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On the Fine Spectra of the Generalized Forward Difference Operator Δ^v Over the Sequence Space ℓ_1

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Abstract : The main purpose of this paper is to determine the fine spectrum of the forward difference operator over the sequence space ℓ_1.

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1 Introduction

Srivastava and Kumar [1] introduced the generalized difference operator Δ_v on the sequence space c_0 as follows: $\Delta_v : c_0 \longrightarrow c_0$ is defined by

$$\Delta_v x = \Delta_v(x_n) = (v_n x_n - v_{n-1} x_{n-1})_{n=0}^{\infty} \quad \text{with} \quad x_{-1} = 0,$$

where (v_k) is either constant or strictly decreasing sequence of positive real numbers satisfying

$$\lim_{k\to\infty} v_k = v > 0 \tag{1.1}$$

and

$$v_0 \leq 2v. \tag{1.2}$$

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In this paper, we introduce a class of a generalized forward difference operator Δ^v on the sequence space ℓ_1 as follows: $\Delta^v : \ell_1 \longrightarrow \ell_1$ is defined by

$$\Delta^v x = \Delta^v(x_n) = (v_n x_n - v_{n+1}x_{n+1})_{n=0}^{\infty}.$$

It is easy to verify that the operator Δ^v can be represented by the matrix,

$$\Delta^v = \begin{bmatrix} v_0 & -v_1 & 0 & 0 & 0 & \cdots \\ 0 & v_1 & -v_2 & 0 & 0 & \cdots \\ 0 & 0 & v_2 & -v_3 & 0 & \cdots \\ 0 & 0 & 0 & v_3 & -v_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

In functional analysis, the spectrum of an operator generalizes the notion of eigenvalues for matrices. The spectrum of an operator over a Banach space is partitioned into three parts, which are the point spectrum, the continuous spectrum and the residual spectrum. The calculation of three parts of the spectrum of an operator is called calculating the fine spectrum of the operator. Several authors have studied the spectrum and fine spectrum of linear operators defined by some particular limitation matrices over some sequence spaces. We introduce knowledge in the existing literature concerning the spectrum and the fine spectrum. The fine spectrum of the Cesaro operator on the sequence space ℓ_p for $(1 < p < \infty)$ has been studied by Gonzalez [2]. Also, Wenger [3] examined the fine spectrum of the integer power of the Cesaro operator over c, and Rhoades [4] generalized this result to the weighted mean methods. Reade [5] worked the spectrum of the Cesaro operator over the sequence space c_0. Okutoyi [6] computed the spectrum of the Cesaro operator over the sequence space bv. The fine spectrum of the Rhally operators on the sequence spaces c_0 and c is studied by Yildirim [7]. The fine spectra of the Cesaro operator over the sequence spaces c_0 and bv_p have determined by Akhmedov and Basar [8, 9]. Akhmedov and Basar [10, 11] have studied the fine spectrum of the difference operator Δ over the sequence spaces ℓ_p, and bv_p, where $(1 \leq p < \infty)$. The fine spectrum of the Zweier matrix as an operator over the sequence spaces ℓ_1 and bv_1 have been examined by Altay and Karakus [12]. Altay and Basar [13, 14] have determined the fine spectrum of the difference operator Δ over the sequence spaces c_0, c and ℓ_p, where $(0 < p < 1)$. The fine spectrum of the difference operator Δ over the sequence spaces ℓ_1 and bv is investigated by Kayaduman and Furkan [15]. Altun and Karakaya [16, 17] has been studied the fine spectra of Lacunary Matrices and Fine spectra of upper triangular double-band matrices. recently, Srivastava and Kumar [1, 18] has been examined the fine spectrum of the generalized difference operator Δ_v over the sequence spaces c_0 and ℓ_1.

In this work, our purpose is to determine the fine spectra of the generalized forward difference operator Δ^v as an operator over the sequence space ℓ_1.

2 Preliminaries

By w, we denote the space of all real or complex valued sequences. Any vector subspace of w is called a sequence space. Let μ and ν be two sequence spaces and $A = (a_{n,k})$ be an infinite matrix operator of real or complex numbers $a_{n,k}$, where $n, k \in \{0, 1, 2, \ldots\}$. We say that A defines a matrix mapping from μ into ν and denote it by $A : \mu \longrightarrow \nu$, if for every sequence $x = \big(x_k\big) \in \mu$ the sequence $Ax = \big((Ax)_n\big)$, the A-transform of x, is in ν, where $(Ax)_n = \sum_{k=0}^{\infty} a_{n,k}x_k$.

