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A New Three-Step Mean Value Iterations with Errors for Asymptotically Nonexpansive Mappings in Banach Spaces

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Abstract : In this paper, we define and study a new three-step iterative scheme inspired by Nilsrakoo and Saejung [Appl. Math. Comput. 181 (2006) 1026–1034]. Many well-known iterations, for examples, Mann-type, Ishikawa-type and Noor-type iterations are covered by the new iterative scheme. Several convergence theorems of this scheme are established for asymptotically nonexpansive mappings in Banach spaces. Our results extend and improve the recent ones announced by Schu [J. Math. Anal. Appl. 158 (1991) 407–413; Bull. Aust. Math. Soc. 43 (1991) 153–159], Xu and Noor [J. Math. Anal. Appl. 267 (2002) 444–453], Suantai [J. Math. Anal. Appl. 311 (2005) 506–517], Nilsrakoo and Saejung [Appl. Math. Comput. 181 (2006) 1026–1034; Appl. Math. Comput. 190 (2007) 1472–1478] and many others.

Keywords : Asymptotically nonexpansive mapping; Uniformly convex Banach space; Mann-type iteration; Ishikawa-type iteration; Three-step iteration.

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1 Introduction

The concept of asymptotically nonexpansiveness was introduced by Goebel and Kirk [1] in 1972. In 2001, Noor [2, 3] have introduced the three-step iterations and studied the approximate solutions of variational inclusion and variational inequalities in Hilbert spaces. Glowinski and Le Tallec [4] used three-step iterative schemes to find the approximate solutions of the elastoviscoplasticity problem, liquid crystal theory and eigenvalue computation. It has been shown in [4] that the three-step iterative scheme gives better numerical results than the two-step and one-step approximate iterations. In 1998, Haubruge et al. [5] studied the convergence analysis of three-step schemes of Glowinski and Le Tallec [4] and applied these schemes to obtain new splitting-type algorithms for solving variation inequalities, separable convex programming and minimization of a sum of convex functions. They also proved that three-step iterations lead to highly parallelized algorithms under certain conditions. Thus we conclude that three-step scheme plays an important and significant part in solving various problems which arise in pure and applied sciences.

In 2002, Xu and Noor [6] introduced and studied a three-step scheme to approximate fixed points of asymptotically nonexpansive mappings. In 2005, Suantai [7] extended their scheme to the modified Noor iterative scheme. Recently, Nilsrakoo and Saejung [8, 9] defined and studied a new three-step mean value iterations to approximate fixed points of asymptotically nonexpansive mappings which is an extension of Suantai's iterative scheme. See [10] and references therein for non-self asymptotically nonexpansive mappings.

Inspired and motivated by these facts, we introduce and study a new iterative scheme with errors for asymptotically nonexpansive mappings. Our results include the Ishikawa, Mann and Noor iterative schemes for solving variational inclusions (inequalities) as spacial cases. The scheme is defined as follows.

Let X be a normed space, C be a nonempty convex subset of X and $T : C \to C$ be a given mapping.

Algorithm 1. For a given $x_1 \in C$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the iterative schemes

$$\begin{aligned} z_n &= \alpha_n^{(1)} T^n x_n + \alpha_n^{(2)} x_n + \lambda_n u_n, \\ y_n &= \beta_n^{(1)} T^n z_n + \beta_n^{(2)} T^n x_n + \beta_n^{(3)} T^{2n} x_n + \beta_n^{(4)} x_n + \mu_n v_n, \qquad n \geq 1, \quad (1.1) \\ x_{n+1} &= \gamma_n^{(1)} T^n y_n + \gamma_n^{(2)} T^n z_n + \gamma_n^{(3)} T^n x_n + \gamma_n^{(4)} T^{2n} z_n + \gamma_n^{(5)} x_n + \nu_n w_n, \end{aligned}$$

where $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are bounded sequences in C and $\{\lambda_n\}$, $\{\mu_n\}$, $\{\nu_n\}$, $\{\alpha_n^{(1)}\}$, $\{\alpha_n^{(2)}\}$, $\{\beta_n^{(1)}\}, \ldots, \{\beta_n^{(4)}\}$,$\{\gamma_n^{(1)}\}, \ldots, \{\gamma_n^{(5)}\}$ are appropriate real sequences in $[0, 1]$ such that $\lambda_n + \alpha_n^{(1)} + \alpha_n^{(2)} = \mu_n + \sum_{i=1}^{4} \beta_n^{(i)} = \nu_n + \sum_{j=1}^{5} \gamma_n^{(j)} = 1$.

If $\lambda_n = \mu_n = \nu_n = \beta_n^{(3)} = \gamma_n^{(4)} \equiv 0$, $\alpha_n^{(2)} = 1 - \alpha_n^{(1)}$, $\beta_n^{(4)} = 1 - \beta_n^{(1)} - \beta_n^{(2)}$ and $\gamma_n^{(5)} = 1 - \gamma_n^{(1)} - \gamma_n^{(2)} - \gamma_n^{(3)}$, then Algorithm 1 reduces to

Algorithm 2. For a given $x_1 \in C$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the iterative schemes

$$\begin{aligned} z_n &= \alpha_n^{(1)} T^n x_n + (1-\alpha_n^{(1)}) x_n, \\ y_n &= \beta_n^{(1)} T^n z_n + \beta_n^{(2)} T^n x_n + (1-\beta_n^{(1)}-\beta_n^{(2)}) x_n, \quad n \geq 1, \\ x_{n+1} &= \gamma_n^{(1)} T^n y_n + \gamma_n^{(2)} T^n z_n + \gamma_n^{(3)} T^n x_n + (1-\gamma_n^{(1)}-\gamma_n^{(2)}-\gamma_n^{(3)}) x_n, \end{aligned} \tag{1.2}$$

where $\{\alpha_n^{(1)}\}$, $\{\beta_n^{(1)}\}$, $\{\beta_n^{(2)}\}$, $\{\beta_n^{(1)}+\beta_n^{(2)}\}$, $\{\gamma_n^{(1)}\}$, $\{\gamma_n^{(2)}\}$, $\{\gamma_n^{(3)}\}$ and $\{\gamma_n^{(1)}+\gamma_n^{(2)}+\gamma_n^{(3)}\}$ are appropriate real sequences in $[0,1]$. The iterative scheme (1.2) is called the *three-step mean value iterative scheme* defined by Nilsrakoo and Saejung [8, 9].

