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On $(LCS)_{2n+1}$-Manifolds Satisfying Certain Conditions on the Concircular Curvature Tensor

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Abstract : We classify Lorentzian concircular structure manifolds, which satisfy the condition $\tilde{C}(\xi, X) \cdot \tilde{C} = 0, \tilde{C}(\xi, X) \cdot R = 0, \tilde{C}(\xi, X) \cdot S = 0$ and $\tilde{C}(\xi, X) \cdot C = 0$.

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1 Introduction

An $(2n+1)$-dimensional Lorentzian manifold M is smooth connected para contact Hausdorff manifold with Lorentzian metric g, that is, M admits a smooth symmetric tensor field g of type $(0, 2)$ such that for each point $p \in M$, the tensor $g_p : T_pM \times T_pM \to R$ is a non degenerate inner

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product of signature $(-,+,\dots,+)$ where T_pM denotes the tangent space of M at p and R is the real number space. In a Lorentzian manifold (M,g) a vector field ρ defined by

$$g(X,\rho) = A(X)$$

for any vector field$X \in \chi(M)$ is said to be *concircular vector field* [1], if

$$(\nabla_X A)(Y) = \alpha\,[g(X,Y) + \omega(X)A(Y)]$$

where α is a non zero scalar function, A is a 1-form and ω is a closed 1-form.

Let M be a Lorentzian manifold admitting a unit time like concircular vector field ξ, called the characteristic vector field of the manifold. Then we have

$$g(\xi,\xi) = -1 \tag{1.1}$$

Since ξ is the unit concircular vector field, there exists a non-zero 1-form such that

$$g(X,\xi) = \eta(X) \tag{1.2}$$

and hence the equation

$$(\nabla_X \eta)(Y) = \alpha\,[g(X,Y) + \eta(X)\eta(Y)] \quad (\alpha \neq 0) \tag{1.3}$$

holds for all vector field X, Y, where ∇ denotes the operator of covariant differentiation with respect to Lorentzian metric g and α is a non zero scalar function satisfying

$$(\nabla_X \alpha) = (X\alpha) = \rho\eta(X), \tag{1.4}$$

where ρ being a scalar function. If we put

$$\phi X = \frac{1}{\alpha}\nabla_X \xi \tag{1.5}$$

Then from (1.3) and (1.5), we have

$$\phi^2 X = X + \eta(X)\xi, \tag{1.6}$$

from which it follows that ϕ is a symmetric $(1,1)$-tensor. Thus the Lorentzian manifold M together with unit time like concircular vector field ξ, it's associate 1-form η and $(1,1)$–tensor field ϕ is said to be Lorentzian concircular structute manifolds (briefly $(LCS)_{2n+1}$-manifold) [2]. In particular if $\alpha = 1$, then the manifold becomes LP-Sasakian structure of Matsumoto [3].

2 Preliminaries

A differentiable manifold M of dimension $(2n+1)$ is called $(LCS)_{2n+1}$-manifold if it admits a $(1,1)$ -tensor ϕ, a contravarient vector field ξ, a covariant vector field η and a Lorentzian metric g which satisfy the following

$$\eta(\xi) = -1, \tag{2.1}$$

$$\phi^2 = I + \eta \otimes \xi, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.3}$$

$$g(X, \xi) = \eta(X), \tag{2.4}$$

$$\phi\xi = 0, \ \eta(\phi X) = 0. \tag{2.5}$$

for all X, Y in TM. Also in a $(LCS)_{2n+1}$-manifold the following relations are satisfied [4].

$$\eta(R(X,Y)Z) = (\alpha^2 - \rho)\left[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\right], \tag{2.6}$$

$$R(X,Y)\xi = (\alpha^2 - \rho)\left[\eta(Y)X - \eta(X)Y\right], \tag{2.7}$$

$$R(\xi, X)Y = (\alpha^2 - \rho)\left[g(X,Y)\xi - \eta(Y)X\right], \tag{2.8}$$

$$R(\xi, X)\xi = (\alpha^2 - \rho)\left[\eta(X)\xi + X\right], \tag{2.9}$$

$$(\nabla_X \phi)(Y) = \alpha\left[g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\right], \tag{2.10}$$

$$S(X, \xi) = 2n(\alpha^2 - \rho)\left[\eta(X)\right], \tag{2.11}$$

$$S(\phi X, \phi Y) = S(X,Y) + 2n(\alpha^2 - \rho)\eta(X)\eta(Y). \tag{2.12}$$

Definition 2.1. A Lorentzian concircular structure manifold is said to be η-*Einstein* [5] if the Ricci operator Q satisfies

$$Q = aId \ + \ b\eta \otimes \xi,$$

where a and b are smooth functions on the manifolds, In particular if $b = 0$, then M is an Einstein manifolds.

