



On Some Coincidence and Common Fixed Point Theorems in G -Cone Metric Spaces¹

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Abstract : In the present paper, some coincidence and common fixed point results were obtained for three mappings, defined on a G -cone metric space, satisfying some special contractive conditions. These results generalize recent well known results in the literature.

Keywords : G -cone metric spaces; Coincidence point; Weakly compatibility; Cone metric spaces.

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1 Introduction

Fixed point theorems play a major role in mathematics such as optimization, mathematical models, economy, military and medicine. So, the metric fixed point theory has been investigated extensively in the past two decades by numerous mathematicians. Some generalizations of a metric space concept have been studied by several authors. These different generalizations have been improved by Gahler [1, 2], by introducing 2-metric spaces, and Dhage [3] by studying the theory of D -metric spaces.

In 2005, Mustafa and Sims [4] introduced a new structure of generalized metric

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spaces which are called G - metric spaces as a generalization of metric spaces. Later, Mustafa et al. [5–7] obtained several fixed point theorems for mappings satisfying different contractive conditions in G - metric spaces.

Further, Huang and Zhang[8] generalized the notion of metric spaces by replacing the real numbers by ordered Banach space and defined the cone metric spaces. They have investigated the convergence in cone metric spaces, introduced the completeness of cone metric spaces and have proved Banach contraction mapping theorem, some other fixed point theorems of contractive type mappings in cone metric spaces using the normality condition. Afterwards, some authors have proved some common fixed point theorems with normal and non-normal cones in these spaces such as [9–13].

Recently, Beg et al. [14] introduced G - cone metric spaces which are generalization of G - metric spaces and cone metric spaces. They proved some topological properties of these spaces such as convergence properties of sequences and completeness. Some fixed point theorems satisfying certain contractive conditions have been also obtained. Jungck [15] introduced the concept of a weakly compatible maps. More recently, several authors have obtained coincidence point results for various mappings on a metric space, by using this concept to obtain fixed point results.

In this present paper, our main purpose is to generalize some results as reported elsewhere [9, 12, 16].

2 Preliminaries

We give some facts and definitions required in the sequel. First we give the definition of a generalized cone metric space.

Let B be a real Banach space and K be a subset of B . K is called a *cone* if it satisfies the following conditions;

1. K is closed, nonempty and $K \neq \{0\}$,
2. $a, b \in \mathbb{R}; a, b \geq 0; x, y \in K \Rightarrow ax + by \in K$, more generally if $a, b, c \in \mathbb{R}$,
 $a, b, c \geq 0, x, y, z \in K \Rightarrow ax + by + cz \in K$,
3. $x \in K$ and $-x \in K \Rightarrow x = 0$.

Given a cone $K \subset E$, we define a partial ordering \leq with respect to K by $x \leq y$ if and only if $y - x \in K$. We write $x < y$ if $x \leq y$ but $x \neq y$; $x \ll y$ if $y - x \in \text{int}K$, where $\text{int}K$ is the interior of K .

There exists two kinds of cones which are normal and non normal cones. The cone K is a *normal cone* if

$$\inf \{ \|x + y\| : x, y \in K \text{ and } \|x\| = \|y\| = 1 \} > 0 \quad (2.1)$$

or equivalently, if there is a number $M > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \Rightarrow \|x\| \leq M \|y\|. \quad (2.2)$$

The least positive number satisfying (2.2) is called the normal constant of K . From (2.1), one can conclude that K is a non normal if and only if there exists sequences $x_n, y_n \in K$ such that

$$0 \leq x_n \leq x_n + y_n, \lim_{n \rightarrow \infty} (x_n + y_n) = 0, \text{ but } \lim_{n \rightarrow \infty} x_n \neq 0$$

Rezapour and Hambarani [13] proved that there were no normal cones with constants $M < 1$ and for each $k > 1$ there are cones with normal constants $M > k$.

Definition 2.1 ([14]). Let X be nonempty set, B be a real Banach space and $K \subset B$ be a cone. Suppose that the mapping $G : X \times X \times X \rightarrow B$ satisfies the followings;

- G1. $G(x, y, z) = 0$; if $x = y = z$,
- G2. $0 < G(x, x, y)$; whenever $x \neq y$, for all $x, y \in X$,
- G3. $G(x, x, y) \leq G(x, y, z)$; whenever $y \neq z$, for all $x, y, z \in X$,
- G4. $G(x, y, z) = G(x, z, y) = G(y, x, z) = \dots$ (Symmetric in all three variables),
- G5. $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.