If $\lambda_n = \mu_n = \nu_n = \beta_n^{(3)} = \gamma_n^{(3)} = \gamma_n^{(4)} \equiv 0$, $\alpha_n^{(2)} = 1-\alpha_n^{(1)}$, $\beta_n^{(4)} = 1-\beta_n^{(1)}-\beta_n^{(2)}$ and $\gamma_n^{(5)} = 1-\gamma_n^{(1)}-\gamma_n^{(2)}$, then Algorithm 1 reduces to

Algorithm 3. For a given $x_1 \in C$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the iterative schemes

$$\begin{aligned} z_n &= \alpha_n^{(1)} T^n x_n + (1-\alpha_n^{(1)}) x_n, \\ y_n &= \beta_n^{(1)} T^n z_n + \beta_n^{(2)} T^n x_n + (1-\beta_n^{(1)}-\beta_n^{(2)}) x_n, \quad n \geq 1, \\ x_{n+1} &= \gamma_n^{(1)} T^n y_n + \gamma_n^{(2)} T^n z_n + (1-\gamma_n^{(1)}-\gamma_n^{(2)}) x_n, \end{aligned} \tag{1.3}$$

where $\{\alpha_n^{(1)}\}$, $\{\beta_n^{(1)}\}$, $\{\beta_n^{(2)}\}$, $\{\beta_n^{(1)}+\beta_n^{(2)}\}$, $\{\gamma_n^{(1)}\}$, $\{\gamma_n^{(2)}\}$ and $\{\gamma_n^{(1)}+\gamma_n^{(2)}\}$ are appropriate real sequences in $[0,1]$. The iterative scheme (1.3) is called the *modified Noor iterations* defined by Suantai [7].

The purpose of this paper is to establish weak and strong convergence theorems of iterative scheme (1.1) for asymptotically nonexpansive mappings in a uniformly convex Banach space. The results presented in this paper extend and improve the corresponding ones announced by Xu and Noor [6], Suantai [7], Nilsrakoo and Saejung [8, 9] and many others.

2 Preliminaries

In this section, we recall the well-known concepts and results. For convenience, we use the notations $\lim_n \equiv \lim_{n\to\infty}$, $\liminf_n \equiv \liminf_{n\to\infty}$ and $\limsup_n \equiv \limsup_{n\to\infty}$. Let C be a nonempty subset of normed space X. A mapping $T : C \to C$ is said to be *asymptotically nonexpansive* if there exists a sequence $\{r_n\}$ in $[0,\infty)$ with $\lim_n r_n = 0$ such that

$$\|T^n x - T^n y\| \leq (1+r_n)\|x-y\|$$

for all $x, y \in C$ and each $n \geq 1$. By passing to the sequence $\{r_n'\}$, we may always assume that $\{r_n\}$ is decreasing, where $r_n' = \sup_{m\geq n} r_m$. If $r_n \equiv 0$, then T is known as a *nonexpansive* mapping.

The mapping T is said to be *uniformly L-Lipschitzian* if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x - y\|$$

for all $x, y \in C$ and each $n \geq 1$. It is easy to see that if T is asymptotically nonexpansive, then it is uniformly L-Lipschitzian with the uniform Lipschitz constant $L = \sup\{1 + r_n : n \geq 1\}$. It is known [1] that if X is a uniformly convex Banach space and T is an asymptotically nonexpansive self-mapping of a nonempty bounded closed convex subset C of X, then $F(T) \neq \emptyset$ where $F(T)$ denotes the set of all fixed points of T.

The mapping $T : C \to C$ with $F(T) \neq \emptyset$ is said to satisfy Condition (A) with respect to the sequence $\{x_n\}$ [11] if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$f(d(x_n, F(T))) \leq \|x_n - Tx_n\|$$

for each $n \geq 1$ where $d(x, F(T)) = \inf\{\|x - y\| : y \in F(T)\}$.

The following example provides an example of asymptotically nonexpansive mapping satisfying Condition (A) which is not nonexpansive.

Example 2.1. *Let $X = \mathbb{R}$, $C = [0, 1]$ and define*

$$T(x) = \begin{cases} 0 & \text{if } x \in [0, 1/2], \\ (3/2)x - 3/4 & \text{if } x \in (1/2, 1]. \end{cases}$$

It is easy to verify that $F(T) = \{0\}$, $T^n x = 0$ for all $x \in C$ and each $n \geq 3$. Therefore, T is asymptotically nonexpansive with $r_1 = 1/2$, $r_2 = 5/4$ and $r_n = 0$ for all $n \geq 3$, but T is not nonexpansive. Also, T satisfies Condition (A) with the function $f(t) = t/4$. Note that $d(x, F(T)) = x$ and $\|x - Tx\| = x - Tx \geq x - (3/4)x = (1/4)x$ for all $x \in C$.

It is well known that every completely continuous mapping satisfies Condition (A) [11]. Thus we shall use Condition (A) instead the complete continuity of the mapping T to study the strong convergence of $\{x_n\}$ defined in (1.1).

In the sequel, the following lemmas are needed to prove our main results.

Lemma 2.2 ([12, Lemma 1])**.** *Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n, \qquad \forall n \geq 1.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_n a_n$ exists.

By Schu's lemma [13, Schu's Lemma], we have the following lemma.

Lemma 2.3 ([8, Lemma 5])**.** *Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences in a uniformly convex Banach space X with $\limsup_n \|x_n\| \leq a$, $\limsup_n \|y_n\| \leq a$ and $\limsup_n \|z_n\| \leq a$ for some $a \geq 0$. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $[0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \geq 1$ and $\lim_n \|\alpha_n x_n + \beta_n y_n + \gamma_n z_n\| = a$. If $\liminf_n \alpha_n > 0$ and $\liminf_n \beta_n > 0$, then $\lim_n \|x_n - y_n\| = 0$.*

By using mathematical induction in Lemma 2.3, we immediately obtain the following.

