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On (k, μ)-Contact Metric Manifolds[1]

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Abstract : The tensor $h = \frac{1}{2}L_\xi \phi$, L denotes the Lie derivative, plays a crucial role to determine the nature of a (k, μ)-contact metric manifold. The object of the present paper is to study (k, μ)-contact metric manifolds for which the tensor h is parallel, recurrent and cyclically parallel. Three-dimensional (k, μ)-contact metric manifolds with η-recurrent Ricci tensor have been studied. Illustrative examples, related to the results obtained in each section, are also given.

Keywords : (k, μ)-contact metric manifolds; η-recurrent Ricci tensor; Locally ϕ-symmetric.
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1 Introduction

In [1], the authors introduced a class of contact metric manifolds for which the characteristic vector field ξ belongs to the (k, μ)-nullity distribution for some real numbers k and μ. Such manifolds are known as (k, μ)-contact metric manifolds. The class of (k, μ)-contact metric manifolds encloses both Sasakian and non-Sasakian manifolds. Before Boeckx [2], two classes of non-Sasakian (k, μ)-contact metric manifolds were known. The first class consists of the unit tangent sphere bundles of spaces of constant curvature, equipped with their natural contact metric structure, and the second class contains all the three-dimensional unimodular Lie groups, except the commutative one, admitting the structure of a left invariant (k, μ)-contact metric manifold [1–3]. A full classification of (k, μ)-contact metric manifolds was given by Boeckx [2]. (k, μ)-contact metric manifolds

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are invariant under D-homothetic transformations. In [4], Sharma showed that if a (k, μ)-contact metric manifold admits a non-zero holomorphically planer conformal vector field, then it is either Sasakian, or, locally isometric to E^3 or $E^{n+1} \times S^n(4)$. Recently, in [5], the authors proved that a non-Sasakian contact metric manifold with η-parallel torsion tensor τ and sectional curvatures of plane sections containing the Reeb vector field, different from 1 at some point, is a (k, μ)-contact manifold. The torsion tensor τ was introduced by Chern and Hamilton [6]. In particular, the authors of [5] also proved that for the standard contact metric structure of the tangent sphere bundle the torsion tensor τ is η-parallel if and only if the manifold is of constant curvature. On a (k, μ)-contact metric manifold the tensor h defined by $h = \frac{1}{2}L_\xi \phi$, L denotes the Lie derivative, plays an important role to determine the nature of the manifold. For instance, the vanishing of h is equivalent to ξ being Killing and the manifold becomes K-contact. It is important to note that η-parallelity of τ need not imply the η-parallelity of h, unless ϕ is η-parallel. In dimension three ϕ is η-parallel and hence the notion of η-parallelity regarding h and τ are equivalent. But this is not true for the manifolds of dimension greater than three [5]. Boeckx and Cho [7] introduced the notion of η-parallel h by taking the vector fields in the contact distribution D, say, where $\eta(X) = 0$, for any arbitrary differentiable vector fields $X \in D$ on the manifold. But in this paper we are concerned with the concept of parallel h by considering arbitrary vector fields on the manifold. In this paper, we like to find necessary and sufficient conditions for a $(2n+1)$-dimensional $(n > 1)$ (k, μ)-contact metric manifold $M^{(2n+1)}$ to have the tensor h as parallel, recurrent and cyclically parallel.

Ricci tensor plays an important role to determine the nature of a contact metric manifold. The notion of η-parallel Ricci tensor was introduced by Kon [8] in the context of Sasakian manifold. In this paper we like to generalize the notion of η-parallel Ricci tensor and study (k, μ)-contact metric manifolds of dimension three with η-recurrent Ricci tensor. We also obtain some interesting corollaries.

After the introduction and preliminaries, we investigate the nature of a $(2n+1)$-dimensional $(n > 1)$ (k, μ)-contact metric manifold, in Section 3, with h as parallel, recurrent and cyclically parallel. In this section, we prove that with each of these conditions the manifold becomes Sasakian. Three-dimensional (k, μ)-contact metric manifolds with η-recurrent Ricci tensor have been studied in Section 4 and it is proved that in this case the manifolds are flat. In this section we also obtain some interesting corollaries regarding three dimensional (k, μ)-contact metric manifolds. Every section is followed by illustrative examples which are related to the results obtained.