Let X and Y be Banach spaces and $T : X \longrightarrow Y$, also be a bounded linear operator. By $R(T)$, we denote the range of T, i.e.,

$$R(T) = \{y \in Y : y = Tx, x \in X\}.$$

By $B(X)$, we denote the set of all bounded linear operator on X into itself. If X is any Banach space and $T \in B(X)$ then the *adjoint* T^* of T is a bounded linear operator on the dual X^* of X defined by $(T^*\psi)(x) = \psi(Tx)$ for all $\psi \in X^*$ and $x \in X$ with $\|T\| = \|T^*\|$.

Let $X \neq \Theta$ be a complex normed space and $T : \mathcal{D}(T) \longrightarrow X$, also be a bounded linear operator with domain $\mathcal{D} \subseteq X$. With T, we associate the operator $T_\lambda = T - \lambda I$, where λ is a complex number and I is the identity operator on $\mathcal{D}(T)$, if T_λ has an inverse, which is linear, we denote it by T_λ^{-1}, that is

$$T_\lambda^{-1} = (T - \lambda I)^{-1}$$

and call it the *resolvent* operator of T.

The name resolvent is appropriate, since T_λ^{-1} helps to solve the equation $T_\lambda x = y$. Thus, $x = T_\lambda^{-1}y$ provided T_λ^{-1} exists. More important, the investigation of properties of T_λ^{-1} will be basic for an understanding of the operator T itself. Naturally, many properties of T_λ and T_λ^{-1} depend on λ, and spectral theory is concerned with those properties. For instance, we shall be interested in the set of all λ in the complex plane such that T_λ^{-1} exists. Boundedness of T_λ^{-1} is another property that will be essential. We shall also ask for what λ the domain of T_λ^{-1} is dense in X, to name just a few aspects. For our investigation of T, T_λ and T_λ^{-1}, we shall need some basic concepts in spectral theory which are given as follows (see [19, pp. 370–371]).

Definition 2.1. Let $X \neq \Theta$ be a complex normed space and $T : \mathcal{D}(T) \longrightarrow X$, be a linear operator with domain $\mathcal{D} \subseteq X$. A *regular* value of T is a complex number λ such that

(R1) T_λ^{-1} exists;

(R2) T_λ^{-1} is bounded;

(R3) T_λ^{-1} is defined on a set which is dense in X.

The *resolvent* set $\rho(T, X)$ of T is the set of all *regular* value λ of T. Its complement $\sigma(T, X) = \mathbb{C} - \rho(T, X)$ in the complex plane $\mathbb{C}$ is called the *spectrum* of T. Furthermore, the spectrum $\sigma(T,X)$ is partitioned into three disjoint sets as follows: The *point spectrum* $\sigma_p(T,X)$ is the set of all $\lambda \in \mathbb{C}$ such that T_λ^{-1} dose not exist. The element of $\sigma_p(T,X)$ is called *eigenvalue* of T. The *continuous spectrum* $\sigma_c(T,X)$ is the set of all $\lambda \in \mathbb{C}$ such that T_λ^{-1} exists and satisfies $(R3)$ but not $(R2)$, that is, T_λ^{-1} is unbounded. The *residual spectrum* $\sigma_r(T,X)$ is the set of all $\lambda \in \mathbb{C}$ such that T_λ^{-1} exists but do not satisfy $(R3)$, that is, the domain of T_λ^{-1} is not dense in X. The condition $(R2)$ may or may not holds good.

Goldberg's classification of operator $T_\lambda = (T-\lambda I)$ (see [20, pp. 58–71]): Let X be a Banach space and $T_\lambda = (T-\lambda I) \in B(X)$, where λ is a complex number. Again let $R(T_\lambda)$ and T_λ^{-1} be denote the range and inverse of the operator T_λ, respectively. Then following possibilities may occur:

(A) $R(T_\lambda) = X$,

(B) $R(T_\lambda) \neq \overline{R(T_\lambda)} = X$,

(C) (C) $\overline{R(T_\lambda)} \neq X$,

and

(1) T_λ is injective and T_λ^{-1} is continuous,

(2) T_λ is injective and T_λ^{-1} is discontinuous,

(3) T_λ is not injective.