Lemma 2.4. *Let $k \geq 2$ and $\{x_n^{(1)}\}, \ldots, \{x_n^{(k)}\}$ be sequences in a uniformly convex Banach space X with $\limsup_n \|x_n^{(i)}\| \leq a$ for each $i \in \{1, 2, \ldots, k\}$ and for some $a \geq 0$. Suppose $\{\alpha_n^{(1)}\}, \ldots, \{\alpha_n^{(k)}\}$ be sequences in $[0,1]$ such that $\sum_{i=1}^k \alpha_n^{(i)} = 1$ and $\lim_n \|\sum_{i=1}^k \alpha_n^{(i)} x_n^{(i)}\| = a$. If $\liminf_n \alpha_n^{(i)} > 0$ and $\liminf_n \alpha_n^{(j)} > 0$, then $\lim_n \|x_n^{(i)} - x_n^{(j)}\| = 0$.*

3 Main Results

In this section, we establish weak and strong convergence theorems of iterative scheme (1.1) for asymptotically nonexpansive mappings. Note that the proof given below is different from that of Xu and Noor, Suantai, Nilsrakoo and Saejung. Throughout this section, we assume that $\sum_{n=1}^\infty \lambda_n < \infty$, $\sum_{n=1}^\infty \mu_n < \infty$ and $\sum_{n=1}^\infty \nu_n < \infty$. In order to prove our main result, the following lemma is needed.

Lemma 3.1. *Let X be a real Banach space, C be a nonempty convex subset of X and $T : C \to C$ be an asymptotically nonexpansive mapping with the nonempty fixed point set $F(T)$ and the sequence $\{r_n\}$ such that $\sum_{n=1}^\infty r_n < \infty$. Let $\{x_n\}$ be the sequence defined by Algorithm 1. Then we have the following conclusions.*

(i) $\lim_n \|x_n - p\|$ *exists for all* $p \in F(T)$.

(ii) $\lim_n d(x_n, F(T))$ *exists.*

(iii) *If* $\liminf_n \gamma_n^{(1)} > 0$, *then* $\lim_n \|y_n - p\| = \lim_n \|x_n - p\|$ *for all* $p \in F(T)$.

(iv) *If* $\liminf_n(\gamma_n^{(1)}\beta_n^{(1)} + \gamma_n^{(2)} + \gamma_n^{(4)}) > 0$, *then* $\lim_n \|z_n - p\| = \lim_n \|x_n - p\|$ *for all* $p \in F(T)$.

(v) *If* $\lim_n \|T^n x_n - x_n\| = 0$, *then* $\lim_n \|Tx_n - x_n\| = 0$.

Proof. (i) Let $p \in F(T)$. For each $n \geq 1$, we note that

$$\begin{aligned}
\|z_n - p\| &= \left\|\alpha_n^{(1)} T^n x_n + \alpha_n^{(2)} x_n + \lambda_n u_n - p\right\| \\
&\leq \alpha_n^{(1)} \|T^n x_n - p\| + \alpha_n^{(2)} \|x_n - p\| + \lambda_n \|u_n - p\| \\
&\leq \alpha_n^{(1)} (1 + r_n) \|x_n - p\| + \alpha_n^{(2)} (1 + r_n) \|x_n - p\| + \lambda_n \|u_n - p\| \\
&\leq (1 + r_n) \|x_n - p\| + \lambda_n \|u_n - p\| \\
&= (1 + r_n) \|x_n - p\| + a_n, \qquad (3.1)
\end{aligned}$$

where $a_n = \lambda_n \|u_n - p\|$. Since $\{u_n\}$ is bounded and $\sum_{n=1}^\infty \lambda_n < \infty$, we see that

$\sum_{n=1}^{\infty} a_n < \infty$. It follows from (3.1) that

$$\begin{aligned}
\|y_n - p\| &\le \beta_n^{(1)}\|T^n z_n - p\| + \beta_n^{(2)}\|T^n x_n - p\| + \beta_n^{(3)}\|T^{2n} x_n - p\| \\
&\quad + \beta_n^{(4)}\|x_n - p\| + \mu_n\|v_n - p\| \\
&\le \beta_n^{(1)}(1 + r_n)\|z_n - p\| \\
&\quad + \left(\beta_n^{(2)}(1 + r_n) + \beta_n^{(3)}(1 + r_{2n}) + \beta_n^{(4)}\right)\|x_n - p\| + \mu_n\|v_n - p\| \\
&\le \beta_n^{(1)}(1 + r_n)((1 + r_n)\|x_n - p\| + a_n) \\
&\quad + (1 + r_n)\left(\beta_n^{(2)} + \beta_n^{(3)} + \beta_n^{(4)}\right)\|x_n - p\| + \mu_n\|v_n - p\| \\
&\le (1 + r_n)^2\left(\beta_n^{(1)} + \beta_n^{(2)} + \beta_n^{(3)} + \beta_n^{(4)}\right)\|x_n - p\| + (1 + r_n)a_n \\
&\quad + \mu_n\|v_n - p\| \\
&\le (1 + r_n)^2\|x_n - p\| + b_n, \qquad (3.2)
\end{aligned}$$

where $b_n = (1 + r_n)a_n + \mu_n\|v_n - p\|$. Since $\{v_n\}$ is bounded, $\sum_{n=1}^{\infty} \mu_n < \infty$ and $\sum_{n=1}^{\infty} a_n < \infty$, we have $\sum_{n=1}^{\infty} b_n < \infty$. Moreover, we see that

$$\begin{aligned}
\|x_{n+1} - p\| &\le \gamma_n^{(1)}(1 + r_n)\|y_n - p\| + \left(\gamma_n^{(2)}(1 + r_n) + \gamma_n^{(4)}(1 + r_{2n})\right)\|z_n - p\| \\
&\quad + \left(\gamma_n^{(3)}(1 + r_n) + \gamma_n^{(5)}\right)\|x_n - p\| + \nu_n\|w_n - p\| \\
&\le \gamma_n^{(1)}(1 + r_n)\left((1 + r_n)^2\|x_n - p\| + b_n\right) \\
&\quad + (1 + r_n)^2\left(\gamma_n^{(2)} + \gamma_n^{(4)}\right)((1 + r_n)\|x_n - p\| + a_n) \\
&\quad + (1 + r_n)^3\left(\gamma_n^{(3)} + \gamma_n^{(5)}\right)\|x_n - p\| + \nu_n\|w_n - p\| \\
&\le (1 + r_n)^3\left(\gamma_n^{(1)} + \cdots + \gamma_n^{(5)}\right)\|x_n - p\| + (1 + r_n)^2(a_n + b_n) \\
&\quad + \nu_n\|w_n - p\| \\
&\le (1 + r_n)^3\|x_n - p\| + c_n, \qquad (3.3)
\end{aligned}$$

where $c_n = (1 + r_n)^2(a_n + b_n) + \nu_n\|w_n - p\|$, so that $\sum_{n=1}^{\infty} c_n < \infty$. By Lemma 2.2, we get $\lim_n \|x_n - p\|$ exists.