2 Preliminaries

Let M be a $(2n+1)$-dimensional C^∞-differentiable manifold. The manifold is said to admit an almost contact metric structure (ϕ, ξ, η, g) if it satisfies the following relations [9]:

$$\phi^2 X = -X + \eta(X)\xi, \eta(\xi) = 1, g(X, \xi) = \eta(X), \tag{2.1}$$

$$\phi\xi = 0, \;\; \eta\phi = 0, \;\; g(X,\phi Y) = -g(\phi X, Y), \;\; g(X,\phi X) = 0, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.3}$$

where ϕ is a tensor field of type $(1,1)$, ξ is a vector field, η is an 1-form and g is a Riemannian metric on M. A manifold equipped with an almost contact metric structure is called an almost contact metric manifold. An almost contact metric manifold is called a contact metric manifold if it satisfies

$$g(X,\phi Y) = d\eta(X,Y).$$

Given a contact metric manifold $M(\phi,\xi,\eta,g)$, we consider a $(1,1)$ tensor field h defined by $h = \frac{1}{2}L_\xi\phi$, where L denotes the Lie differentiation. h is a symmetric operator and satisfies $h\phi = -\phi h$. If λ is an eigenvalue of h with eigenvector X, then $-\lambda$ is also an eigenvalue of h with eigenvector ϕX. Again, we have $\mathrm{tr}h = \mathrm{tr}\phi h = 0$, and $h\xi = 0$. Moreover, if ∇ denotes the Riemannian connection of g, then the following relation holds [1]:

$$\nabla_X\xi = -\phi X - \phi hX, \;\; (\nabla_X\eta)Y = g(X + hX, \phi Y). \tag{2.4}$$

The vector field ξ is a Killing vector field with respect to g if and only if $h = 0$. A contact metric manifold $M(\phi,\xi,\eta,g)$ for which ξ is a Killing vector is said to be a K-contact manifold. A K-contact structure on M gives rise to an almost complex structure on the product $M \times \mathbb{R}$. If this almost complex structure is integrable, the contact metric manifold is said to be Sasakian. Equivalently, a contact metric manifold is said to be Sasakian if and only if

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y$$

holds for all X, Y, where R denotes the Riemannian curvature tensor of the manifold M. The (k,μ)-nullity distribution of a contact metric manifold $M(\phi,\xi,\eta,g)$ is a distribution [1]

$$\begin{aligned} N(k,\mu) : p \to N_p(k,\mu) \\ = \{Z \in T_p(M) : R(X,Y)Z = k(g(Y,Z)X - g(X,Z)Y) \\ + \mu(g(Y,Z)hX - g(X,Z)hY)\}, \end{aligned} \tag{2.5}$$

for any $X, Y \in T_pM$. Hence, if the characteristic vector field ξ belongs to the (k,μ)-nullity distribution, we have

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY]. \tag{2.6}$$

A contact metric manifold with ξ belonging to (k,μ)-nullity distribution is called a (k,μ)-contact metric manifold. If $k = 1$, $\mu = 0$, then the manifold becomes Sasakian [1]. In particular, if $\mu = 0$, then the notion of (k,μ)-nullity distribution reduces to k-nullity distribution introduced by Tanno [10]. A contact metric manifold with ξ belonging to k-nullity distribution is known as $N(k)$-contact metric manifold.

In a $(2n+1)$-dimensional (k,μ)-contact metric manifold we have the following [1]:

$$h^2 = (k-1)\phi^2, \ \ k \leq 1. \tag{2.7}$$

$$(\nabla_X \phi)(Y) = g(X+hX, Y)\xi - \eta(Y)(X+hX). \tag{2.8}$$

$$Q\phi - \phi Q = 2(2(n-1)+\mu)h\phi. \tag{2.9}$$

Also, for a contact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$ with $\xi \in N(k,\mu)$, the Ricci operator Q is given by [1]

$$QX = [2(n-1)-n\mu]X + [2(n-1)+\mu]hX + [2(1-n)+n(2k+\mu)]\eta(X)\xi, \ \ n \geq 1. \tag{2.10}$$

$$\begin{aligned} S(X,Y) = {} & [2(n-1)-n\mu]g(X,Y) + [2(n-1)+\mu]g(hX,Y) \\ & + [2(1-n)+n(2k+\mu)]\eta(X)\eta(Y), \ n \geq 1. \end{aligned} \tag{2.11}$$

$$S(X,\xi) = 2nk\eta(X), \tag{2.12}$$

where S is the Ricci tensor of the manifold.