If these possibilities are combined in all possible ways, nine different states are created. These are labelled by: $A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2$ and C_3. If λ is a complex number such that $T_\lambda \in A_1$ *or* $T_\lambda \in B_1$, then λ is in the resolvent set $\rho(T,X)$ of T on X. The other classifications give rise to the fine spectrum of T. We use $\lambda \in B_2\sigma(T,X)$ means the operator $T_\lambda \in B_2$, i.e. $R(T_\lambda) \neq \overline{R(T_\lambda)} = X$ and T_λ is injective but T_λ^{-1} is discontinuous. Similarly others.

Lemma 2.2 ([20, pp. 59])**.** *A linear operator T has a dense range if and only if the adjoint T^* is one to one.*

Lemma 2.3 ([20, pp. 60])**.** *The adjoint operator T^* is onto if and and only if T has a bounded inverse.*

Lemma 2.4. *The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(\ell_1)$ from ℓ_1 to itself if and only if the supremum of ℓ_1 norms of the columns of A is bounded.*

3 Main Results

In this section, we compute spectrum, the point spectrum, the continuous spectrum and the residual spectrum of the generalized forward difference operator Δ^v over the sequence space ℓ_1.

Theorem 3.1. *The operator $\Delta^v : \ell_1 \longrightarrow \ell_1$ is a bounded linear operator and*

$$\|\Delta^v\| = 2\sup_k(v_k).$$

Proof. It is elementary. □

Theorem 3.2. *Point spectrum of the operator Δ^v over ℓ_1 is given by*

$$\sigma_p(\Delta^v, \ell_1) = \{\lambda \in C : |\lambda - v| < v\}.$$

Proof. The proof of this theorem is divided into two cases.

Cases(i): Suppose (v_k) is a constant sequence, say $v_k = v$ for all k. Consider $\Delta^v x = \lambda x$, for $x \neq \mathbf{0} = (0, 0, 0, \ldots)$ in ℓ_1, which gives

$$\begin{aligned} v_0x_0 - v_1x_1 &= \lambda x_o \\ v_1x_1 - v_2x_2 &= \lambda x_1 \\ v_2x_2 - v_3x_3 &= \lambda x_2 \\ &\vdots \\ v_kx_k - v_{k+1}x_{k+1} &= \lambda x_k \\ &\vdots \end{aligned}$$

If $x_0 = 0$, then $x_k = 0$ for all k. Hence $x_0 \neq 0$ and solving the equation above, we get

$$x_k = \left(\frac{v - \lambda}{v}\right)^k x_0, \quad k \in \mathbf{N}.$$

Hence $\lambda \in \sigma_p(\Delta^v, \ell_1)$ if and only if $|\lambda - v| < v$.

Cases(ii): Suppose (v_k) is a strictly decreasing sequence. Consider $\Delta^v x = \lambda x$, for $x \neq \mathbf{0} = (0, 0, 0, \ldots)$ in ℓ_1, which gives system of equations above, solving this equations, we get

$$x_n = \prod_{i=1}^{n} \left(\frac{v_{i-1} - \lambda}{v_i}\right) x_0 \quad \text{for all } n \in \mathbf{N}.$$

Now suppose $\lambda \in C$ with $|\lambda - v| < v$, then $\lim_{n\to\infty} |\frac{v_{n-1}-\lambda}{v_n}| < 1$. Therefore

$$\lim_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| = \lim_{n\to\infty} \left|\frac{v_n - \lambda}{v_{n+1}}\right| < 1.$$

This means that $(x_n) \in \ell_1$, and consequently

$$\{\lambda \in C : |\lambda - v| < v\} \subseteq \sigma_p(\Delta^v, \ell_1).$$

Conversely it is required to show

$$\sigma_p(\Delta^v, \ell_1) \subseteq \{\lambda \in C : |\lambda - v| < v\}.$$

Let $\lambda \in C$ with $|\lambda - v| \geq v$. Clearly, $\lambda = v$ as well as $\lambda = v_k$, for all k do not satisfied. So, $\lambda \neq v$ and $\lambda \neq v_k$, for all k. Then $\lim_{n\to\infty} |\frac{v_{n-1}-\lambda}{v_n}| \geq 1$. This means that $|\frac{v_{n-1}-\lambda}{v_n}| \geq 1$ for large n, and consequently

$$\lim_{n\to\infty} |x_n| = \lim_{n\to\infty} \left|\frac{(v_0-\lambda)(v_1-\lambda)\cdots(v_{n-1}-\lambda)}{v_1 v_2 \cdots v_n}\right| x_0 \neq 0.$$