Then G is called a *generalized cone metric* on X , and X is called a generalized cone metric space or more specifically a G -cone metric space. It is obvious that the concept of a G -cone metric space is more general than a G - metric space and a cone metric space.

Definition 2.2 ([14]). Let X be a G -cone metric space and $\{x_n\}$ be a sequence in X and $x \in X$. We say that $\{x_n\}$ is a

1. *Convergent sequence* if for every $c \in B$ with $0 \ll c$, there is $N \in \mathbb{N}$ such that for all $m, n > N$, $G(x_m, x_n, x) \ll c$ for some fixed x in X .
2. *Cauchy sequence* if for every $c \in B$ with $0 \ll c$, there is $N \in \mathbb{N}$ such that for all $m, n, l > N$, $G(x_m, x_n, x_l) \ll c$.

A G -cone metric space X is said to be *complete* if every Cauchy sequence in X is convergent in X .

Proposition 2.3 ([17]). Let (X, G) be a G -cone metric space, $u, v, w \in X$. Then

- i. If $u \ll v$ and $v \ll w$, then $u \ll w$.
- ii. If $u \leq v$ and $v \ll w$, then $u \ll w$.
- iii. If $0 \leq u \ll c$ for each $c \in \text{int}K$, then $u = 0$.
- iv. If $c \in \text{int}K$, $0 \leq a_n$ and $a_n \rightarrow 0$, then there exists n_0 such that for all $n > n_0$, it follows that $a_n \ll c$.
- v. $\text{int}K + \text{int}K \subset \text{int}K$, $\lambda \text{int}K \subset \text{int}K (\lambda > 0)$.

Throughout the paper, we assume that B is a real Banach space and K is a non normal cone in B with $\text{int}K \neq \emptyset$. By this way, we uniquely determine the limit of a sequence.

Proposition 2.4 ([14]). *Let X be a G -cone metric space. Then the following statements are equivalent;*

- i. $\{x_n\}$ converges to x .*
- ii. $G(x_n, x_n, x) \rightarrow 0$ as $n \rightarrow \infty$.*
- iii. $G(x_n, x, x) \rightarrow 0$ as $n \rightarrow \infty$.*
- iv. $G(x_m, x_n, x) \rightarrow 0$ as $m, n \rightarrow \infty$.*

The following lemmas are about the topological structure of G -cone metric space and these lemmas have been proved in [14], so we give them without the proofs.

Lemma 2.5. *Let X be a G -cone metric space, $\{x_m\}$, $\{y_n\}$ and $\{z_l\}$ be sequences in X such that $x_m \rightarrow x$, $y_n \rightarrow y$ and $z_l \rightarrow z$, then $G(x_m, y_n, z_l) \rightarrow G(x, y, z)$ as $m, n, l \rightarrow \infty$.*

Lemma 2.6. *Let $\{x_n\}$ be sequence in a G -cone metric space X and $x \in X$. If $\{x_n\}$ converges to x and $\{x_n\}$ converges to y , then $x = y$.*

Lemma 2.7. *Let $\{x_n\}$ be sequence in a G -cone metric space X and if $\{x_n\}$ converges to $x \in X$, then $G(x_m, x_n, x) \rightarrow 0$ as $m, n \rightarrow \infty$.*

Lemma 2.8. *Let $\{x_n\}$ be sequence in a G -cone metric space X and $x \in X$. If $\{x_n\}$ converges to x , then $\{x_n\}$ is a Cauchy sequence.*

Lemma 2.9. *Let $\{x_n\}$ be sequence in a G -cone metric space X and if $\{x_n\}$ is a Cauchy sequence in X , then $G(x_m, x_n, x_l) \rightarrow 0$ as $m, n, l \rightarrow \infty$.*

Remark 2.10 ([10]). *If B is a real Banach space with cone K and if $a \leq \lambda a$, where $a \in K$ and $0 < \lambda < 1$, then $a = 0$.*

Definition 2.11. Let f and g be self mappings of a set X . If $w = fx = gx$ for some x in X , then x is called a *coincidence point* of f and g and w is called a point of coincidence of f and g .

