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An Equivalent Definition for the Backwards Itô Integral[1]

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Abstract : This paper aims to give an equivalent definition of backwards Itô integral that considers a full division on a compact interval $[a, b]$.

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1 Introduction

In [1], Toh et al. defined his *forward* Itô-McShane integral using the generalized Riemann approach. This approach was discovered independently by Kurzweil [2] and Henstock [3] in 1950s. They used non-uniform meshes (meshes that vary from point to point) instead of the uniform meshes as depicted in the usual Riemann approach. The power of this approach lies in the fact that it can integrate highly oscillatory functions which the usual Riemann approach fails to handle. The *forward* Itô-McShane integral uses forward filtration. Moreover, the δ-fine division is *belated* in the sense that the associated points (or *tags*) are always on the left endpoints of the subintervals. This preserves the adaptedness property of

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the integral with respect to the forward filtration. Furthermore, the δ-fine belated division is partial since a full division may not exist.

In this paper, we define *backwards Itô integral* using the same approach. But this time, it is with reversed time frame, that is, if all processes start at time $t = 0$ and progress to a later time t, here, we fix a time T and then proceed backwards to some earlier time s. Furthermore, *backwards δ-fine division* and *backwards filtration* were used. Backwards δ-fine division in a sense that we choose the right endpoint of the subintervals as tags so that our integral can assume the adaptedness property with respect to the backwards filtration. Here, backwards δ-fine division is partial since a full division may not exist. Now, note that the forward and backward integrals of a process with respect to a Brownian motion have different values. This is due to the fact that a Brownian motion is not of bounded variation. Thus, the forward and backwards stochastic integrals are not equivalent.

Now, one might ask if there is a corresponding integral that considers a full division on an interval $[a, b]$. The answer is affirmative. In [4], Boonpogkrong and Chew defined *integral* which considers a homogeneous McShane δ-fine full division on an interval $[a, b]$, that is, *all interval-point pairs that cover the interval* $[a, b]$ *are McShane δ-fine.* Here, we define *McShane backwards Itô integral* with a full division that is composed of backwards δ-fine and McShane δ-fine partial divisions. Unlike the backwards Itô integral, this integral yet to be defined does not assume adaptedness property. In this paper, we shall prove that these two integrals are equivalent.

2 Preliminaries

Throughout this note, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_0^+$ the set of non-negative real numbers, $\mathbb{N}$ the set of positive integers and $(\Omega, \mathcal{G}, \mathbb{P})$ denotes a probability space. Let $\{\mathcal{G}^s : 0 \le s \le T\}$ be a family of sub σ-algebras of $\mathcal{G}$. Then $\{\mathcal{G}^s : 0 \le s \le T\}$ is called a backwards filtration if $\mathcal{G}^t \subseteq \mathcal{G}^s$ for all $0 \le s < t \le T$. If in addition, $\{\mathcal{G}^s : 0 \le s \le T\}$ satisfies the following condition: (1) $\mathcal{G}^T$ contains all sets of $\mathbb{P}$ -measure zero in $\mathcal{G}$; and (2) for each $s \in [0, T]$, $\mathcal{G}^s = \mathcal{G}^{s-}$ where $\mathcal{G}^{s-} = \bigcap_{\varepsilon>0} \mathcal{G}^{s-\varepsilon}$. Then $\{\mathcal{G}^s : 0 \le s \le T\}$ is called a *standard backwards filtration*. We often write $\{\mathcal{G}^s\}$ instead of $\{\mathcal{G}^s : 0 \le s \le T\}$. See [5].

A *stochastic process* f or simply *process* is a function $f : \Omega \times [0, T] \to \mathbb{R}$, where $[0, T]$ is an interval in $\mathbb{R}_0^+$ and $f(\cdot, s)$ is $\mathcal{G}^s$-measurable for each $s \in [0, T]$. A process $f = \{f_s : s \in [0, T]\}$ is said to be *adapted* to the standard backwards filtration $\{\mathcal{G}^s\}$ if f_s is $\mathcal{G}^s$-measurable for each $s \in [0, T]$. Let $B = \left\{B_t : t \in \mathbb{R}_0^+\right\}$ be a standard Brownian motion (BM). Let $\sigma(B_u : s \le u \le T)$ be the smallest σ-algebra generated by $\{B_u : s \le u \le T\}$. This is the smallest σ-algebra containing the information about the structure of BM on $[s, T]$.

