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Some Generalized Difference Sequence Spaces

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Abstract : The idea of difference sequences was introduced by Kizmaz [1]. In this paper, we define some new sequence spaces and give some topological properties of these new spaces. The results which we give in this paper are more general than those of Kizmaz [1], Et and Esi [3], Basarir [6] and Et *et al.*[2].

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1 Introduction

Let w denote the set of all sequences of complex numbers. Let l_∞, c and c_0 be the Banach spaces of bounded, convergent and null sequences $x = (x_k)$ normed by as usual by $\|x\|_\infty = \sup_k |x_k|$.

Recently, Kizmaz [1] defined the sequence spaces

$$l_\infty(\Delta) = \Big\{x = (x_k) : \Delta x \in l_\infty\Big\},$$

$$c(\Delta) = \Big\{x = (x_k) : \Delta x \in c\Big\},$$

and

$$c_0(\Delta) = \Big\{x = (x_k) : \Delta x \in c_0\Big\}$$

where $\Delta x = \big(x_k - x_{k+1}\big)$. These are Banach spaces with the norm

$$\|x\|_\Delta = |x_1| + \|\Delta x\|_\infty .$$

Let $p = (p_k)$ be a sequence of real numbers such that $p_k > 0$ for all k and $\sup_k p_k = H < \infty$, $v = (v_k)$ be any fixed sequence of nonzero complex numbers and $m \in \mathbb{N}$ be fixed. This assumption is made throughout the rest of this paper. Now we define the following sequence sets

$$
\begin{aligned}
l_\infty(\Delta_v^m, s, p) &= \Big\{x = (x_k) \in w : \sup_k k^{-s} |\Delta_v^m x_k|^{p_k} < \infty,\ s \geq 0\Big\},\\
c(\Delta_v^m, s, p) &= \Big\{x = (x_k) \in w : k^{-s} |\Delta_v^m x_k - L|^{p_k} \to 0,\ (k \to \infty),\\
&\qquad s \geq 0,\ \text{for some } L\Big\},\\
c_0(\Delta_v^m, s, p) &= \Big\{x = (x_k) \in w : k^{-s} |\Delta_v^m x_k|^{p_k} \to 0,\ (k \to \infty),\ s \geq 0\Big\},
\end{aligned}
$$

where

$$\Delta_v^o x_k = (v_k x_k), \Delta_v x_k = (v_k x_k - v_{k+1} x_{k+1}), \Delta_v^m x_k = (\Delta_v^{m-1} x_k - \Delta_v^{m-1} x_{k+1})$$

and so that

$$\Delta_v^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} v_{k+i} x_{k+i}.$$

We get the following sequence spaces from the above sequence spaces by choosing some of the special p, m, s and v. Some examples :

If $s = 0$, $m = 1$, $v = (1, 1, 1, ...)$ and $p_k = 1$ for all k, we have $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$, which were defined by Kizmaz [1].

If $s = 0$ and $p_k = 1$ for all k, we have the following sequence spaces which were defined by Et and Esi [3]

$$
\begin{aligned}
l_\infty(\Delta_v^m) &= \Big\{x = (x_k) \in w : \Delta_v^m x \in l_\infty\Big\},\\
c(\Delta_v^m) &= \Big\{x = (x_k) \in w : \Delta_v^m x \in c\Big\},\\
c_0(\Delta_v^m) &= \Big\{x = (x_k) \in w : \Delta_v^m x \in c_0\Big\}.
\end{aligned}
$$

If $s = 0$, $m = 0$ and $v = (1, 1, 1, ...)$, we have the following sequence spaces which were defined by Maddox [4].

$$
\begin{aligned}
l_\infty(p) &= \Big\{x = (x_k) \in w : \sup_k |x_k|^{p_k} < \infty\Big\},\\
c(p) &= \Big\{x = (x_k) \in w : |x_k - L|^{p_k} \to 0, (k \to \infty),\ \text{for some } L\Big\},\\
c_0(p) &= \Big\{x = (x_k) \in w : |x_k|^{p_k} \to 0, (k \to \infty)\Big\}.
\end{aligned}
$$

If $m = 0, v = (1, 1, 1, ...)$, we have the following sequence spaces which were defined by Basarir [6]

$$
\begin{aligned}
l_\infty(p, s) &= \Big\{x = (x_k) \in w : \sup_k k^{-s} |x_k|^{p_k} < \infty,\ s \geq 0\Big\},\\
c(p, s) &= \Big\{x = (x_k) \in w : k^{-s} |x_k - L|^{p_k} \to 0, (k \to \infty), \text{for some } L,\ s \geq 0\Big\},\\
c_0(p, s) &= \Big\{x = (x_k) \in w : k^{-s} |x_k|^{p_k} \to 0, (k \to \infty),\ s \geq 0\Big\}.
\end{aligned}
$$

If $s = 0, m = 0$ and $p_k = v_k = 1$ for all k, we have l_∞, c and c_0.