Definition 2.12. The self-mappings f and g of a set X are said to be *weakly compatible* if they commute at their coincidence points, that is, if $fu = gu$ for some $u \in X$, then $fgu = gfu$.

Proposition 2.13 ([12]). *Let f and g be weakly compatible self mappings of a set X . If f and g have a unique point of coincidence, that is, $w = fx = gx$, then w is the unique common fixed point of f and g .*

3 Main Results

Theorem 3.1. *Let (X, G) be a G -cone metric space. Suppose that the mappings $f, g, T : X \rightarrow X$ satisfy the following conditions:*

i.

$$G(Tx, fy, fz) \leq aG(gx, gy, gz) + bG(gx, Tx, Tx) + cG(gy, fy, fy) + dG(gz, fz, fz) \tag{3.1}$$

for all $x, y, z \in X$, where $a + b + c + d < 1$.

ii. $TX \cup fX \subset gX$ and

iii. gX is G -complete subspace of X .

Then f, T and g have a unique point of coincidence. Moreover,

iv. If the pairs $\{T, g\}$ and $\{f, g\}$ are weakly compatible, then f, T and g have a unique common fixed point.

Proof. Assume that x_0 be an arbitrary element in X and choose a point x_1 in X such that $gx_1 = Tx_0$. Similarly, choose a point x_2 in X such that $gx_2 = fx_1$. Continuing this process having chosen x_n in X , we obtain x_{n+1} in X such that

$$\begin{aligned} gx_{2k+1} &= Tx_{2k} \\ gx_{2k+2} &= fx_{2k+1} \end{aligned} \tag{3.2}$$

for $k \geq 0$, this can be done by property (ii). Since the equation (3.1) holds we have

$$\begin{aligned} G(gx_{2k+1}, gx_{2k+2}, gx_{2k+2}) &= G(Tx_{2k}, fx_{2k+1}, fx_{2k+1}) \\ &\leq aG(gx_{2k}, gx_{2k+1}, gx_{2k+1}) + bG(gx_{2k}, Tx_{2k}, Tx_{2k}) \\ &\quad + cG(gx_{2k+1}, fx_{2k+1}, fx_{2k+1}) + dG(gx_{2k+1}, fx_{2k+1}, fx_{2k+1}) \end{aligned}$$

that is,

$$\begin{aligned} G(gx_{2k+1}, gx_{2k+2}, gx_{2k+2}) &\leq aG(gx_{2k}, gx_{2k+1}, gx_{2k+1}) + bG(gx_{2k}, gx_{2k+1}, gx_{2k+1}) \\ &\quad + cG(gx_{2k+1}, gx_{2k+2}, gx_{2k+2}) + dG(gx_{2k+1}, gx_{2k+2}, gx_{2k+2}), \\ G(gx_{2k+1}, gx_{2k+2}, gx_{2k+2}) &\leq \frac{a+b}{(1-(c+d))} G(gx_{2k}, gx_{2k+1}, gx_{2k+1}) \end{aligned}$$

which implies that,

$$G(gx_{2k+1}, gx_{2k+2}, gx_{2k+2}) \leq \lambda G(gx_{2k}, gx_{2k+1}, gx_{2k+1})$$

where $0 \leq \lambda = \frac{a+b}{1-(c+d)} < 1$. Now for all $n \in N$, we have

$$G(gx_n, gx_{n+1}, gx_{n+1}) \leq \lambda G(gx_{n-1}, gx_n, gx_n) \leq \dots \leq \lambda^n G(gx_0, gx_1, gx_1).$$

For $m > n$,

$$\begin{aligned} G(gx_n, gx_m, gx_m) &\leq G(gx_n, gx_{n+1}, gx_{n+1}) + G(gx_{n+1}, gx_{n+2}, gx_{n+2}) \\ &\quad + \cdots + G(gx_{m-1}, gx_m, gx_m) \\ &\leq (\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1}) G(gx_0, gx_1, gx_1) \\ &\leq \frac{\lambda^n}{1-\lambda} G(gx_0, gx_1, gx_1). \end{aligned}$$