Throughout this note, we assume that the standard backwards filtration $\{\mathcal{G}^s\}$ is the family of σ-algebras $\sigma(B_u : s \le u \le T)$. This family is then called the *natural backwards filtration* of B. Let $(\Omega, \mathcal{G}, \{\mathcal{G}^s\}, \mathbb{P})$ be a *standard backwards* *filtering space.* We write $L^p(\Omega)$ for $L^p(\Omega, \mathcal{G}, \mathbb{P})$ where $f \in L^p(\Omega)$ if $\mathbb{E}(f)^p < \infty$. For $f \in L^1(\Omega)$, let $\mathbb{E}(f)$ denote the expectation of f, that is, $\mathbb{E}(f) = \int_\Omega f d\mathbb{P}$. The conditional expectation of f given $\mathcal{G}^s$ is the random variable $\mathbb{E}(f|\mathcal{G}^s)$.

The following lemmas are well-known.

Lemma 2.1. *Let $a_1, a_2 \in \mathbb{R}$ and $f, g \in L^1(\Omega)$, that is, the expectation of f and g exist. Then for each $s, t \in [0,T]$ with $s \leq t$, so that $\mathcal{G}^t \subseteq \mathcal{G}^s$ we have*

(i) $\mathbb{E}(a_1 f + a_2 g|\mathcal{G}^s) = a_1\mathbb{E}(f|\mathcal{G}^s) + a_2\mathbb{E}(g|\mathcal{G}^s)$ *a.s.;*

(ii) $\mathbb{E}\left(\mathbb{E}\left(f|\mathcal{G}^s\right)\right) = \mathbb{E}\left(f\right)$ *a.s.;*

(iii) $\mathbb{E}\left(fg|\mathcal{G}^s\right) = f\mathbb{E}\left(g|\mathcal{G}^s\right)$ *a.s., if f is $\mathcal{G}^s$-measurable.*

Lemma 2.2. *Let $u, v \in [0,T]$. Then*

(i) $\mathbb{E}(B_v - B_u)^2 = |v-u|$;

(ii) $\mathbb{E}(B_v - B_u)^4 = 3(v-u)^2$;

(iii) $\mathbb{E}(B_v - B_u|\mathcal{G}^v) = 0$, *where $u \leq v$;*

(iv) $\mathbb{E}(B_u|\mathcal{G}^v) = B_v$, *where $u \leq v$;*

(v) $\mathbb{E}\left[f(B_v - B_u)^2\right] = -\mathbb{E}\left[f\left(B_v^2 - B_u^2\right)\right] = \mathbb{E}(f)(v-u)$, *whenever f is $\mathcal{G}^v$-measurable and $u \leq v$. In particular,*

$$\mathbb{E}(B_v - B_u)^2 = -\mathbb{E}(B_v^2 - B_u^2).$$

(vi) $\mathbb{E}[f(B_v - B_u)(B_t - B_s)] = 0$, *whenever $(u,v]$ and $(s,t]$ are disjoint subintervals of $[0,T]$ and f is $\mathcal{G}^v$-measurable. In particular,*

$$\mathbb{E}[(B_v - B_u)(B_t - B_s)] = 0.$$

3 Backwards Itô Integral

In this section, we shall present the backwards Itô integral and its related results.

Let δ be a positive function on $(0,T]$. A *finite* collection $D = \{((u_i, \xi_i], \xi_i)\}_{i=1}^n$ of interval-point pairs is said to be a *backwards partial division* of $[0,T]$ if $\{(u_i, \xi_i]\}_{i=1}^n$ is a finite collection of disjoint subintervals of $(0,T]$. An interval-point pair $((u,\xi],\xi)$ is said to be *backwards δ-fine* if $(u,\xi] \subseteq (\xi - \delta(\xi), \xi]$, whenever $(u,\xi] \subseteq (0,T]$ and $\xi \in (0,T]$. We call $D = \{((u_i,\xi_i],\xi_i)\}_{i=1}^n$ a *backwards δ-fine partial division of* $[0,T]$ if D is a *backwards partial division* of $[0,T]$ and for each i, the interval-point pair $((u_i,\xi_i],\xi_i)$ is *backwards δ-fine.*

Let $\delta > 0$. One may not be able to find a *full* division that covers the entire interval $(0,T]$. For example, take $\delta(\xi) = \frac{\xi}{2}$. Then the interval $(0,T]$ cannot be covered by any *finite* collection of backwards δ-fine intervals.

Let $\eta > 0$ be given, a backwards δ-fine partial division D is said to *fail to cover* $(0, T]$ *by at most Lebesgue measure* η if

$$\left| T - \sum_{i=1}^{n} (\xi_i - u_i) \right| \leq \eta.$$

We are now ready to define the *backwards Itô integral.*

Definition 3.1. Let $f = \{f_s : s \in [0, T]\}$ be a process adapted to the standard backwards filtering space $(\Omega, \mathcal{G}, \{\mathcal{G}^s\}, \mathbf{P})$. Then f is said to be *backwards Itô integrable* on $[0, T]$ if there exists an $A \in L^2(\Omega)$ such that for any $\varepsilon > 0$, there exist a positive function δ on $(0, T]$ and a positive number η such that for any backwards δ-fine partial division $D = \{((u_i, \xi_i], \xi_i)\}_{i=1}^n$ of $[0, T]$ that fails to cover $(0, T]$ by at most Lebesgue measure η we have

$$\mathbb{E}\left(|S(f, D, \delta, \eta) - A|^2 \right) \leq \varepsilon, \tag{3.1}$$

where $S(f, D, \delta, \eta) = \sum_{i=1}^n f_{\xi_i} (B_{\xi_i} - B_{u_i})$. We denote A by $(BI) \int_0^T f_t dB_t$.