If $s = 0$ we have $l_\infty(p)(\Delta_v^m)$, $c(p)(\Delta_v^m)$, $c_0(p)(\Delta_v^m)$ which were defined by Et *et al.*[2].

2 Main Results

In this section we examine some topological properties of the sequence spaces $l_\infty(\Delta_v^m, s, p)$, $c(\Delta_v^m, s, p)$ and $c_0(\Delta_v^m, s, p)$ and investigate some inclusion relations between these spaces.

Theorem 2.1 *The following statements are hold :*

(i) $c_0(\Delta_v^m, s) \subset c(\Delta_v^m, s) \subset l_\infty(\Delta_v^m, s)$ *and the inclusion is strict.*

(ii) $X(\Delta_v^m, s, p) \subset X(\Delta_v^{m+1}, s, p)$ *does not hold in general for any* $X = l_\infty$, c *and* c_0.

Proof. (i) Inclusion relation of these spaces is trivial and the inclusion is strict, for example, if we choose $s = 0$, $x = (1,0,1,0,...)$ and $v = (1,1,1,...)$, then $\Delta_v^m x_k = (-1)^{k+1} 2^{m-1}$ and so $x \in l_\infty(\Delta_v^m, s)$, but $x \notin c(\Delta_v^m, s)$.

(ii) Let $v = (1,1,1,...)$, $p = (p_k)$ and $x = (x_k)$ given by

$$\begin{array}{lll} p_k = 1 & x_k = k^2 & \text{if } k \text{ is odd,} \\ p_k = 2 & x_k = k & \text{if } k \text{ is even,} \end{array}$$

since for $k \geq 1$, $\left|\Delta_v^0 x_k\right|^{p_k} = |x_k|^{p_k} = k^2$, $k^{-3}\left|\Delta_v^0 x_k\right|^{p_k} = k^{-1} \to 0 \ (k \to \infty)$ and for $j \geq 1$

$$|\Delta_v x_{2j}|^{p_{2j}} = \left(2j^2 + 2j + 1\right)^2, (2j)^{-3}|\Delta_v x_{2j}|^{p_{2j}} \geq 2j \to \infty \ (j \to \infty).$$

Now, we can see that $x \in c_0\left(\Delta_v^0, 3, p\right)$ and $x \notin l_\infty\left(\Delta_v^0, 3, p\right)$, which imply that $X(\Delta_v^m, s, p)$ is not a subset of $X\left(\Delta_v^{m+1}, s, p\right)$ for any $X = l_\infty, c$ and c_0 with $m = 0$ and $s = 3$.

If X is a linear space over the field $\mathbb{C}$, then a paranorm on X is a function $g : g(\Theta) = 0$, where $\Theta = (0,0,0,...)$, $g(-x) = g(x)$, $g(x+y) \leq g(x) + g(y)$ and $|\lambda - \lambda_0| \to 0$, $g(x - x_0)$ imply $g(\lambda x - \lambda_0 x_0) \to 0$, where $\lambda, \lambda_0 \in \mathbb{C}$ and $x, x_0 \in X$. A paranormed space is a linear space X with a paranorm g and is written (X, g).□

Theorem 2.2 $c_0(\Delta_v^m, s, p)$, $c(\Delta_v^m, s, p)$ *and* $l_\infty(\Delta_v^m, s, p)$ *are linear spaces over the complex field* $\mathbb{C}$.

Proof. Suppose that

$$M = \max\left(1, \sup_k p_k = H\right).$$

Since $\frac{p_k}{M} \leq 1$, we have for all k, (*See Maddox* [5])

$$|\Delta_v^m(x_k + y_k)|^{p_k/M} \leq (|\Delta_v^m x_k|^{p_k/M} + |\Delta_v^m y_k|^{p_k/M}) \tag{2}$$

and $\forall\lambda\in\mathbb{C}$ $(See\ Maddox\ [4], p.346)$

$$|\lambda|^{p_k/M}\leq\max\left(1,|\lambda|\right). \tag{3}$$

Now the linearity follows from (2) and (3). □

Theorem 2.3 *$c_0\left(\Delta_v^m,s,p\right)$ is a linear topological space over the complex field paranormed by g defined by*

$$g\left(x\right)=\sum_{i=1}^{m}|x_i|+\sup_k k^{-s/M}\left|\Delta_v^m x_k\right|^{p_k/M}$$

where

$$M=\max\left(1,\sup_k p_k=H\right).$$

$l_\infty\left(\Delta_v^m,s,p\right)$ is paranormed by g if $\inf p_k=\vartheta>0$.