Let $0 \ll \varepsilon$ be given. Choose a natural number n_0 such that

$$\frac{\lambda^n}{1-\lambda} G(gx_0, gx_1, gx_1) \ll \varepsilon$$

for all $n \geq n_0$. Thus,

$$G(gx_n, gx_m, gx_m) \ll \varepsilon$$

for $m > n \geq n_0$. Therefore $\{gx_n\}$ is a G -Cauchy sequence. By property (iii) we know that gX is G -complete subspace of X then there exists a point v in gX such that $gx_n \rightarrow v$ as $n \rightarrow \infty$. Consequently, there is an u in X such that $gu = v$. We claim that, $gu = fu$. Let choose a natural number n_1 such that

$$\begin{aligned} G(gu, gx_{2n+1}, gx_{2n+1}) &\ll \frac{\varepsilon}{3} (1 - (c + d)), \\ G(gx_{2n}, gu, gu) &\ll \frac{\varepsilon}{3a} (1 - (c + d)), \\ G(gx_{2n}, gx_{2n+1}, gx_{2n+1}) &\ll \frac{\varepsilon}{3b} (1 - (c + d)), \end{aligned}$$

for all $n \geq n_1$. Hence

$$\begin{aligned} G(gu, fu, fu) &\leq G(gu, gx_{2n+1}, gx_{2n+1}) + G(gx_{2n+1}, fu, fu) \\ &= G(gu, gx_{2n+1}, gx_{2n+1}) + G(Tx_{2n}, fu, fu) \end{aligned}$$

by using inequality (3.1) we obtain that

$$\begin{aligned} G(gu, fu, fu) &\leq G(gu, gx_{2n+1}, gx_{2n+1}) + aG(gx_{2n}, gu, gu) \\ &\quad + bG(gx_{2n}, Tx_{2n}, Tx_{2n}) + cG(gu, fu, fu) + dG(gu, fu, fu) \\ &= G(gu, gx_{2n+1}, gx_{2n+1}) + aG(gx_{2n}, gu, gu) \\ &\quad + bG(gx_{2n}, gx_{2n+1}, gx_{2n+1}) + (c + d)G(gu, fu, fu) \end{aligned}$$

which implies that

$$\begin{aligned} G(gu, fu, fu) &\leq \frac{1}{1-(c+d)} G(gu, gx_{2n+1}, gx_{2n+1}) + \frac{a}{1-(c+d)} G(gx_{2n}, gu, gu) \\ &\quad + \frac{b}{1-(c+d)} G(gx_{2n}, gx_{2n+1}, gx_{2n+1}) \\ &\ll \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

for all $n \geq n_1$. Thus, $G(gu, fu, fu) \ll \frac{\varepsilon}{m}$ for all $m \geq 1$. So $\frac{\varepsilon}{m} - G(gu, fu, fu) \in \text{int}K \subset K$ for all $m \geq 1$. Since $\frac{\varepsilon}{m} \rightarrow 0$ as $m \rightarrow \infty$ and K is closed, $-G(gu, fu, fu) \in K$. But $G(gu, fu, fu) \in K$. Therefore, $G(gu, fu, fu) = 0$. Hence $gu = fu = v$. Similarly to show $gu = Tu$ we have $G(Tu, gu, gu) = G(Tu, fu, fu)$ and using (3.1),

$$\begin{aligned} G(Tu, fu, fu) &\leq aG(gu, gu, gu) + bG(gu, Tu, Tu) + cG(gu, fu, fu) \\ &\quad + dG(gu, fu, fu) \\ &\leq bG(fu, Tu, Tu) = bG(Tu, Tu, fu) \\ &\leq b[G(Tu, fu, fu) + G(fu, Tu, fu)], \\ (1 - 2b)G(Tu, fu, fu) &\leq 0 \end{aligned}$$

and this implies that $G(Tu, fu, fu) = 0$ that is, $fu = gu$ and $fu = Tu$. So,

$$gu = fu = Tu = v$$

this shows us that f, T and g have a common point of coincidence in X .