It is not difficult to see that the backwards Itô integral $(BI) \int_0^T f_t dB_t$ is unique up to a set of $\mathbf{P}$-measure zero by using the fact that if D is a backwards δ-fine partial division, then D is a backwards δ_i-fine partial division that fails to cover $(0, T]$ by at most a set of measure η_i for $i = 1, 2$, where $\delta = \min\{\delta_1, \delta_2\}$ and $\eta = \min\{\eta_1, \eta_2\}$ whenever D_1 and D_2 are backwards δ_i-fine partial division of $[0, T]$ that fail to cover $(0, T]$ by at most a set of measure η_i for $i = 1, 2$.

The following results are standard in the classical Henstock integration theory and the proof has been ommited, see [6, 7]. In the succeeding discussions, we always assume that $[a, b] \subseteq [0, T]$.

Proposition 3.2 (Cauchy Criterion)**.** *Let f be an adapted process on $[a, b]$. Then f is backwards Itô integrable on $[a, b]$ if and only if for each $\varepsilon > 0$, there exist a positive function δ on $[a, b]$ and a positive constant η such that*

$$\mathbb{E}\left(|S(f, D_1, \delta, \eta) - S(f, D_2, \delta, \eta)|^2 \right) \leq \varepsilon, \tag{3.2}$$

whenever D_1 and D_2 are backwards δ-fine partial divisions of $[a, b]$ that fail to cover $[a, b]$ by at most a set of measure η.

Lemma 3.3 (Henstock's Lemma)**.** *Let f be backwards Itô integrable on $[a, b]$ and $F(u, v) = \int_u^v f_t dB_t$ for any $(u, v] \subseteq [a, b]$. Then for every $\varepsilon > 0$, there exists a positive function δ on $[a, b]$ such that*

$$\mathbb{E}\left(\sum_{i=1}^{n} |f_{\xi_i}(B_{\xi_i} - B_{u_i}) - F(u_i, \xi_i)|^2 \right) \leq \varepsilon, \tag{3.3}$$

whenever $D = \{((u_i, \xi_i], \xi_i)\}_{i=1}^n$ is a backwards δ-fine partial division of $[a, b]$.

Theorem 3.4 (Itô Isometry)**.** *Let f be backwards Itô integrable on $[0,T]$. Then $\mathbb{E}[f_t^2]$ is Lebesgue integrable on $[0,T]$ and*

$$\mathbb{E}\left(\int_0^T f_t dB_t\right)^2 = \int_0^T \mathbb{E}[f_t^2]dt.$$

Definition 3.5. Let $F = \{F_s : s \in [0,T]\}$ be a stochastic process. Then the process F is said to have an *AC^2- property* if for each $\varepsilon > 0$, there exists $\eta > 0$ such that whenever $\{(u_i, v_i]\}_{i=1}^n$ is a finite collection of disjoint subintervals of $[0,T]$ with $\sum_{i=1}^n |v_i - u_i| \leq \eta$, we have

$$\mathbb{E}\left(\left|\sum_{i=1}^n F(u_i, v_i)\right|^2\right) \leq \varepsilon.$$

We omit the proof of the following theorem since it is standard in the theory of Henstock integration.

Theorem 3.6. *Let f be backwards Itô integrable on $[0,T]$ with $\Phi(u,T) = \int_u^T f_t dB_t$. Then Φ has the AC^2- property.*

4 McShane Backwards Itô Integral

In this section, we shall discuss the McShane Backwards Itô integral and investigate its properties.

Let δ be a positive function on $(0,T]$, $(u,v] \subseteq (0,T]$ and $\xi \in (0,T]$. An interval-point pair $((u,v],\xi)$ is said to be *McShane δ-fine* if $(u,v] \subseteq (\xi - \delta(\xi), \xi + \delta(\xi))$. Then $D = \{((u_i,v_i],\xi_i)\}_{i=1}^n$ is said to be a *McShane δ-fine partial division of* $[0,T]$ if $\{(u_i,v_i]\}_{i=1}^n$ is a finite collection of disjoint subintervals of $[0,T]$ and for each i, $((u_i,v_i],\xi_i)$ is McShane δ-fine. If in addition, $\bigcup_{i=1}^n (u_i,v_i] = (0,T]$, then D is a *McShane δ-fine (full) division* of $[0,T]$. Note that ξ may not belong to $(u,v]$. It is not difficult to see that if D is a backwards δ-fine partial division of $[0,T]$ then D is a McShane δ-fine partial division of $[0,T]$. However, the converse is not true.