Proof. One can easily see that $g\left(\Theta\right)=0$ and $g\left(-x\right)=g\left(x\right)$. The subadditivity of g follows from (2). Let $\lambda\in\mathbb{C}$, $x\in c_0\left(\Delta_v^m,s,p\right)$. The continuity of product follows from the following inequality.

$$g\left(\lambda x\right)\leq\max\left(1,|\lambda|^{H/M}\right)\ g\left(x\right).$$

□

Theorem 2.4 *Let $0<p_k\leq q_k\leq 1$ then $l_\infty\left(\Delta_v^m,s,q\right)$ is a closed subspace of $l_\infty\left(\Delta_v^m,s,p\right)$.*

Proof. Let $x\in l_\infty\left(\Delta_v^m,s,q\right)$. Then $\exists$ a constant $A>1$ such that

$$k^{-s}\left|\Delta_v^m x_k\right|^{q_k}\leq A \qquad (\forall k)$$

and so

$$k^{-s}\left|\Delta_v^m x_k\right|^{p_k}\leq A \qquad (\forall k).$$

Thus $x\in l_\infty\left(\Delta_v^m,s,p\right)$. To show that $l_\infty\left(\Delta_v^m,s,q\right)$ is closed, suppose that $x^i\in l_\infty\left(\Delta_v^m,s,q\right)$ and $x^i\to x\in l_\infty\left(\Delta_v^m,s,p\right)$. Then for every $0<\varepsilon<1$, $\exists\,\mathbb{N}$ such that $\forall k$

$$k^{-s}\left|\Delta_v^m\left(x_k^i-x_k\right)\right|^{p_k}<\varepsilon \qquad (\forall i>N)$$

Now,

$$k^{-s}\left|\Delta_v^m x_k^i-x_k\right|^{q_k}<k^{-s}\left|\Delta_v^m x_k^i-x_k\right|^{p_k}<\varepsilon \qquad (\forall i>N).$$

Therefore $x\in l_\infty(\Delta_v^m,s,q)$. This completes the proof. □

Theorem 2.5 *Let $v=(v_k)$ and $u=(u_k)$ be any fixed sequences of nonzero complex numbers, then* *(i)* *If* $\sup_k k^m \left|v_k^{-1}u_k\right| < \infty$, *then* $l_\infty\left(\Delta_v^m, s\right) \subset l_\infty\left(\Delta_u^m, s\right)$,

(ii) *If* $k^m \left|v_k^{-1}u_k\right| \to L \ \ (k \to \infty)$, *for some* L, *then* $c\left(\Delta_v^m, s\right) \subset c\left(\Delta_u^m, s\right)$,

(iii) *If* $k^m \left|v_k^{-1}u_k\right| \to 0 \ \ (k \to \infty)$, *then* $c_0\left(\Delta_v^m, s\right) \subset c_0\left(\Delta_u^m, s\right)$.

Proof. (i) Suppose that $\sup_k k^m \left|v_k^{-1}u_k\right| < \infty$ and $x \in l_\infty\left(\Delta_v^m, s\right)$. Since

$$\begin{aligned}
k^{-s}\left|\Delta_u^m x_k\right| &= k^{-s}\left|\Delta^{m-1}\left(\Delta_u x_k\right)\right| = k^{-s}\left|\sum_{i=0}^{m-1}(-1)^i\binom{m-1}{i}\Delta\left(x_{k+i}u_{k+i}\right)\right| \\
&= k^{-s}\left|\sum_{i=0}^{m-1}(-1)^i\binom{m-1}{i}\left(x_{k+i}u_{k+i} - x_{k+i+1}u_{k+i+1}\right)\right| \\
&\le k^{-s}\left(\sum_{i=0}^{m-1}(-1)^i\binom{m-1}{i}(k+i)^m\left|v_{k+i}^{-1}u_{k+i}\right|(k+i)^{-m}v_{k+i}x_{k+i}\right) \\
&\quad + k^{-s}\left(\sum_{i=0}^{m-1}(-1)^i\binom{m-1}{i}(k+i+1)^m\left|v_{k+i+1}^{-1}u_{k+i+1}\right|\right. \\
&\quad \left.\times (k+i+1)^{-m}\left|v_{k+i+1}x_{k+i+1}\right|\right).
\end{aligned}$$

So, we obtain $x \in l_\infty\left(\Delta_u^m, s\right)$. This completes the proof. □

If we take $u = (1, 1, 1, ...)$ in Theorem 2.5, then we have the Corollary 2.6.