(ii) It follows from (i) that $\{x_n\}$ is bounded. Using (3.1), we have

$$\begin{aligned}
\|z_n - p\| &\le (1 + r_n)\left(\alpha_n^{(1)} + \alpha_n^{(2)}\right)\|x_n - p\| + \lambda_n\|u_n - p\| \\
&\le (1 + r_n)\left(\alpha_n^{(1)} + \alpha_n^{(2)}\right)\|x_n - p\| + (1 + r_n)\lambda_n(\|u_n - x_n\| + \|x_n - p\|) \\
&= (1 + r_n)\|x_n - p\| + a'_n \qquad (3.4)
\end{aligned}$$

for all $p \in F(T)$, where $a'_n = (1 + r_n)\lambda_n\|u_n - x_n\|$. Since $\{u_n\}$ and $\{x_n\}$ are bounded, $\lim_n r_n = 0$ and $\sum_{n=1}^{\infty} \lambda_n < \infty$, we see that $\sum_{n=1}^{\infty} a'_n < \infty$. Note that $\{a'_n\}$ does not depend on p. Again, by continuing this process, we may obtain a sequence of nonnegative real numbers $\{c'_n\}$ such that it is independent of p, $\sum_{n=1}^{\infty} c'_n < \infty$ and

$$\|x_{n+1} - p\| \leq (1 + r_n)^3 \|x_n - p\| + c'_n \tag{3.5}$$

for all $n \geq 1$ and $p \in F(T)$. Taking infimum over all p in $F(T)$, we obtain

$$d(x_{n+1}, F(T)) \leq (1 + r_n)^3 d(x_n, F(T)) + c'_n.$$

Again, by Lemma 2.2, we get $\lim_n d(x_n, F(T))$ exists.

(iii) Let $p \in F(T)$. Since $\lim_n \|x_n - p\|$ exists, it follows from (3.2) that $\limsup_n \|y_n - p\| \leq \lim_n \|x_n - p\|$. Also, by (3.2) and (3.3) we have

$$\begin{aligned}
\|x_{n+1} - p\| &\leq \gamma_n^{(1)}(1 + r_n)\|y_n - p\| + \left(\gamma_n^{(2)}(1 + r_n) + \gamma_n^{(4)}(1 + r_{2n})\right) \|z_n - p\| \\
&\quad + \left(\gamma_n^{(3)}(1 + r_n) + \gamma_n^{(5)}\right) \|x_n - p\| + \nu_n \|w_n - p\| \\
&\leq (1 + r_n) \left[\gamma_n^{(1)} \|y_n - p\| + \left(\gamma_n^{(2)} + \gamma_n^{(4)}\right) ((1 + r_n)\|x_n - p\| + a_n) \right. \\
&\quad \left. + \left(1 - \gamma_n^{(1)} - \gamma_n^{(2)} - \gamma_n^{(4)}\right) \|x_n - p\|\right] + \nu_n \|w_n - p\| \\
&\leq (1 + r_n)^2 \left[\gamma_n^{(1)} \|y_n - p\| + \left(1 - \gamma_n^{(1)}\right) \|x_n - p\|\right] + a''_n
\end{aligned}$$

for all $n \geq 1$, where $a''_n = (1 + r_n)a_n + \nu_n \|w_n - p\|$. Since $\liminf_n \gamma_n^{(1)} > 0$, then

$$\frac{\|x_{n+1} - p\| - (1 + r_n)^2 \|x_n - p\|}{\gamma_n^{(1)}(1 + r_n)^2} + \|x_n - p\| \leq \|y_n - p\| + \frac{a''_n}{\gamma_n^{(1)}(1 + r_n)^2}$$

for sufficiently large numbers n. By taking $\liminf_n$ in both sides, we obtain

$$\lim_n \|x_n - p\| \leq \liminf_n \|y_n - p\|.$$

(iv) It follows from (3.1) that $\limsup_n \|z_n - p\| \leq \lim_n \|x_n - p\|$. For convenience, we take $\beta_n = \gamma_n^{(1)} \beta_n^{(1)} + \gamma_n^{(2)} + \gamma_n^{(4)}$. Using (3.2) and (3.3), we obtain

$$\begin{aligned}
\|x_{n+1} - p\| &\leq \gamma_n^{(1)}(1 + r_n)\|y_n - p\| + \left(\gamma_n^{(2)}(1 + r_n) + \gamma_n^{(4)}(1 + r_{2n})\right) \|z_n - p\| \\
&\quad + \left(\gamma_n^{(3)}(1 + r_n) + \gamma_n^{(5)}\right) \|x_n - p\| + \nu_n \|w_n - p\| \\
&\leq \gamma_n^{(1)}(1 + r_n)^2 \left[\beta_n^{(1)} \|z_n - p\| + \left(\beta_n^{(2)} + \beta_n^{(3)} + \beta_n^{(4)}\right) \|x_n - p\|\right] \\
&\quad + (1 + r_n)\mu_n \|v_n - p\| + \left(\gamma_n^{(2)} + \gamma_n^{(4)}\right) (1 + r_n)^2 \|z_n - p\| \\
&\quad + \left(\gamma_n^{(3)} + \gamma_n^{(5)}\right) (1 + r_n)^2 \|x_n - p\| + \nu_n \|w_n - p\| \\
&\leq (1 + r_n)^2 \left(\left[\gamma_n^{(1)} \beta_n^{(1)} + \gamma_n^{(2)} + \gamma_n^{(4)}\right] \|z_n - p\| \right. \\
&\quad \left. + \left[\gamma_n^{(1)} \left(\beta_n^{(2)} + \beta_n^{(3)} + \beta_n^{(4)}\right) + \gamma_n^{(3)} + \gamma_n^{(5)}\right] \|x_n - p\|\right) + b''_n
\end{aligned}$$

$$\begin{aligned}&\leq (1+r_n)^2\left(\beta_n\|z_n-p\|+\left[\gamma_n^{(1)}\left(1-\beta_n^{(1)}\right)\right.\right.\\&\quad\left.\left.+\left(1-\gamma_n^{(1)}-\gamma_n^{(2)}-\gamma_n^{(4)}\right)\right]\|x_n-p\|\right)+b_n''\\&=(1+r_n)^2(\beta_n\|z_n-p\|+(1-\beta_n)\|x_n-p\|)+b_n''\end{aligned}$$