This shows that $\sigma_p(\Delta^v, \ell_1) \subseteq \{\lambda \in C : |\lambda - v| < v\}$. And this completes the proof. □

If $T : \ell_1 \longrightarrow \ell_1$ is a bounded linear operator with matrix A, then it is known that the adjoint operator $T^* : \ell_1^* \longrightarrow \ell_1^*$ is defined by the transpose of the matrix A. The dual space of ℓ_1 is isomorphic to ℓ_∞, the space of all bounded sequences, with the norm $\|x\| = \sup_k |x_k|$. We now obtain spectrum of the dual operator $(\Delta^v)^*$ of Δ^v over the space ℓ_1^*.

Theorem 3.3. *The point spectrum of the operator over* ℓ_1^* *is*

$$\sigma_p((\Delta^v)^*, c_0^*) = \emptyset.$$

Proof. The proof of this theorem is divided into two cases.

Cases(i): Suppose (v_k) is a constant sequence, say $v_k = v$ for all k. Consider $(\Delta^v)^* f = \lambda f$, for $f \neq \mathbf{0} = (0,0,0,\ldots)$ in $\ell_1^* \cong \ell_\infty$, where

$$(\Delta^v)^* = \begin{bmatrix} v_0 & 0 & 0 & 0 & 0 & \cdots \\ -v_1 & v_1 & 0 & 0 & 0 & \cdots \\ 0 & -v_2 & v_2 & 0 & 0 & \cdots \\ 0 & 0 & -v_2 & v_3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad and \quad f = \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \end{bmatrix}$$

this gives

$$\begin{aligned} v_0 f_0 &= \lambda f_o \\ -v_1 f_0 + v_1 f_1 &= \lambda f_1 \\ -v_2 f_1 + v_2 f_2 &= \lambda f_2 \\ &\vdots \\ -v_k f_{k-1} + v_k f_k &= \lambda f_k \\ &\vdots \end{aligned}$$

Let f_m be the first non-zero entry of the sequence (f_n). So we get $-vfm_1 + vf_m = \lambda f_m$ which implies $\lambda = v$ and from the equation $-vf_m + vf_{m+1} = \lambda f_{m+1}$ we get $f_m = 0$, which is a contradiction to our assumption. Therefore,

$$\sigma_p((\Delta^v)^*, \ell_1^*) = \emptyset.$$

Cases(ii): Suppose (v_k) is a strictly decreasing sequence. Consider $(\Delta^v)^* f = \lambda f$, for $f \neq \mathbf{0} = (0, 0, 0, \ldots)$ in $\ell_1^* \cong \ell_\infty$, which gives above system of equations. Hence, for all $\lambda \notin \{v_0, v_1, v_2, \ldots\}$, we have $f_k = 0$ for all k, which is a contradiction. So $\lambda \notin \sigma_p((\Delta^v)^*, \ell_1^*)$. This shows that

$$\sigma_p((\Delta^v)^*, \ell_1^*) \subseteq \{v_0, v_1, v_2, \ldots\}.$$

Let $\lambda = v_m$ for some m. Then $f_0 = f_1 = \cdots = f_{m-1} = 0$. Now if $f_m = 0$, then $f_k = 0$ for all k, which is a contradiction. Also if $f_m \neq 0$, then

$$f_{k+1} = \frac{v_{k+1}}{v_{k+1} - v_m} f_k, \quad \text{for all } k \geq m,$$

and hence,

$$\left| \frac{f_{k+1}}{f_k} \right| = \left| \frac{v_{k+1}}{v_{k+1} - v_m} \right| > 1 \quad \text{for all } k \geq m,$$

since $v_0 \leq 2v$. Then, $f \notin \ell_1^*$. Thus $\sigma_p((\Delta^v)^*, \ell_1^*) = \emptyset$. □

Theorem 3.4. *For any $\lambda \in C$, $\Delta^v_\lambda : \ell_1 \longrightarrow \ell_1$ has a dence range.*

Proof. By Theorem 3.3, $\sigma_p((\Delta^v)^*, \ell_1^*) = \emptyset$. Hence $(\Delta^v)^* - \lambda I$ is one to one for all λ. By applying Lemma 2.2, we get the result. □

Corollary 3.5. *Residual spectrum $\sigma_r(\Delta^v, \ell_1)$ of operator Δ^v over ℓ_1 is*

$$\sigma_r(\Delta^v, \ell_1) = \emptyset$$

Theorem 3.6. *The spectrum of Δ^v on ℓ_1 is given by*

$$\sigma(\Delta^v, \ell_1) = \{\lambda \in C : |\lambda - v| \leq v\}.$$

Proof. The proof of this theorem is divided into two cases.