Let δ be a positive function on $(0,T]$ and η a positive number. A finite collection $D = \{((u_i,v_i],\xi_i)\}_{i=1}^n$ of interval-point pairs is said to be *$\eta - \delta$- fine division of* $[0,T]$ if $D = D_1 \cup D_2$ where D_1 is backwards δ-fine partial division and D_2 is McShane δ-fine partial division of $[0,T]$ with $(D_2)\sum(v_i - u_i) \leq \eta$. Given any positive function δ on $(0,T]$, the $\eta - \delta$-fine division D of $[0,T]$ always exists, since we can find a *full McShane δ-fine division* D_2 of $(0,T] \setminus \bigcup_{(u,v]\in D_1}(u,v]$.

Now, D is said to be an *$\eta - \delta$-fine partial division* of $[0,T]$ if D is a subset of some $\eta - \delta$-fine full division D' of $[0,T]$. It is not difficult to see that an $\eta - \delta$-fine partial division D of $[0,T]$ is a McShane partial division but not a backwards δ-fine partial division of $[0,T]$.

We are now ready to define the *McShane backwards Itô integral.*

Definition 4.1. Let $f : \Omega \times [0,T] \to \mathbb{R}$ be an $L^2(\Omega)$ process, i.e., $\mathbb{E}(|f_s|^2) < \infty$ and adapted to the standard backwards filtering space $(\Omega, \mathcal{G}, \{\mathcal{G}^s\}, \mathbf{P})$. Then f is said to be *McShane backwards Itô integrable* on $[0,T]$ if there exists an $A \in L^2(\Omega)$ such that for any $\varepsilon > 0$, there exist a positive function δ on $[0,T]$ and a positive number η such that for any $\eta - \delta$- fine *division* $D = \{((u_i, v_i], \xi_i)\}_{i=1}^n$ *of* $[0,T]$, we have

$$\mathbb{E}\left(\left|\overline{S}(f, D, \eta - \delta) - F\right|^2\right) \leq \varepsilon \tag{4.1}$$

where $\overline{S}(f, D, \eta - \delta) = \sum_{i=1}^n \mathbb{E}\left(f_{\xi_i} | \mathcal{G}^{v_i}\right)\left(B_{v_i} - B_{u_i}\right).$

Denote the integral F by $(BM)\int_0^T f_t dB_t$. It is not difficult to show that the integral is unique up to a set of $\mathbf{P}$-measure zero.

The following results and their proofs are standard in the theory of Henstock integration, see [6, 7]. Hence, the proofs are omitted.

Proposition 4.2 (Cauchy Criterion)**.** *Let f be an adapted process on $[0,T]$. Then f is McShane backwards Itô integrable on $[0,T]$ if and only if for each $\varepsilon > 0$, there exist a positive function δ on $[0,T]$ and a positive constant η such that for every $\eta - \delta$-fine divisions D_1 and D_2 of $[0,T]$, we have*

$$\mathbb{E}\left(\left|\overline{S}(f, D_1, \eta - \delta) - \overline{S}(f, D_2, \eta - \delta)\right|^2\right) \leq \varepsilon. \tag{4.2}$$

Proposition 4.3. *Let f be an adapted process on $[0,T]$. Then f is McShane backwards Itô integrable on $[0,T]$ if and only if there exist $A \in L^2(\Omega)$, a decreasing sequence $\{\delta_n(\xi)\}$ of positive functions defined on $[0,T]$, and a decresing sequence $\{\eta_n\}$ of positive numbers, that is , $0 < \delta_{n+1}(\xi) \leq \delta_n(\xi)$ and $0 < \eta_{n+1} \leq \eta_n$ for all n and all $\xi \in [0,T]$, such that for every sequence $\{D_n\}$ where D_n is an $\eta_n - \delta_n$-fine division of $[0,T]$ we have*

$$\lim_{n\to\infty} \mathbb{E}\left(\left|\overline{S}(f, D_n, \eta_n - \delta_n) - A\right|^2\right) = 0. \tag{4.3}$$

Proposition 4.4. *Let f be McShane backwards Itô integrable on $[a,c]$ and $[c,b]$. Then f is McShane backwards Itô integrable on $[a,b]$ and further*

$$\int_a^b f_t dB_t = \int_a^c f_t dB_t + \int_c^b f_t dB_t.$$

Proposition 4.5. *If f is McShane backwards Itô integrable on $[a,b]$, then f is McShane backwards Itô integrable on any subinterval $[c,d]$ of $[a,b]$.*

5 Some Stochastic Properties

In this section, we will derive some stochastic properties of the McShane backwards Itô integral. The proofs are based on the ideas of the corresponding forward Itô integral using non-uniform Riemann approach, see [1, 4, 8].