Let (M, g) be an n-dimensional Riemannian manifold, then the Concircular curvature tensor C and the Weyl Conformal curvature tensor C are defined by [6]:

$$\tilde{C}(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}\left[g(Y,Z)X - g(X,Z)Y\right], \tag{2.13}$$

$$\begin{aligned} C(X,Y)Z = R(X,Y)Z - \frac{1}{(n-2)}[&S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ &- g(X,Z)QY] + \frac{r}{n(n-1)}\left[g(Y,Z)X - g(X,Z)Y\right], \end{aligned} \tag{2.14}$$

for all $X, Y, Z \in TM$, respectively, where r is the scalar curvature of M.

3 Main Results

In this section, we obtain a necessary and sufficient condition for $(LCS)_{2n+1}$-manifolds satisfying the derivation conditions $\tilde{C}(\xi, X)\cdot\tilde{C} = 0, \tilde{C}(\xi, X)\cdot R = 0, \tilde{C}(\xi, X)\cdot S = 0$ and $\tilde{C}(\xi, X)\cdot C = 0$.

Theorem 3.1. *An $(2n+1)$-dimensional Lorentzian concircular structure manifold M satisfies*

$$\tilde{C}(\xi, X)\cdot\tilde{C} = 0$$

if and only if either the scalar curvature r of M is $r = 2n(2n+1)$ or M is locally isometric to the Hyperbolic sphere $H^{2n+1}(\rho - \alpha^2)$.

Proof. In a Lorenzian concircular structure manifold M, we have

$$\tilde{C}(\xi, Y)Y = \left[(\alpha^2 - \rho) - \frac{r}{n(n-1)}\right]\{g(X,Y)\xi - \eta(X)Y\}, \tag{3.1}$$

$$\tilde{C}(X, Y)\xi = \left[(\alpha^2 - \rho) - \frac{r}{n(n-1)}\right]\{\eta(Y)X - \eta(X)Y\}. \tag{3.2}$$

The condition $\tilde{C}(\xi, X)\cdot\tilde{C} = 0$ implies that

$$\tilde{C}(\xi, U)\tilde{C}(X, Y)\xi - \tilde{C}(\tilde{C}(\xi, U)X, Y)\xi - \tilde{C}(X, \tilde{C}(\xi, U)Y)\xi = 0.$$

In view of (3.2), we get

$$\begin{aligned} 0 = \left[(\alpha^2 - \rho) - \frac{r}{n(n+1)}\right] &\times [g(U, \tilde{C}(X,Y)\xi)\xi - \tilde{C}(X,Y)\xi\eta(U) \\ &- g(U,X)\tilde{C}(\xi,Y)\xi + \eta(X)\tilde{C}(U,Y)\xi - g(U,Y)\tilde{C}(X,\xi)\xi \\ &+ \eta(Y)\tilde{C}(X,U)\xi - \tilde{C}(X,Y)U]. \end{aligned}$$

Using (3.1), we have

$$\begin{aligned} 0 = &\left[(\alpha^2 - \rho) - \frac{r}{n(n+1)}\right] \\ &\times \left[\tilde{C}(X,Y)U - \left((\alpha^2 - \rho) - \frac{r}{n(n+1)}\right)(g(U,Y)X - g(U,X)Y)\right]. \end{aligned}$$

Therefore either the scalar curvature $r = 2n(2n+1)(\alpha^2 - \rho)$ or

$$\left[\tilde{C}(X,Y)U - \left((\alpha^2 - \rho) - \frac{r}{n(n+1)}\right)(g(U,Y)X - g(U,X)Y)\right] = 0.$$

In view of (2.13), we get

$$R(X,Y)U = (\alpha^2 - \rho)\,[g(Y,U)X - g(X,U)Y]$$

This equation implies that M is of constant curvature $(\rho - \alpha^2)$. Consequently it is locally isometric to the Hyperbolic space $H^{2n+1}(\rho - \alpha^2)$.

Conversely, if it has the scalar curvature $r = 2n(2n+1)(\alpha^2 - \rho)$ then from (3.2) it follows that $\tilde{C}(\xi, X) = 0$. Similarly, in the second case, since constant $r = 2n(2n+1)(\alpha^2 - \rho)$, therefore again we get $\tilde{C}(\xi, X) = 0$. □

Using the fact $\tilde{C}(\xi, X) \cdot R = 0$, $\tilde{C}(\xi, X)$ is acting as a derivation, we have the following a corollary.

Corollary 3.2. *An $(2n+1)$-dimensional Lorentzian concircular structure manifold M satisfies*

$$\tilde{C}(\xi, X) \cdot R = 0$$

if and only if either M is locally isometric to the Hyperbolic sphere $H^{2n+1}(\rho - \alpha^2)$ or M has the scalar curvature $r = 2n(2n+1)$.