Cases(i): Suppose (v_k) is a constant sequence, say $v_k = v$ for all k, and let $f \in \ell_\infty$. Consider $(\Delta^v_\lambda)^* x = f$. Then we have the linear system of equations

$$\begin{aligned} (v-\lambda)x_0 &= f_o \\ -vx_0 + (v-\lambda)x_1 &= f_1 \\ -vx_1 + (v-\lambda)x_2 &= f_2 \\ &\vdots \\ -vf_{k-1} + (v-\lambda)x_k &= f_k \\ &\vdots \end{aligned}$$

solving the equations, we get

$$x_k = \frac{1}{v-\lambda}\sum_{i=0}^{k}\left(\frac{v}{v-\lambda}\right)^{k-i} f_i$$

for all k. Therefore

$$|x_k| \leq \frac{1}{|v-\lambda|}\sum_{i=0}^{\infty}\left|\frac{v}{v-\lambda}\right|^{i} ||f||_\infty.$$

Now for $|v| < |\lambda - v|$, we can see that

$$\|x\|_\infty \leq \frac{1}{|v-\lambda| - |v|}\|f\|_\infty.$$

Hence, for $|v| < |\lambda - v|$, $(\Delta_\lambda^v)^*$ is onto, and by Lemma 2.3, Δ_λ^v has a bounded inverse. This means that

$$\sigma_c(\Delta^v, \ell_1) \subseteq \{\lambda \in C : |\lambda - v| \leq v\}.$$

Combining this with Theorem 3.2 and Corollary 3.5, we get

$$\{\lambda \in C : |\lambda - v| < v\} \subseteq \sigma(\Delta^v, \ell_1) \subseteq \{\lambda \in C : |\lambda - v| \leq v\}.$$

Since the spectrum of any bounded operator is closed, we have

$$\sigma(\Delta^v, \ell_1) = \{\lambda \in C : |\lambda - v| \leq v\}.$$

Cases(ii): Suppose (v_k) is a strictly decreasing sequence, and let $f \in \ell_\infty$. Consider $(\Delta_\lambda^v)^* x = f$. Then we have the linear system of equations

$$\begin{aligned}
(v_0 - \lambda)x_0 &= f_o \\
-v_1 x_0 + (v_1 - \lambda)x_1 &= f_1 \\
-v_2 x_1 + (v_2 - \lambda)x_2 &= f_2 \\
&\vdots \\
-v_k f_{k-1} + (v_k - \lambda)x_k &= f_k \\
&\vdots
\end{aligned}$$

solving the equations, for $x = (x_k)$ in terms of f, we get

$$\begin{aligned}
x_k = {} & \frac{v_1 v_2 \cdots v_k}{(v_0-\lambda)(v_1-\lambda)\cdots(v_k-\lambda)} f_0 + \frac{v_2 v_3 \cdots v_k}{(v_1-\lambda)(v_2-\lambda)\cdots(v_k-\lambda)} f_1 \\
& + \cdots + \frac{v_k}{(v_{k-1}-\lambda)(v_k-\lambda)} f_{k-1} + \frac{1}{v_k - \lambda} f_k, \quad \text{for all } k.
\end{aligned}$$

Then $|x_k| \leq S_k \|f\|_\infty$, where

$$S_k = \frac{1}{|v_k - \lambda|} + \frac{v_k}{|v_{k-1} - \lambda||v_k - \lambda|} + \frac{v_{k-1}v_k}{|v_k - \lambda||v_{k-1} - \lambda||v_{k-2} - \lambda|} + \cdots + \frac{v_1 v_2 \cdots v_k}{|v_0 - \lambda||v_1 - \lambda| \cdots |v_k - \lambda|}.$$

Clearly, each S_k is finite. Now we prove that $\sup_k S_k$ is finite. Since

$$\lim_{n\to\infty} |\frac{v_k}{v_{k-1} - \lambda}| = \frac{v}{|v - \lambda|} = p < 1.$$