for all $n\geq 1$, where $b_n''=(1+r_n)\mu_n\|v_n-p\|+\nu_n\|w_n-p\|$. Since $\liminf_n\beta_n>0$, we have

$$\frac{\|x_{n+1}-p\|-(1+r_n)^2\|x_n-p\|}{\beta_n(1+r_n)^2}+\|x_n-p\|\leq\|z_n-p\|+\frac{b_n''}{\beta_n(1+r_n)^2}$$

for sufficiently large numbers n. By taking $\liminf_n$ in both sides, we get

$$\lim_n\|x_n-p\|\leq\liminf_n\|z_n-p\|.$$

(v) Using (1.1), we have

$$\|z_n-x_n\|\leq\alpha_n^{(1)}\|T^nx_n-x_n\|+\lambda_n\|u_n-x_n\|\to 0, \tag{3.6}$$

$$\begin{aligned}\|T^nz_n-x_n\|&\leq\|T^nz_n-T^nx_n\|+\|T^nx_n-x_n\|\\&\leq(1+r_n)\|z_n-x_n\|+\|T^nx_n-x_n\|\to 0,\end{aligned}$$

$$\begin{aligned}\|y_n-x_n\|&\leq\beta_n^{(1)}\|T^nz_n-x_n\|+\beta_n^{(2)}\|T^nx_n-x_n\|\\&\quad+\beta_n^{(3)}\|T^{2n}x_n-x_n\|+\mu_n\|v_n-x_n\|\\&\leq\beta_n^{(1)}\|T^nz_n-x_n\|+\left(\beta_n^{(2)}+\beta_n^{(3)}(2+r_n)\right)\|T^nx_n-x_n\|\\&\quad+\mu_n\|v_n-x_n\|\to 0,\end{aligned}\tag{3.7}$$

$$\begin{aligned}\|T^ny_n-x_n\|&\leq\|T^ny_n-T^nx_n\|+\|T^nx_n-x_n\|\\&\leq(1+r_n)\|y_n-x_n\|+\|T^nx_n-x_n\|\to 0\end{aligned}$$

and so

$$\begin{aligned}\|x_{n+1}-x_n\|&\leq\gamma_n^{(1)}\|T^ny_n-x_n\|+\left(\gamma_n^{(4)}(1+r_n)+\gamma_n^{(2)}\right)\|T^nz_n-x_n\|\\&\quad+\left(\gamma_n^{(3)}+\gamma_n^{(4)}\right)\|T^nx_n-x_n\|+\nu_n\|w_n-x_n\|\to 0.\end{aligned}\tag{3.8}$$

Also,

$$\begin{aligned}\|Tx_n-x_n\|&\leq\|x_{n+1}-x_n\|+\|T^{n+1}x_{n+1}-x_{n+1}\|+\|T^{n+1}x_n-T^{n+1}x_{n+1}\|\\&\quad+\|Tx_n-T^{n+1}x_n\|\\&\leq(2+r_{n+1})\|x_{n+1}-x_n\|+\|T^{n+1}x_{n+1}-x_{n+1}\|\\&\quad+(1+r_1)\|x_n-T^nx_n\|.\end{aligned}$$

This together with (3.8) implies that $\lim_n\|Tx_n-x_n\|=0$. This completes the proof. □

The following lemmas are the important ingredients for proving our main results.

Lemma 3.2. *Let X be a uniformly convex Banach space, C be a nonempty convex subset of X and $T : C \to C$ be an asymptotically nonexpansive mapping with the nonempty fixed point set $F(T)$ and the sequence $\{r_n\}$ such that $\sum_{n=1}^{\infty} r_n < \infty$. Let $\{x_n\}$ be the sequence defined by Algorithm 1 and $\gamma_n = \gamma_n^{(1)} + \gamma_n^{(2)} + \gamma_n^{(3)} + \gamma_n^{(4)}$ for all $n \geq 1$. Then we have the following conclusions.*

(i) *If* $0 < \liminf_n \gamma_n^{(1)} \leq \limsup_n \gamma_n < 1$, *then* $\lim_n \|T^n y_n - x_n\| = 0$.

(ii) *If* $0 < \liminf_n \gamma_n^{(2)} \leq \limsup_n \gamma_n < 1$, *then* $\lim_n \|T^n z_n - x_n\| = 0$.

(iii) *If* $0 < \liminf_n \gamma_n^{(3)} \leq \limsup_n \gamma_n < 1$, *then* $\lim_n \|T^n x_n - x_n\| = 0$.

(iv) *If* $\liminf_n \gamma_n^{(1)} > 0$ *and* $0 < \liminf_n \beta_n^{(1)} \leq \limsup_n(\beta_n^{(1)} + \beta_n^{(2)} + \beta_n^{(3)}) < 1$, *then* $\lim_n \|T^n z_n - x_n\| = 0$.

(v) *If* $\liminf_n(\gamma_n^{(1)}\beta_n^{(1)} + \gamma_n^{(2)} + \gamma_n^{(4)}) > 0$ *and* $0 < \liminf_n \alpha_n^{(1)} \leq \limsup_n \alpha_n^{(1)} < 1$, *then* $\lim_n \|T^n x_n - x_n\| = 0$.