Theorem 3.3. *Let (M, g) be an$(2n+1)$-dimensional Riemannian manifold. Then $R \cdot \tilde{C} = R \cdot R$*

Proof. Let $X, Y, U, V, W \in TM$. Then

$$\begin{aligned}(R(X,Y)\tilde{C})(U,V,W) &= R(X,Y)\tilde{C}(U,V)W - \tilde{C}(R(X,Y)U,V)W \\ &\quad - \tilde{C}(U, R(X,Y)V)W - \tilde{C}(U,V)R(X,Y)W.\end{aligned}$$

From (2.13) and symmetric properties of the curvature tensor R, we have

$$\begin{aligned}(R(X,Y)\tilde{C})(U,V,W) &= R(X,Y)R(U,V)W - R(R(X,Y)U,V)W \\ &\quad - R(U, R(X,Y)V)W - R(U,V)R(X,Y)W \\ &= (R(X,Y) \cdot R)(U,V,W).\end{aligned}$$

which proves the Theorem 3.3. □

Theorem 3.4. *An $(2n+1)$-dimensional Lorentzian concircular structure manifold M satisfies*

$$\tilde{C}(\xi, X) \cdot S = 0$$

if and only if either M has the scalar curvature $r = 2n(2n+1)(\alpha^2 - \rho)$ or is an Einstein manifold with the scalar curvature $r = 2n(2n+1)(\alpha^2 - \rho)$.

Proof. The condition $\tilde{C}(\xi, X) \cdot S = 0$ implies that

$$S(\tilde{C}(\xi, X)Y, \xi) + S(Y, \tilde{C}(\xi, X)\xi) = 0.$$

In view of (3.2), it gives

$$\left[(\alpha^2 - \rho) - \frac{r}{n(n-1)}\right] \times [g(X,Y)S(\xi,\xi) - S(X,\xi)\eta(Y) + S(Y,\xi)\eta(X) + S(X,\xi)] = 0.$$

By the use of (2.11), we have

$$\left[(\alpha^2 - \rho) - \frac{r}{n(n-1)}\right] \left[S(X,Y) - 2n(\alpha^2 - \rho)g(X,Y)\right] = 0.$$

Therefore, either the scalar curvature of M is $r = 2n(2n+1)(\alpha^2 - \rho)$ or $S(X,Y) = 2n(\alpha^2 - \rho)g(X,Y)$ which implies that M is an Einstein manifold with the scalar curvature $r = 2n(2n+1)(\alpha^2 - \rho)$, which proves the Theorem 3.4. □

Theorem 3.5. *An $(2n+1)$-dimensional Lorentzian concircular structure manifold M satisfies*

$$\tilde{C}(\xi, X) \cdot C = 0$$

if and only if either M has the scalar curvature $r = 2n(2n+1)(\alpha^2 - \rho)$ or is an η-Einstein manifold.

Proof. The condition $\tilde{C}(\xi, X) \cdot C = 0$ implies that

$$\left[\tilde{C}(\xi,U)C(X,Y)W - C(\tilde{C}(\xi,U)X,Y)W - C(X,\tilde{C}(,U)Y)W\right] = 0$$

Thus in view of (3.2) gives

$$\begin{aligned} 0 = \left[(\alpha^2 - \rho) - \frac{r}{n(n-1)}\right] &\times [C(X,Y)W,U)\xi - \eta(C(X,Y)W)U \\ &- g(U,X)C(\xi,Y)W + \eta(X)C(Y,U)W - g(U,Y)C(X,\xi)W \\ &+ \eta(Y)C(X,U)W + \eta(W)C(X,Y)W - g(U,W)C(X,Y)\xi]. \end{aligned}$$

So either the scalar curvature of M is $r = 2n(2n+1)(\alpha^2 - \rho)$ or the equation

$$\begin{aligned} 0 = C(X,Y)W,U)\xi &- \eta(C(X,Y)W)U - g(U,X)C(\xi,Y)W \\ &+ \eta(X)C(Y,U)W - g(U,Y)C(X,\xi)W + \eta(Y)C(X,U)W \\ &+ \eta(W)C(X,Y)W - g(U,W)C(X,Y)\xi \end{aligned}$$

holds on M. Taking the inner product of this equation with ξ, we get

$$\begin{aligned} 0 = &-C(X,Y,W,U) - C(X,Y)W)\eta(U) - g(U,X)\eta(C(\xi,Y)W) \\ &+ \eta(X)\eta(C(Y,U)W) - g(U,Y)\eta(C(X,\xi)W) \\ &+ \eta(Y)\eta(C(X,U)W) + \eta(W)\eta(C(X,Y)W) - g(U,W)\eta(C(X,Y)\xi). \end{aligned} \tag{3.3}$$

Using (2.6), (2.11) and (2.14) in (3.3), we get

$$S(Y,W) = \left[(\alpha^2-\rho) - \frac{r}{2n(2n+1)}\right] g(Y,W) + \left[(\alpha^2-\rho) + \frac{r}{2n(2n+1)}\right] \eta(Y)\eta(W),$$

which proves the Theorem 3.5. □

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