Then, there exists $k \in N$ such that $\frac{v_n}{|v_{n-1}-\lambda|} < p_0 < 1$, for all $n \geq k+1$ and so we get

$$S_{n+k} \leq \frac{1}{|v_{n+k} - \lambda|} \times \left(\frac{v_1 v_2 \cdots v_k}{|v_0 - \lambda||v_1 - \lambda| \cdots |v_{k-1} - \lambda|} p_0^n + \frac{v_2 v_3 \cdots v_k}{|v_1 - \lambda| \cdots |v_{k-1} - \lambda|} p_0^{n-1} + \cdots + p_0 + 1 \right).$$

If we put $M = \max\{\frac{v_j v_{j+1} \cdots v_k}{|v_{j-1}-\lambda||v_j-\lambda| \cdots |v_k-\lambda|} : 1 \leq j \leq k\}$, then we have

$$S_{n+k} \leq \frac{M}{|v_{n+k} - \lambda|}\left(1 + p_0 + p_0^2 + \cdots + p_0^n\right) \leq \frac{M}{|v_{n+k} - \lambda|}\left(1 + p_0 + p_0^2 + \cdots\right).$$

But, for large n, we have $\frac{1}{|v_{n+k}-\lambda|} < d < \frac{1}{v}$ and so $S_{n+k} \leq \frac{Md}{1-p_0}$, for all $n \geq k+1$. Thus, $\sup_k S_k < \infty$. This shows that $||x||_\infty \leq \sup_k S_k ||f||_\infty < \infty$. Therefore $x \in \ell_\infty$. Hence, for $v < |\lambda - v|$, $(\Delta_\lambda^v)^*$ is onto, and by Lemma 2.3, Δ_λ^v has a bounded inverse. This means that

$$\sigma_c(\Delta^v, \ell_1) \subseteq \{\lambda \in C : |\lambda - v| \leq v\}.$$

Combining this with Theorem 3.2 and Corollary 3.5, we get

$$\{\lambda \in C : |\lambda - v| < v\} \subseteq \sigma(\Delta^v, \ell_1) \subseteq \{\lambda \in C : |\lambda - v| \leq v\}.$$

Since the spectrum of any bounded operator is closed, we have

$$\sigma(\Delta^v, \ell_1) = \{\lambda \in C : |\lambda - v| \leq v\}.$$

□

Theorem 3.7. *Continuous spectrum $\sigma_c(\Delta^v, \ell_1)$ of operator Δ^v over ℓ_1 is*

$$\sigma_c(\Delta^v, \ell_1) = \{\lambda \in C : |\lambda - v| = v\}.$$

Proof. Since $\sigma_r(\Delta^v, \ell_1) = \emptyset$, $\sigma_p(\Delta^v, \ell_1) = \{\lambda \in C : |\lambda - v| < v\}$ and $\sigma(\Delta^v, \ell_1)$ is the disjoint union of the parts $\sigma_p(\Delta^v, \ell_1)$, $\sigma_r(\Delta^v, \ell_1)$ and $\sigma_c(\Delta^v, \ell_1)$, we deduce that $\sigma_c(\Delta^v, \ell_1) = \{\lambda \in C : |\lambda - v| = v\}$. □

Theorem 3.8. *If* $|\lambda - v| < v$, *then* $\lambda \in A_3\sigma(\Delta^v, \ell_1)$.

Proof. Let $|\lambda - v| < v$. Then by Theorem 3.2, $\lambda \in (\mathbf{3})$ it remains to prove that Δ^v_λ is surjective when $|\lambda - v| < v$. Let $y = (y_0, y_1, y_2, \dots) \in \ell_1$ and consider the equation $\Delta^v_\lambda x = y$. Then we have the linear system of equations

$$\begin{aligned}(v_0 - \lambda)x_0 - v_1 x_1 &= y_o\\ (v_1 - \lambda)x_1 - v_2 x_2 &= y_1\\ (v_2 - \lambda)x_2 - v_3 x_3 &= y_2\\ &\vdots\\ (v_k - \lambda)x_k - v_{k+1} x_{k+1} &= y_k\\ &\vdots\end{aligned}$$

