* (x_k) *defined by*

$$(y_k) = (x_1, x_2, x_4, x_3, x_9, x_5, x_{16}, x_6, x_{25}, x_7, ...).$$

Then (y_k) *does not belong to* $[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_Z$.

Hence the spaces $[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_Z$ are not symmetric in general. □

Property 3.20. *The space* $[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_0$ *is not convergence free.*

Proof. The proof of the result follows from the following example.

Example 3.21. *Let* $M_k(x) = x$, $p_k = k$, $v_k = k$, *for all* $k \in \mathbf{N}$. *and* $q(x) = |x|$. *Let* $\xi = (2^r)$ *be a lacunary sequence for all* $k \in \mathbf{N}$. *Consider a sequence* (x_k) *defined by*

$$x_k = \begin{cases} 2, & \text{if } k \text{ is even;} \\ 0, & \text{if } k \text{ is odd.} \end{cases}$$

Then (x_k) *belongs to* $[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_0$, *for* $m = 2$. *Consider the sequence* (y_k) *defined by*

$$y_k = \begin{cases} k^2, & \text{if } k \text{ is even;} \\ 0, & \text{if } k \text{ is odd.} \end{cases}$$

Then (y_k) *deos not belong to* $[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_0$.

Hence the space $[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_0$ is not convergence free. □

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