Proof. Let $p \in F(T)$. It follows from Lemma 3.1(i) that $\lim_n \|x_n - p\|$ exists. Let $\lim_n \|x_n - p\| = a$ for some $a \geq 0$. Since $\sum_{n=1}^{\infty} \nu_n < \infty$ and $\|x_n - p + \nu_n(w_n - x_n)\| \leq \|x_n - p\| + \nu_n\|w_n - x_n\|$, we have

$$\limsup_n \|x_n - p + \nu_n(w_n - x_n)\| \leq a. \tag{3.9}$$

Also,

$$\limsup_n \|T^n x_n - p\| \leq \limsup_n (1 + r_n)\|x_n - p\| = \lim_n \|x_n - p\| = a.$$

Next, we observe that $\|T^n x_n - p + \nu_n(w_n - x_n)\| \leq \|T^n x_n - p\| + \nu_n\|w_n - x_n\|$. Thus,

$$\limsup_n \|T^n x_n - p + \nu_n(w_n - x_n)\| \leq a. \tag{3.10}$$

It follows from (3.2) that $\|y_n - p\| \leq (1 + r_n)^2\|x_n - p\| + b_n$ for all $n \geq 1$, where $\{b_n\}$ is a nonnegative real sequence such that $\sum_{n=1}^{\infty} b_n < \infty$. Taking $\limsup_n$ in both sides, we obtain

$$\limsup_n \|y_n - p\| \leq \limsup_n ((1 + r_n)^2\|x_n - p\| + b_n) = \lim_n \|x_n - p\| = a.$$

So that

$$\limsup_n \|T^n y_n - p\| \leq \limsup_n (1 + r_n)\|y_n - p\| = \limsup_n \|y_n - p\| \leq a.$$

Thus, we have

$$\limsup_n \|T^n y_n - p + \nu_n(w_n - x_n)\| \leq a. \tag{3.11}$$

By using (3.1) and the similar method as above, we have

$$\limsup_n \|T^n z_n - p + \nu_n(w_n - x_n)\| \le a \tag{3.12}$$

and

$$\limsup_n \|T^{2n} z_n - p + \nu_n(w_n - x_n)\| \le a. \tag{3.13}$$

Since $\gamma_n = \gamma_n^{(1)} + \gamma_n^{(2)} + \gamma_n^{(3)} + \gamma_n^{(4)}$, we note that

$$\begin{aligned}
a &= \lim_n \|x_{n+1} - p\| \\
&= \lim_n \left\|\gamma_n^{(1)} T^n y_n + \gamma_n^{(2)} T^n z_n + \gamma_n^{(3)} T^n x_n + \gamma_n^{(4)} T^{2n} z_n + \gamma_n^{(5)} x_n + \nu_n w_n - p\right\| \\
&= \lim_n \Big\|\gamma_n^{(1)} T^n y_n + \gamma_n^{(2)} T^n z_n + \gamma_n^{(3)} T^n x_n + \gamma_n^{(4)} T^{2n} z_n \\
&\qquad - \left(\gamma_n^{(1)} + \gamma_n^{(2)} + \gamma_n^{(3)} + \gamma_n^{(4)}\right) p + (1 - \gamma_n)(x_n - p) + \nu_n w_n - \nu_n x_n\Big\| \\
&= \lim_n \Big\|\gamma_n^{(1)} (T^n y_n - p) + \gamma_n^{(2)} (T^n z_n - p) + \gamma_n^{(3)} (T^n x_n - p) \\
&\qquad + \gamma_n^{(4)} (T^{2n} z_n - p) + (1 - \gamma_n)(x_n - p) - \nu_n x_n + \nu_n w_n\Big\| \\
&= \lim_n \Big\|\gamma_n^{(1)} (T^n y_n - p) + \gamma_n^{(2)} (T^n z_n - p) + \gamma_n^{(3)} (T^n x_n - p) + \gamma_n^{(4)} (T^{2n} z_n - p) \\
&\qquad + \gamma_n \nu_n w_n - \gamma_n \nu_n x_n + (1 - \gamma_n)(x_n - p) - \nu_n x_n + \nu_n w_n + \gamma_n \nu_n x_n - \gamma_n \nu_n w_n\Big\| \\
&= \lim_n \Big\|\gamma_n^{(1)} (T^n y_n - p + \nu_n(w_n - x_n)) + \gamma_n^{(2)} (T^n z_n - p + \nu_n(w_n - x_n)) \\
&\qquad + \gamma_n^{(3)} (T^n x_n - p + \nu_n(w_n - x_n)) + \gamma_n^{(4)} (T^{2n} z_n - p + \nu_n(w_n - x_n)) \\
&\qquad + (1 - \gamma_n)(x_n - p + \nu_n(w_n - x_n))\Big\|.
\end{aligned}$$

This together with (3.9)-(3.13) and Lemma 2.4 implies that (i), (ii) and (iii) are satisfied. Next, we shall prove (iv). Since $\liminf_n \gamma_n^{(1)} > 0$, it follows from Lemma 3.1(iii) that $\lim_n \|y_n - p\| = a$. From (3.1) we know that $\limsup_n \|z_n - p\| \le a$ and hence

$$\limsup_n \|T^n z_n - p\| \le \limsup_n (1 + r_n)\|z_n - p\| \le a.$$

Thus, we have

$$\limsup_n \|T^n z_n - p + \mu_n(v_n - x_n)\| \le a. \tag{3.14}$$

Also, we observe that

$$\limsup_n \|x_n - p + \mu_n(v_n - x_n)\| \le a, \tag{3.15}$$

$$\limsup_{n} \|T^n x_n - p + \mu_n(v_n - x_n)\| \leq a \tag{3.16}$$

and

$$\limsup_{n} \|T^{2n} x_n - p + \mu_n(v_n - x_n)\| \leq a. \tag{3.17}$$

Next, we note that

$$\begin{aligned} a = & \lim_n \|y_n - p\| = \lim_n \left\| \beta_n^{(1)} T^n z_n + \beta_n^{(2)} T^n x_n + \beta_n^{(3)} T^{2n} x_n + \beta_n^{(4)} x_n + \mu_n v_n - p \right\| \\ = & \lim_n \Big\| \beta_n^{(1)} \left(T^n z_n - p + \mu_n(v_n - x_n)\right) + \beta_n^{(2)} \left(T^n x_n - p + \mu_n(v_n - x_n)\right) \\ & + \beta_n^{(3)} (T^{2n} x_n - p + \mu_n(v_n - x_n)) \\ & + \left(1 - \beta_n^{(1)} - \beta_n^{(2)} - \beta_n^{(3)}\right) (x_n - p + \mu_n(v_n - x_n)) \Big\| . \end{aligned} \tag{3.18}$$

This together with (3.14)-(3.17) and Lemma 2.4 implies that $\lim_n \|T^n z_n - x_n\| = 0$. This completes the proof of (iv). By using the same argument as in proof of (iv), we can get (v). □

Lemma 3.3. *Let X, C, γ_n and T be as in Lemma 3.2 and $\{x_n\}$ be the sequence defined by Algorithm 1 such that the parameters satisfy one of the following control conditions:*

(C1) $0 < \liminf_n \gamma_n^{(1)} \leq \limsup_n \gamma_n < 1$ *and one of the following holds:*