Now, set $x_0 = 0$ and by solving these equations, we get $x_1 = -\frac{1}{v_1} y_0$ and

$$x_k = \frac{-1}{v_k}\left(\sum_{i=0}^{k-2}\left[\prod_{j=i+1}^{k-1}\big(1 - \frac{\lambda}{v_j}\big)\right] y_i + y_{k-1}\right) \quad \text{for all } k \geq 2.$$

Then $\sum_k |x_k| \leq \sum_k S_k |y_k|$, where

$$S_k = \frac{1}{v_{k+1}} + \frac{1}{v_{k+2}}\frac{|v_{k+1} - \lambda|}{v_{k+1}} + \frac{1}{v_{k+3}}\frac{|v_{k+1} - \lambda|}{v_{k+1}}\frac{|v_{k+2} - \lambda|}{v_{k+2}} + \cdots, \quad \text{for all } k.$$

Let

$$\begin{aligned}S_{n,k} = &\frac{1}{v_{k+1}} + \frac{1}{v_{k+2}}\frac{|v_{k+1} - \lambda|}{v_{k+1}} + \frac{1}{v_{k+3}}\frac{|v_{k+1} - \lambda|}{v_{k+1}}\frac{|v_{k+2} - \lambda|}{v_{k+2}}\\ &+ \cdots + \frac{1}{v_{k+n+1}}\frac{|v_{k+1} - \lambda|}{v_{k+1}}\frac{|v_{k+2} - \lambda|}{v_{k+2}}\cdots\frac{|v_{k+n} - \lambda|}{v_{k+n}} \quad \text{for all } k, n.\end{aligned}$$

Then

$$S_n = \lim_{k\to\infty} S_{n,k} = \frac{1}{v} + \frac{|v - \lambda|}{v^2} + \frac{|v - \lambda|^2}{v^3} + \cdots + \frac{|v - \lambda|^n}{v^{n+1}}.$$

Now for $|\lambda - v| < v$, we can see that

$$S = \lim_{n\to\infty} S_n = \frac{1}{v} + \frac{|v - \lambda|}{v^2} + \frac{|v - \lambda|^2}{v^3} + \cdots < \infty,$$

hence (S_k) is a sequence of positive real numbers which has a lim S. Therefore, (S_k) is bounded and $\sup_k S_k < \infty$. Thus

$$\sum_k |x_k| \leq \sup_k S_k \sum_k |y_k| < \infty.$$

This shows that $x \in \ell_1$. □

Theorem 3.9. *Let* v_k *be a constant sequence and* $|\lambda - v| = v$. *Then* $\lambda \in B_2\sigma(\Delta^v, \ell_1)$.

Proof. Suppose $v_k = v$ for all k. By Theorem 3.7, $\lambda \in A_2 \cup B_2$. To prove $\lambda \in B_2$, we need to show that Δ^v is not surjective when λ satisfies $|\lambda - v| = v$. Define $y = (y_0, y_1, y_2, \dots) \in \ell_1$ by

$$y_k = \left(\frac{v-\lambda}{v}\right)^k \frac{1}{k^2+1}.$$

Suppose $x \in \ell_1$ with $\Delta^v_\lambda x = y$. Then we have the linear system equations

$$\begin{aligned}
(v-\lambda)x_0 - vx_1 &= 1\\
(v-\lambda)x_1 - vx_2 &= \left(\frac{v-\lambda}{v}\right)\frac{1}{1^2+1}\\
(v-\lambda)x_2 - vx_3 &= \left(\frac{v-\lambda}{v}\right)^2\frac{1}{2^2+1}\\
&\vdots
\end{aligned}$$

solving x_n by means of x_0, we get

$$x_n - \left(\frac{v-\lambda}{v}\right)^n x_0 = -\frac{1}{v}\left(\frac{v-\lambda}{v}\right)^{n-1}\left(1+\frac{1}{2}+\frac{1}{5}+\cdots+\frac{1}{(n-1)^2+1}\right).$$

Now, By taking absolute value of both sides and using the triangle inequality we get

$$\frac{1}{v}\left(1+\frac{1}{2}+\frac{1}{5}+\cdots+\frac{1}{(n-1)^2+1}\right) \le |x_0| + |x_n|.$$

Then we have $\lim_{n\to\infty}|x_n| \neq 0$, which contradicts the fact that $x \in \ell_1$. Hence, there is no $x \in \ell_1$ satisfying $\Delta^v_\lambda x = y$. So, Δ^v_λ is not surjective. □

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