Lemma 5.1. *Let $f : \Omega\times[0,T] \longrightarrow \mathbb{R}$ be McShane backwards Itô integrable on $[0,T]$ and $F(A) = \int_A f_t dB_t$, where A is a left-open subinterval of $(0,T]$. Let $J = (c,d]$ and $K = (u,v]$ be two disjoint left-open subintervals of $(0,T]$. Then*

(i) F has the orthogonal increment property, that is, $\mathbb{E}(F(J)F(K)) = 0$;

(ii) $\mathbb{E}(B(J)F(K)) = 0$, where $B(J)$ is the increment of BM on the subinterval J; and

(iii) $\mathbb{E}\left\{\left(\mathbb{E}\left(f_\xi|\mathcal{G}^d\right)B(I) - F(I)\right)\left(\mathbb{E}\left(f_\eta|\mathcal{G}^v\right)B(J) - F(J)\right)\right\} = 0$ where $\xi,\eta \in (0,T]$.

Proof. Let $\xi_i, \xi_j \in [0,T]$. By Lemma 2.1(ii) and Lemma 2.2(iii), we have

$$\mathbb{E}\left\{\mathbb{E}\left(f_{\xi_i}|\mathcal{G}^{v_i}\right)(B_{v_i} - B_{u_i})\mathbb{E}\left(f_{\xi_j}|\mathcal{G}^{v_j}\right)(B_{v_j} - B_{u_j})\right\} = 0 \tag{5.1}$$

whenever $(u_i, v_i], (u_j, v_j]$ are disjoint subintervals of $[0,T]$ with $v_i < u_j$. Similarly, if $I = (c,d]$ and $(u_i, v_i]$ are disjoint subintervals of $(0,T]$ with $v_i < c$ then

$$\mathbb{E}\left\{B(I)\mathbb{E}\left(f_{\xi_i}|\mathcal{G}^{v_i}\right)(B_{v_i} - B_{u_i})\right\} = 0. \tag{5.2}$$

Now, let $D(I) = \{((u_i, v_i], \xi_i)\}_{i=1}^n$ and $D(J) = \{((u_j, v_j], \xi_j)\}_{j=1}^m$ be an $\eta - \delta$-fine divisions of I and J respectively, where I and J are disjoint subintervals of $(0,T]$. Then for every positive function δ_n of $[0,T]$ and positive number η_n, we have

$$\mathbb{E}\left[\overline{S}\left(f, D(I), \eta_n - \delta_n\right)\cdot\overline{S}\left(f, D(J), \eta_n - \delta_n\right)\right] = 0. \tag{5.3}$$

So by Proposition 4.3, we may choose decreasing sequences $\{\delta_n\}$ and $\{\eta_n\}$ such that for every pair of sequences $\{D_n(I)\}$ and $\{D_n(J)\}$ with $D_n(I)$ and $D_n(J)$ both $\delta_n - \eta_n$-fine divisions of I, J respectively, we have

$$\lim_{n\to\infty}\mathbb{E}\left(\left|\overline{S}(f, D_n(I), \eta_n - \delta_n) - F(I)\right|^2\right) = 0$$

and

$$\lim_{n\to\infty}\mathbb{E}\left(\left|\overline{S}(f, D_n(J), \eta_n - \delta_n) - F(J)\right|^2\right) = 0.$$

Hence, $\mathbb{E}\left((F(I)F(J))\right) = 0$. Similary, $\mathbb{E}\left((B(I)F(J))\right) = 0$. Hence, (i) and (ii) hold. Furthermore, (iii) follows directly from (i) and (ii). □

The following lemma can be easily obtained from the Lemma 5.1 above.

Lemma 5.2. *Let $f : \Omega \times [0,T] \longrightarrow \mathbb{R}$ be McShane backwards Itô integrable on $[0,T]$ and $F(u,v) = \int_u^v f_t dB_t$. Let $D = \{((u_i, v_i], \xi_i)\}_{i=1}^n$ be an $\eta - \delta$-fine partial division of $[0,T]$. Then*

(i) $$\mathbb{E}\left[(D)\sum_{i=1}^n \mathbb{E}\left(f_{\xi_i}|\mathcal{G}^{v_i}\right)(B_{v_i} - B_{u_i})\right]^2 = (D)\sum_{i=1}^n \mathbb{E}\left[\mathbb{E}\left(f_{\xi_i}|\mathcal{G}^{v_i}\right)\right]^2 (v_i - u_i);$$

(ii) $$\mathbb{E}\left[(D)\sum_{i=1}^{n}\{\mathbb{E}\left(f_{\xi_i}|\mathcal{G}^{v_i}\right)(B_{v_i}-B_{u_i})-F(u_i,v_i)\}\right]^2$$
$$=\mathbb{E}\left[(D)\sum_{i=1}^{n}\{\mathbb{E}\left(f_{\xi_i}|\mathcal{G}^{v_i}\right)(B_{v_i}-B_{u_i})-F(u_i,v_i)\}^2\right].$$

Henstock's Lemma is an important tool to achieve our goal. The proof follows from Lemma 5.2 (ii) above.