Now, we show that f, T and g have a unique common point of coincidence in X . For this purpose, assume that there exists another coincidence point v^* in X such that

$$gu^* = fu^* = Tu^* = v^*$$

for some u^* in X . Now,

$$\begin{aligned} G(v^*, v, v) &= G(Tv^*, fv, fv) \\ &\leq aG(gv^*, gv, gv) + bG(gv^*, Tv^*, Tv^*) + (c + d)G(gv, fv, fv) \end{aligned}$$

therefore, the result $v = v^*$ can be seen easily. Since the property (iv) holds then we have

$$gv = gfu = fgu = fv, Tv = Tgu = gTu = gv.$$

It implies that, $Tv = gv = fv = w$. Hence, w is a point of coincidence of f, T and g , so $w = v$ by uniqueness. By using Proposition 2.13 we obtain that v is the unique common fixed point of f, T and g . \square

Theorem 3.2. *Let (X, G) be a G -cone metric space and the mappings $f, g : X \rightarrow X$ satisfy:*

i.

$$\begin{aligned} G(fx, fy, fz) &\leq a\{G(fx, gy, gz) + G(gx, fy, fz) + G(fx, gx, gx) \\ &\quad + G(fy, gy, gy)\} \end{aligned} \tag{3.3}$$

for all $x, y, z \in X$, where $a \in [0, 1/4)$ is constant,

ii. $fX \subset gX$ and

iii. gX is G -complete subspace of X .

Then f and g have a unique point of coincidence. Moreover,

iv. If $\{f, g\}$ is weakly compatible pair then f and g have a unique common fixed point.

Proof. By using the same method as in Theorem 3.1 we can obtain the proof. \square

We now give an example to illustrate Theorem 3.2.

Example 3.3. Let $E = \mathbb{R}$ and $K = \{x \in E : x \geq 0\} \subset E$ be a cone. Let $X = [0, 1]$ and $G : X \times X \times X \rightarrow E$ be defined by $G(x, y, z) = d(x, y) + d(y, z) + d(z, x)$, where $d(x, y) = |x - y|$. Then (X, G) is a G -cone metric space. Define

$$\begin{aligned} f : X &\rightarrow X & g : X &\rightarrow X \\ x &\rightarrow fx = \frac{x}{16} & x &\rightarrow gx = \frac{x}{2} \end{aligned}$$

for all $x \in X$. Then

- (1) f and g are weakly compatible,
- (2) $fX \subset gX$ and gX is a G -complete subspace of X ,
- (3)

$$\begin{aligned} G(fx, fy, fz) &= d(fx, fy) + d(fy, fz) + d(fz, fx) \\ &= \frac{1}{16} \{|x - y| + |y - z| + |z - x|\}. \end{aligned}$$

From this equation we obtain the following

$$\begin{aligned} \frac{1}{16} \{|x - y| + |y - z| + |z - x|\} &= \frac{1}{8} \left\{ \left| \frac{x}{2} - \frac{y}{2} \right| + \left| \frac{y}{2} - \frac{z}{2} \right| + \left| \frac{z}{2} - \frac{x}{2} \right| \right\} \\ &\leq \frac{1}{8} \left\{ \left| \frac{x}{2} - \frac{y}{8} \right| + \left| \frac{y}{8} - \frac{y}{2} \right| + \left| \frac{y}{2} - \frac{z}{2} \right| + \left| \frac{z}{2} - \frac{x}{8} \right| \right. \\ &\quad \left. + \left| \frac{x}{8} - \frac{x}{2} \right| + \left| \frac{x}{2} - \frac{x}{8} \right| + \left| \frac{x}{8} - \frac{y}{2} \right| + \left| \frac{y}{2} - \frac{y}{2} \right| \right. \\ &\quad \left. + \left| \frac{y}{2} - \frac{y}{8} \right| + \left| \frac{y}{8} - \frac{z}{8} \right| + \left| \frac{z}{8} - \frac{x}{2} \right| + \left| \frac{x}{2} - \frac{x}{2} \right| \right\} \end{aligned}$$

that is,

$$\begin{aligned} G(fx, fy, fz) &\leq \frac{1}{8} \{G(fx, gy, gz) + Gg(fx, fy, fz) + G(fx, gx, gx) \\ &\quad + G(fy, gy, gy)\}, \end{aligned}$$

- (4) $f0 = g0 = 0$.

Since Theorem 3.2 holds, f and g have a unique common fixed point which is 0.

Now, we further improve Theorem 3.2 as follows.