- *(a)* $\limsup_n (\beta_n^{(1)} + \beta_n^{(2)} + \beta_n^{(3)}) < 1$*;*
- *(b)* $\liminf_n \gamma_n^{(2)} > 0$ *and* $\limsup_n (\beta_n^{(2)} + 2\beta_n^{(3)}) < 1$*;*

(C2) $\liminf_n \gamma_n^{(1)} > 0$ *and one of the following holds:*

- *(a)* $0 < \liminf_n \beta_n^{(1)} \leq \limsup_n (\beta_n^{(1)} + \beta_n^{(2)} + \beta_n^{(3)}) < 1$ *and* $\limsup_n \alpha_n^{(1)} < 1$*;*
- *(b)* $0 < \liminf_n \beta_n^{(2)} \leq \limsup_n (\beta_n^{(1)} + \beta_n^{(2)} + \beta_n^{(3)}) < 1$*;*

(C3) $0 < \liminf_n \gamma_n^{(2)} \leq \limsup_n \gamma_n < 1$ *and* $\limsup_n \alpha_n^{(1)} < 1$*;*

(C4) $0 < \liminf_n \gamma_n^{(3)} \leq \limsup_n \gamma_n < 1$*;*

(C5) $\liminf_n (\gamma_n^{(1)} \beta_n^{(1)} + \beta_n^{(2)} + \gamma_n^{(4)}) > 0$ *and* $0 < \liminf_n \alpha_n^{(1)} \leq \limsup_n \alpha_n^{(1)} < 1$.

Then $\lim_n \|T^n x_n - x_n\| = 0$*, and so by Lemma 3.1(v),* $\lim_n \|T x_n - x_n\| = 0$.

Proof. By using (1.1), we have

$$\begin{aligned} \|T^n x_n - x_n\| & \leq \|T^n x_n - T^n y_n\| + \|T^n y_n - x_n\| \\ & \leq (1 + r_n)\|x_n - y_n\| + \|T^n y_n - x_n\| \\ & \leq (1 + r_n) \Big(\beta_n^{(1)} \|T^n z_n - x_n\| + \beta_n^{(2)} \|T^n x_n - x_n\| \\ & \quad + \beta_n^{(3)} \|T^{2n} x_n - x_n\| + \mu_n \|v_n - x_n\| \Big) + \|T^n y_n - x_n\| \end{aligned} \tag{3.19}$$

and

$$\begin{aligned}\|T^n x_n - x_n\| &\le \|T^n x_n - T^n z_n\| + \|T^n z_n - x_n\| \\ &\le (1+r_n)\|x_n - z_n\| + \|T^n z_n - x_n\| \\ &\le (1+r_n)\left(\alpha_n^{(1)}\|T^n x_n - x_n\| + \lambda_n\|u_n - x_n\|\right) \\ &\quad + \|T^n z_n - x_n\|. \qquad (3.20)\end{aligned}$$

Proof of $(C1-a)$. It follows from Lemma 3.2(i) that $\lim_n \|T^n y_n - x_n\| = 0$. To show that $\lim_n \|T^n x_n - x_n\| = 0$, let $\{m_j\}$ be a subsequence of $\{n\}$. It suffices to show that there is a subsequence $\{n_k\}$ of $\{m_j\}$ such that $\lim_k \|T^{n_k} x_{n_k} - x_{n_k}\| = 0$. Take $p \in F(T)$ and consider Lemma 3.2.

If $\liminf_j \beta_{m_j}^{(1)} > 0$ and $\liminf_j \beta_{m_j}^{(3)} > 0$, then it follows from (3.14)-(3.18) and Lemma 2.4 that $\lim_j \|T^{m_j} z_{m_j} - x_{m_j}\| = \lim_j \|T^{2m_j} x_{m_j} - x_{m_j}\| = 0$.

If $\liminf_j \beta_{m_j}^{(1)} > 0$ and $\liminf_j \beta_{m_j}^{(3)} = 0$, again by (3.14)-(3.18) and Lemma 2.4 we obtain $\lim_j \|T^{m_j} z_{m_j} - x_{m_j}\| = 0$ and we may extract subsequence $\{n_k\}$ of $\{m_j\}$ such that $\lim_k \beta_{n_k}^{(3)} = 0$.

If $\liminf_j \beta_{m_j}^{(1)} = 0$, then we may extract subsequence $\{n_i'\}$ of $\{m_j\}$ such that $\lim_i \beta_{n_i'}^{(1)} = 0$. Now if $\liminf_i \beta_{n_i'}^{(3)} > 0$, then $\lim_i \|T^{2n_i'} x_{n_i'} - x_{n_i'}\| = 0$ and if $\liminf_i \beta_{n_i'}^{(3)} = 0$, then we may extract subsequence $\{n_k\}$ of $\{n_i'\}$ such that $\lim_k \beta_{n_k}^{(3)} = 0$.

In any case, it follows from (3.19) that

$$\lim_k (1 - (1+r_{n_k})\beta_{n_k}^{(2)})\|T^{n_k} x_{n_k} - x_{n_k}\| = 0$$

for some subsequence $\{n_k\}$ of $\{m_j\}$. Since $\limsup_n \beta_n^{(2)} < 1$, then $\lim_k \|T^{n_k} x_{n_k} - x_{n_k}\| = 0$. By Double Extract Subsequence Principle, we obtain $\lim_n \|T^n x_n - x_n\| = 0$.

Proof of $(C1-b)$. It follows from (3.19) that

$$\begin{aligned}\|T^n x_n - x_n\| \le (1+r_n)&\left(\beta_n^{(1)}\|T^n z_n - x_n\| + \left(\beta_n^{(2)} + \beta_n^{(3)}(2+r_n)\right)\|T^n x_n - x_n\|\right. \\ &\left. + \mu_n\|v_n - x_n\|\right) + \|T^n y_n - x_n\|.\end{aligned}$$

By using (i) and (ii) of Lemma 3.2, we obtain

$$\lim_n \left(1 - (1+r_n)\left(\beta_n^{(2)} + \beta_n^{(3)}(2+r_n)\right)\right)\|T^n x_n - x_n\| = 0.$$

Since $\limsup_n(\beta_n^{(2)} + 2\beta_n^{(3)}) < 1$, then $\lim_n \|T^n x_n - x_n\| = 0$.