Lemma 5.3 (Henstock's Lemma). *Let f be McShane backwards Itô integrable on $[a,b]$ and $F(u,v)=\int_u^v f_t dB_t$ for each $(u,v]\subset[a,b]$. Then for every $\varepsilon>0$, there exist a positive function $\delta(\xi)$ on $[a,b]$ and a positive number η such that whenever $D=\{((u_i,v_i],\xi_i)\}_{i=1}^n$ is an $\eta-\delta$-fine* partial *division of $[a,b]$ we have*

$$\mathbb{E}\left((D)\sum_{i=1}^{n}\left\{\mathbb{E}\left(f_{\xi_i}|\mathcal{G}^{v_i}\right)(B_{v_i}-B_{u_i})-F(u_i,v_i)\right\}^2\right)\le\varepsilon.$$

Theorem 5.4. *Let f is McShane backwards Itô integrable on $[0,T]$ with primitive $F(s,b)=\int_s^b f_t dB_t$. Then F has AC^2-property on $[0,T]$.*

Proof. Henstock's Lemma says, for each $\varepsilon>0$ there exist $\delta(\xi)>0$ on $[a,b]$ and $\eta>0$ such that whenever $D=\{((u_i,v_i],\xi)\}_{i=1}^n$ is an $\eta-\delta$-fine *partial* division of $[a,b]$ we have

$$\mathbb{E}\left(\left|(D)\sum_{i=1}^{n}\{\mathbb{E}\left(f_{\xi_i}|\mathcal{G}^{v_i}\right)(B_{v_i}-B_{u_i})-F(u_i,v_i)\}\right|^2\right)\le\frac{\varepsilon}{4}. \tag{5.4}$$

Hence,

$$\mathbb{E}\left(\left|(D)\sum_{i=1}^{n}F(u_i,v_i)\right|^2\right)\le 2(D)\sum_{i=1}^{n}\left[\mathbb{E}\left(f_{\xi_i}\right)^2\right](v_i-u_i)+\frac{\varepsilon}{2}. \tag{5.5}$$

Let $D'=\{((s_i,t_i],\xi_i)\}_{i=1}^n$ be a fixed $\eta-\delta$-fine division of $[0,T]$ with $M\ge\max_{1\le i\le n}\{\mathbb{E}(f_{\xi_i}^2)\}$. Choose $\eta<\frac{\varepsilon}{2(M+1)}$. Let $P=\{(x_i,y_i]\}_{i=1}^m$ be a finite collection of disjoint subintervals of $(0,T]$ with $(P)\sum_{i=1}^m|y_i-x_i|<\eta$. Let $\{(a_k,b_k]\}_{k=1}^q$ be the common refinement of $\{(s_i,t_i]\}_{i=1}^n$ and $\{(x_i,y_i]\}_{i=1}^m$ on $\bigcup_{i=1}^m(x_i,y_i]$. Then $\bigcup_k(a_k,b_k]=\bigcup_i(x_i,y_i]$. If $(a_k,b_k]$ is a subinterval of $(s_i,t_i]$ then we choose ξ_i as an associate tag of $(a_k,b_k]$, denote it by η_k. From this construction, we obtain a new $\eta-\delta$-fine partial division $D''=\{((a_k,b_k],\eta_k)\}_{k=1}^q$ of $[0,T]$. Therefore,

$$\begin{aligned}\mathbb{E}\left(\left|(P)\sum_{i=1}^{m}F(x_i,y_i)\right|^2\right)&\le\mathbb{E}\left(\left|(D'')\sum_{k=1}^{q}F(a_k,b_k)\right|^2\right)\\&\le 2(D'')\sum_{k=1}^{q}\mathbb{E}\left(f_{\eta_k}^2\right)(b_k-a_k)+\frac{\varepsilon}{2}\\&\le 2M\sum_{k=1}^{q}(b_k-a_k)+\frac{\varepsilon}{2}\\&<\varepsilon.\end{aligned}$$