Theorem 3.4. *Let (X, G) be a G -cone metric space. Suppose that the mappings $T, f, g : X \rightarrow X$ satisfy the following contractive condition:*

$$G(Tx, fy, fz) \leq a \{G(Tx, gy, gz) + G(gx, fy, fz) + G(Tx, gx, gx) + G(fy, gy, gy)\} \tag{3.4}$$

for all $x, y, z \in X$, where $a \in [0, 1/4)$ is constant. If properties (ii) and (iii) in Theorem 3.1 hold, then f, T and g have a unique point of coincidence. Moreover, if property (iv) in Theorem 3.1 holds then f, T and g have a unique common fixed point.

Proof. By using the same technique as in Theorem 3.1 we get the proof. □

Theorem 3.5. *Let (X, G) be a G - cone metric space. Suppose that the mappings $f, g, T : X \rightarrow X$ satisfy the following contractive condition:*

$$G(Tx, fy, fy) \leq a \{G(gx, fy, fy) + G(gy, Tx, Tx)\} \tag{3.5}$$

for all $x, y, \in X$, where $a \in [0, 1/2)$. If properties (ii) and (iii) in Theorem 3.1 hold, then f, T and g have a unique point of coincidence. Moreover, if property (iv) in Theorem 3.1 holds, then f, T and g have a unique common fixed point.

Proof. With the similar proof as in Theorem 3.1 we obtain the desired result. □

Example 3.6. *Let $E = R^2$ and $K = \{(x, y) \in R^2 : x, y \geq 0\}$ be a cone. Let $X = [0, 1]$ and $G : X \times X \times X \rightarrow E$ be defined by $G(x, y, z) = d(x, y) + d(y, z) + d(z, x)$, where $d(x, y) = (|x - y|, \alpha |x - y|)$. (X, G) is a G -cone metric space. Define*

$$\begin{array}{lll} f : X \rightarrow X & g : X \rightarrow X & T : X \rightarrow X \\ x \rightarrow fx = \frac{x}{8} & x \rightarrow gx = \frac{x}{2} & x \rightarrow Tx = \frac{x}{4}. \end{array}$$

Then,

- (1) $\{T, g\}$ and $\{f, g\}$ are weakly compatible pairs,
- (2) $TX \cup fX \subset gX$, and
- (3)

$$\begin{aligned} G(Tx, fy, fy) &= d(Tx, fy) + d(fy, fy) + d(fy, Tx) \\ &= (2|Tx - fy|, 2\alpha|Tx - fy|) \\ &= \frac{1}{4}(|2x - y|, \alpha|2x - y|) \end{aligned}$$

and

$$\begin{aligned} G(gx, fy, fy) &= d(gx, fy) + d(fy, fy) + d(fy, gx) \\ &= (2|gx - fy|, 2\alpha|gx - fy|) \\ &= \frac{1}{4}(|4x - y|, \alpha|4x - y|), \end{aligned}$$

$$\begin{aligned}
 G(gy, Tx, Tx) &= d(gy, Tx) + d(Tx, Tx) + d(Tx, gy) \\
 &= (2|gy - Tx|, 2\alpha|gy - Tx|) \\
 &= \frac{1}{4}(|4y - 2x|, \alpha|4y - 2x|),
 \end{aligned}$$

$$\begin{aligned}
 G(gx, fy, fy) + G(gy, Tx, Tx) \\
 = \frac{1}{4}((|4x - y| + |4y - 2x|), \alpha(|4x - y| + |4y - 2x|)).
 \end{aligned}$$

Now,

$$\begin{aligned}
 G(Tx, fy, fy) &= \frac{1}{4}(|2x - y|, \alpha|2x - y|) \\
 &\leq \frac{1}{4}(|6x - y|, \alpha|6x - y|) \\
 &\leq \frac{1}{4}(|y - 2x - 4x + 4y|, \alpha|y - 2x - 4x + 4y|) \\
 &\leq \frac{1}{4}(|4x - y| + |4y - 2x|, \alpha|4x - y| + |4y - 2x|),
 \end{aligned}$$

$$G(Tx, fy, fy) \leq \frac{1}{4}(G(gx, fy, fy) + G(gy, Tx, Tx))$$

where $a = \frac{1}{4} \in [0, 1/2)$.

$$(4) T0 = f0 = g0 = 0.$$

Since all the conditions of Theorem 3.5 hold, we can see that T , f and g have a common fixed point.

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