Proof of $(C2-a)$. By Lemma 3.2(iv), $\lim_n \|T^n z_n - x_n\| = 0$. Using (3.20), we have $\lim_n (1 - (1+r_n)\alpha_n^{(1)})\|T^n x_n - x_n\| = 0$. Since $\limsup_n \alpha_n^{(1)} < 1$, then $\lim_n \|T^n x_n - x_n\| = 0$.

$(C2-b)$ is immediate consequence of (3.14)-(3.18) and Lemma 2.4. Also, $(C3)$ follows from Lemma 3.2(ii) and (3.20). Finally, $(C4)$ and $(C5)$ follow respectively from (iii) and (v) of Lemma 3.2. This completes the proof of lemma. □

Now, we state and prove the strong convergence theorem.

Theorem 3.4. *Let X be a uniformly convex Banach space, C be a nonempty closed convex subset of X and $T : C \to C$ be an asymptotically nonexpansive mapping with the nonempty fixed point set $F(T)$ and the sequence $\{r_n\}$ such that $\sum_{n=1}^{\infty} r_n < \infty$. Let $\{x_n\}$ be as in Algorithm 1 satisfying one of the control conditions in Lemma 3.3. If T satisfies Condition (A) with respect to the sequence $\{x_n\}$, then $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to a fixed point of T.*

Proof. By Lemma 3.1(ii), we know that $\lim_n d(x_n, F(T))$ exists. Now by combined effect Condition (A) and Lemma 3.3, we get

$$\lim_n f(d(x_n, F(T))) \leq \lim_n \|Tx_n - x_n\| = 0.$$

Since f is nondecreasing function with $f(r) > 0$ for all $r \in (0, \infty)$ and $f(0) = 0$, we have $\lim_n d(x_n, F(T)) = 0$. Then there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $\{y_k\}$ in $F(T)$ such that $\|x_{n_k} - y_k\| < 1/2^k$. It follows from the proof of Tan and Xu [12] that $\{y_k\}$ is a Cauchy sequence in $F(T)$ and so $y_k \to y$ for some $y \in F(T)$. It follows that $x_{n_k} \to y$. Since $\lim_n \|x_n - y\|$ exists, then $x_n \to y$. By (3.6) and (3.7) we have $\lim_n \|y_n - x_n\| = \lim_n \|z_n - x_n\| = 0$. It follows that $\lim_n y_n = \lim_n z_n = y$. This completes the proof. □

In the next result, we prove weak convergence of Algorithm 1 for asymptotically nonexpansive mapping in a uniformly convex Banach space. To do this, we need the following lemmas.

Lemma 3.5 ([14, Lemma 1.6]). *Let X be a uniformly convex Banach space, C be a nonempty closed convex subset of X and $T : C \to C$ be an asymptotically nonexpansive mapping. Then $(I - T)$ is demiclosed at 0, i.e., if $x_n \to x$ weakly and $x_n - Tx_n \to 0$ strongly, then $x \in F(T)$.*

A Banach space X is said to satisfy Opial's condition [15] if for any sequence $\{x_n\}$ in X, $x_n \to x$ weakly implies that $\limsup_n \|x_n - x\| < \limsup_n \|x_n - y\|$ for all $y \in X$ with $y \neq x$.

Lemma 3.6 ([7, Lemma 2.7]). *Let X be a Banach space which satisfies Opial's condition and let $\{x_n\}$ be a sequence in X. Let $u, v \in X$ be such that $\lim_n \|x_n - u\|$ and $\lim_n \|x_n - v\|$ exist. If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ which converge weakly to u and v, respectively, then $u = v$.*

Theorem 3.7. *Let X be a uniformly convex Banach space which satisfies Opial's condition, C be a nonempty closed convex subset of X and $T : C \to C$ be an asymptotically nonexpansive mapping with the nonempty fixed point set $F(T)$ and* *the sequence* $\{r_n\}$ *such that* $\sum_{n=1}^{\infty} r_n < \infty$. *Let* $\{x_n\}$ *be as in Algorithm 1 satisfying one of the control conditions in Lemma 3.3. Then* $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *converge weakly to a fixed point of* T.

Proof. It follows from Lemma 3.3 that $\lim_n \|Tx_n - x_n\| = 0$. Since X is uniformly convex and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to u$ weakly as $k \to \infty$. By Lemma 3.5, we have $u \in F(T)$. Suppose that $\{x_{m_k}\}$ be an arbitrary subsequence of $\{x_n\}$ converging weakly to v. Again, from Lemma 3.5, $v \in F(T)$. By Lemma 3.1(i), $\lim_n \|x_n - u\|$ and $\lim_n \|x_n - v\|$ exist. It follows from Lemma 3.6 that $u = v$. Therefore $\{x_n\}$ converges weakly to u. By (3.6) and (3.7) we have $\lim_n \|y_n - x_n\| = \lim_n \|z_n - x_n\| = 0$. It follows that $\{y_n\}$ and $\{z_n\}$ converge weakly to u. □

Remark 3.8. *When* $\lambda_n = \mu_n = \nu_n = \beta_n^{(3)} = \gamma_n^{(4)} \equiv 0$, $\alpha_n^{(2)} = 1 - \alpha_n^{(1)}$, $\beta_n^{(4)} = 1 - \beta_n^{(1)} - \beta_n^{(2)}$ *and* $\gamma_n^{(5)} = 1 - \gamma_n^{(1)} - \gamma_n^{(2)} - \gamma_n^{(3)}$ *in Theorem 3.4 and Theorem 3.7, we obtain weak and strong convergence theorems of Algorithm 2 [9, Theorem 9 and Theorem 10].*

Remark 3.9. *When* $\lambda_n = \mu_n = \nu_n = \beta_n^{(3)} = \gamma_n^{(3)} = \gamma_n^{(4)} \equiv 0$, $\alpha_n^{(2)} = 1 - \alpha_n^{(1)}$, $\beta_n^{(4)} = 1 - \beta_n^{(1)} - \beta_n^{(2)}$ *and* $\gamma_n^{(5)} = 1 - \gamma_n^{(1)} - \gamma_n^{(2)}$ *in Theorem 3.4 and Theorem 3.7, we obtain weak and strong convergence theorems of Algorithm 3 [7, Theorem 2.3 and Theorem 2.8] without the restrictions* $\liminf_n b_n > 0$ *(*b_n *replaced by* $\beta_n^{(1)}$ *in our definition) and the boundedness of* C.

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