The proof is now complete. □

6 The Equivalence of Two Integrals

In this section, we will establish that *backwards Itô-integrability* and *McShane backwards Itô-integrability* are equivalent. First, we make some observations. Let $D_1 = \{((u_i, v_i], \xi_i)\}_{i=1}^n$ and $D_2 = \{((s_j, t_j], \eta_j)\}_{j=1}^m$ be two McShane δ-fine partial divisions of $[0, T]$. Let $\{(x_k, y_k]\}_{k=1}^p$ be the refinement of D_1 and D_2. If $(x_k, y_k] \subset (u_i, v_i] \cap (s_i, t_i]$, then choose $\gamma_k = \xi_i$ or η_j. Thus, $D = \{((x_k, y_k], \gamma_k)\}_{k=1}^p$ is a McShane δ-fine partial division of $[0, T]$. We remark that if D_1 and D_2 are backwards δ-fine partial division, then D may not be a backwards δ-fine partial division of $[0, T]$ instead D is a McShane δ-fine partial division of $[0, T]$. Now note that if D_1 and D_2 are $\eta - \delta$-fine partial divisions of $[0, T]$, then D is a McShane δ-fine partial division of $[0, T]$ but may not be an $\eta - \delta$-fine partial division of $[0, T]$.

6.1 M-integrable

To achieve our goal, we will need the notion of *M-integral*.

Definition 6.1 ([4]). Let $f : \Omega \times [0, T] \longrightarrow \mathbb{R}$ be an $L^1(\Omega)$-process. Then f is said to be *M-integrable on* $[0, T]$, if for each $\varepsilon > 0$ there exist a positive function δ defined on $[0, T]$ such that whenever $D = \{((u_i, v_i], \xi_i)\}_{i=1}^n$ and $D' = \{((u_i, v_i], \eta_i)\}_{i=1}^n$ are two McShane δ-fine partial divisions of $[0, T]$ we have

$$\sum_{i=1}^{n} \mathbb{E}\left|(f_{\xi_i} - f_{\eta_i})(v_i - u_i)\right| \leq \varepsilon.$$

The following result is true. See [9].

Theorem 6.2 ([9]). *Let $f : \Omega \times [0, T] \to \mathbb{R}$ be an L^1-process. If f is M-integrable on $[0, T]$ if and only if $\mathbb{E}(|f_t|)$ is Lebesgue integrable on $[0, T]$ and their integrals are equal.*

Theorem 6.3. *Let $f : \Omega \times [0, T] \to \mathbb{R}$ be an $L^2(\Omega)$-process. If f is backwards Itô-integrable on $[0, T]$ if and only if f^2 is M-integrable on $[0, T]$.*

Proof. ($\Longrightarrow$) Let f is backwards Itô-integrable on $[0, T]$. Then Theorem 3.4 says that $\mathbb{E}[f_t^2]$ is Lebesgue integrable on $[0, T]$. It follows by Theorem 6.2 above, that f^2 is M-integrable on $[0, T]$.
($\Longrightarrow$) For the converse, assume that f^2 is M-integrable on $[0, T]$. By Definition 6.1, for each $\varepsilon > 0$ there exist a positive function δ defined on $[0, T]$ such that whenever $D'_0 = \{(u_i, v_i], \xi_i\}_{i=1}^n$ and $D''_0 = \{((u_i, v_i], \eta_i)\}_{i=1}^n$ are two McShane δ-fine partial divisions of $[0, T]$ we have

$$\sum_{i=1}^{n} \mathbb{E}\left|(f_{\xi_i}^2 - f_{\eta_i}^2)(v_i - u_i)\right| \leq \varepsilon. \tag{6.1}$$

Now, by using the Cauchy Criterion (Proposition 3.2) we will show that if f^2 is M-integrable on $[0,T]$ then f is backwards Itô-integrable on $[0,T]$.

Let $D_1 = \{((u_i,\xi_i],\xi_i)\}_{i=1}^n$ and $D_2 = \{((u_j,\eta_j],\eta_j)\}_{j=1}^m$ are two backwards δ-fine partial divisions of $[0,T]$ that fail to cover $[0,T]$ by at most a set of measure η. As in the refinement construction, we can get $D = \{((x_k,y_k],\gamma_k)\}_{k=1}^p$ a McShane δ-fine division of $[0,T]$. From this new refinement McShane division D of $[0,T]$, we can get two arbitrary McShane divisions $D' = \{((u_k,v_k],\xi_k)\}_{k=1}^p$ and $D'' = \{((u_k,v_k],\eta_k)\}_{k=1}^p$, where ξ_k comes from the set of tags of D_1 and η_k comes from the set of tags of D_2. Therefore,

$$\mathbb{E}\Big(\Big|(D_1)\sum_{i=1}^n f_{\xi_i}(B_{\xi_i}-B_{u_i}) - (D_2)\sum_{j=1}^m f_{\eta_j}(B_{\eta_j}-B_{u_j})\Big|^2\Big) \le \varepsilon.$$

The proof is now complete. □

Theorem 6.4. *Let $f:\Omega\times[0,T]\longrightarrow\mathbb{R}$ be an $L^2(\Omega)$-process. If f^2 is M-integrable on $[0,T]$ then f is McShane backwards Itô-integrable on $[0,T]$.*