Corollary 2.6 *The following statements are hold :*

(i) *If* $\sup_k k^m \left|v_k^{-1}\right| < \infty$, *then* $l_\infty\left(\Delta_v^m, s\right) \subset l_\infty\left(\Delta^m, s\right)$,

(ii) *If* $k^m \left|v_k^{-1}\right| \to L \ \ (k \to \infty)$ *for some* L, *then* $c\left(\Delta_v^m, s\right) \subset c\left(\Delta^m, s\right)$,

(iii) *If* $k^m \left|v_k^{-1}\right| \to 0 \ \ (k \to \infty)$, *then* $c_0\left(\Delta_v^m, s\right) \subset c_0\left(\Delta^m, s\right)$.

If we take $v = (1, 1, 1, ...)$ in Theorem 2.5, then we have the Corollary 2.7.

Corollary 2.7 *The following statements are hold :*

(i) *If* $\sup_k k^m \left|u_k\right| < \infty$, *then* $l_\infty\left(\Delta^m, s\right) \subset l_\infty\left(\Delta_u^m, s\right)$,

(ii) *If* $k^m \left|u_k\right| \to L \ \ (k \to \infty)$ *for some* L, *then* $c\left(\Delta^m, s\right) \subset c\left(\Delta_u^m, s\right)$,

(iii) *If* $k^m \left|u_k\right| \to 0 \ \ (k \to \infty)$, *then* $c_0\left(\Delta^m, s\right) \subset c_0\left(\Delta_u^m, s\right)$.

For a subspace Ψ of a linear space is said to be sequence algebra if $x, y \in \Psi$ implies that $x.y = (x_k y_k) \in \Psi$, see Kamptan and Gupta [7]. It is well-known that the sequence space c_0 is sequence algebra. A sequence E is said to be solid (or normal) if $(\lambda_k x_k) \in E$, whenever $(x_k) \in E$ for all sequences of scalars (λ_k) with $|\lambda_k| \le 1$.

Proposition 2.8 *For $X = l_\infty, c$ and c_0, then we obtain*

(i) $X(\Delta_v^m, s, p)$ *is not sequence algebra, in general.*

(ii) $X(\Delta_v^m, s, p)$ *is not solid, in general.*

Proof. (i) This result is clear the following example.

Example 2.9 Let $p_k = 1, v_k = \frac{1}{k^2}, x_k = k^2$ and $y_k = k^2$ for all k. Then we have $x, y \in c_0(\Delta, 0, p)$ but $x.y \notin c_0(\Delta, 0, p)$ with $m = 1$ and $s = 0$.

(ii) This result is clear the following example.

Example 2.10 Let $p_k = 1, \lambda_k = (-1)^k, v_k = \frac{1}{k^2}$ and $x_k = k^2$ for all k, then by $\Delta_v x = (1, 1, 1, ...)$ and $|\Delta_v \lambda_k x_k| = (-2, 0, -2, 0...)$, we have $x \in c(\Delta_v^m, s, p)$ but $\lambda x \notin c(\Delta_v^m, s, p)$ with $m = 1$ and $s = 0$.

The following proposition's proof is a routine verification.

Proposition 2.11 *For $X = l_\infty, c$ and c_0, the we obtain*

(i) $s_1 \leq s_2$ *implies* $X(\Delta_v^m, s_1, p) \subset X(\Delta_v^m, s_2, p)$,

(ii) Let $0 < \inf p_k \leq p_k \leq 1$, *then* $X(\Delta_v^m, s, p) \subset X(\Delta_v^m, s)$,

(iii) Let $1 \leq p_k \leq \sup_k p_k < \infty$, *then* $X(\Delta_v^m, s) \subset X(\Delta_v^m, s, p)$,

(iv) Let $0 < p_k \leq q_k$ *and* $\left(\frac{q_k}{p_k}\right)$ *be bounded, then* $X(\Delta_v^m, s, q) \subset X(\Delta_v^m, s, p)$.

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