Proof. Suppose f^2 is M-integrable on $[0,T]$. Then $|f^2|$ is M-integrable on $[0,T]$. Observe that $f^+ = \{f_t^+ : t\in[0,T]\} = \frac{|f|+f}{2}$ and $f^- = |f| - f^+$. Thus, $(f^+)^2$ and $(f^-)^2$ are M-integrable on $[0,T]$. Therefore, in the following proof we may assume that f is nonnegative. Now, we will prove this result via Cauchy criterion. Let $D_1 = \{((u_i,v_i],\xi_i)\}_{i=1}^n$ and $D_2 = \{((u_j,v_j],\eta_j)\}_{j=1}^m$ are two backwards $\eta-\delta$-fine divisions of $[0,T]$. By the refinement procedure, as in the proof above, we can obtain two arbitrary McShane divisions $D' = \{((u_k,v_k],\xi_k)\}_{k=1}^p$ and $D'' = \{((u_k,v_k],\eta_k)\}_{k=1}^p$, where ξ_k comes from the set of tags of D_1 and η_k comes from the set of tags of D_2. Therefore,

$$\begin{aligned}\mathbb{E}\Big(\Big|(D_1)\sum_{i=1}^n \mathbb{E}\left[f_{\xi_i}|\mathcal{G}^{\xi_i}\right](B_{\xi_i}-B_{u_i}) - (D_2)\sum_{j=1}^m \mathbb{E}\left[f_{\eta_j}|\mathcal{G}^{\eta_j}\right](B_{\eta_j}-B_{u_j})\Big|^2\Big)&\\ \le \sum_{k=1}^p \mathbb{E}\left|(f_{\xi_k}^2 - f_{\eta_k}^2)(v_k-u_k)\right|&\\ \le \varepsilon.&\end{aligned}$$

Therefore, f is McShane backwards Itô-integrable on $[0,T]$. The proof is now complete. □

As a consequence of Theorem 6.3 and Theorem 6.4 we have the following corollary.

Corollary 6.5. *Let $f:\Omega\times[0,T]\to\mathbb{R}$ be an $L^2(\Omega)$-process. If f is backwards-Itô integrable on $[0,T]$ then f is McShane backwards Itô-integrable on $[0,T]$.*

We shall prove that the converse of the above corollary is true.

Theorem 6.6. *Let $f : \Omega \times [0,T] \to \mathbb{R}$ be an $L^2(\Omega)$-process. If f is McShane backwards Itô-integrable then f is backwards-Itô integrable on $[0,T]$.*

Proof. Let f be McShane backwards Itô-integrable with primitive $F(u,v)$ on $[0,T]$ and let $\varepsilon > 0$. Then by Henstock's lemma, there exist a positive function δ on $[0,T]$ and a positive number η such that for every η−δ-fine division $D = \{((u_i, v_i], \xi_i)\}_{i=1}^n$ of $[0,T]$ we have

$$\mathbb{E}\left((D)\sum_{i=1}^{n}\left\{\mathbb{E}\left(f_{\xi_i}|\mathcal{G}^{v_i}\right)(B_{v_i}-B_{u_i}) - F(u_i,v_i)\right\}^2\right) \leq \frac{\varepsilon}{4}. \tag{6.2}$$

Let $D_1 = \{((u_i, \xi_i], \xi_i)\}_{i=1}^n$ be a backwards δ-fine partial division of $[0,T]$ that fails to cover $[0,T]$ by measure at most η. Then D_1 is an $\eta - \delta$-fine partial division of $[0,T]$.

Let $\{((u_j, v_j], \xi_j)\}_{j=1}^N$ be a finite collection of subintervals of $(0,T]$, such that

$$\bigcup_{j=1}^{N}(u_j, v_j] = (0,T] \setminus \bigcup_{i=1}^{n}(u_i, \xi_i].$$

Then

$$\sum_{j=1}^{N}(v_j - u_j) < \eta.$$

By Theorem 5.4,

$$\mathbb{E}\left(\left|\sum_{j=1}^{N} F(u_j, v_j)\right|^2\right) \leq \frac{\varepsilon}{4}.$$

Therefore,

$$\mathbb{E}\left(\left|(D_1)\sum_{i=1}^{n}\mathbb{E}\left(f_{\xi_i}|\mathcal{G}^{\xi_i}\right)(B_{\xi_i} - B_{u_i}) - F(0,T)\right|^2\right) < \varepsilon.$$

The proof is now complete. □

Finally, by Theorem 6.6 and Corollary 6.5 we have the following result.

Corollary 6.7. *Let $f : \Omega \times [0,T] \to \mathbb{R}$ be an $L^2(\Omega)$-process. Then f is McShane backwards Itô-integrable if and only if f is backwards-Itô integrable on $[0,T]$.*

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