$$r = 2n(2n-2+k-n\mu), \tag{2.13}$$

where r is the scalar curvature of the manifold. Also

$$(\nabla_X h)Y = [(1-k)g(X,\phi Y) + g(X,h\phi Y)]\xi + \eta(Y)h(\phi X + \phi hX) - \mu\eta(X)\phi hY. \tag{2.14}$$

Lemma 2.1 ([11]). *A contact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$ with $R(X,Y)\xi = 0$, for all vector fields X, Y on the manifold and $n > 1$, is locally isometric to the Riemannian product $E^{n+1} \times S^n(4)$, and for $n = 1$ the manifold is flat.*

Lemma 2.2 ([12]). *Let M^{2n+1} be a contact metric manifold with harmonic curvature tensor and ξ belonging to the (k,μ)-nullity distribution. Then M is either*

- *(i) an Einstein Sasakian manifold, or,*
- *(ii) an η-Einstein manifold, or,*
- *(iii) locally isometric to the Riemannian product $E^{n+1} \times S^n(4)$ including a flat contact metric structure for $n = 1$.*

Lemma 2.3 ([13]). *Let M^3 be a contact metric manifold with contact metric structure (ϕ,ξ,η,g). Then the following conditions are equivalent:*

- *(i) M^3 is $\eta-$Einstein;*
- *(ii) $Q\phi = \phi Q$.*

Lemma 2.4 ([13]). *Let M^3 be a contact metric manifold with $\phi Q = Q\phi$. Then M^3 is locally ϕ-symmetric if and only if the scalar curvature of the manifold is constant.*

In this connection we mention that a contact metric manifold is called η-Einstein if its Ricci tensor S satisfies

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \tag{2.15}$$

where a and b are scalars, for all differentiable vector fields X, Y.

A contact metric manifold is called ϕ-symmetric if its curvature tensor R satisfies $\phi^2(\nabla_W R)(X, Y)Z = 0$, for the differentiable vector fields X, Y, Z, W. If X, Y, Z, W are orthogonal to ξ, then it is called locally ϕ-symmetric. The notion of locally ϕ-symmetric manifolds was introduced by Takahashi [14] in the context of Sasakian manifolds.

3 (k, μ)-Contact Metric Manifolds with h as Parallel, Recurrent and Cyclically Parallel

Definition 3.1. The tensor h on a (k, μ)-contact metric manifold is called *parallel* if it satisfies

$$g((\nabla_X h)Y, Z) = 0 \tag{3.1}$$

for all differentiable vector fields on the manifold.

In view of (2.14), (3.1) yields

$$[(1-k)g(X, \phi Y) + g(X, h\phi Y)]\eta(Z) + \eta(Y)g(h(\phi X + \phi hX), Z) - \mu\eta(X)g((\phi hY), Z). \tag{3.2}$$

Putting $Z = \xi$ and using (2.2), we get from (3.2)

$$[(1-k)g(X, \phi Y) + g(X, h\phi Y)] = 0.$$

The above equation can be written as

$$(k-1)g(X, \phi Y) = g(hX, \phi Y).$$

Replacing ϕY by W in the above equation, we obtain

$$(k-1)g(X, W) = g(hX, W).$$

In the above equation putting $X = W = e_i$, where $\{e_i\}$ is a orthonormal basis of the tangent space at each point of the manifold, and taking summation over i, $i = 1, 2, 3, ..., 2n+1$, $(n > 1)$, and using $\mathrm{tr}h = 0$, we obtain

$$(2n+1)(k-1) = 0.$$

Hence, $k = 1$. Consequently, the manifold is Sasakian.

Conversely, suppose that the manifold is Sasakian. Then $k = 1$ and $h = 0$, which trivially implies that $g((\nabla_X h)Y, Z) = 0$. From the above discussion, we are in a position to state the following:

Theorem 3.2. *The tensor h of a $(2n+1)$-dimensional $(n>1)$ (k,μ)-contact metric manifold is parallel if and only if the manifold is Sasakian.*

Definition 3.3. We call the tensor h of a (k,μ)-contact metric manifold as *recurrent* if there exists an $1-$form A on the manifold such that

$$g((\nabla_X h)Y,Z) = A(X)g(hY,Z)$$

for all differentiable vector fields on the manifold. If the 1-form vanishes identically on the manifold, then h is parallel.

Let us consider a $(2n+1)$-dimensional $(n>1)$ (k,μ)-contact metric manifold with recurrent h. By (2.2) and (2.14), we get for $Z=\xi$,

$$[(1-k)g(X,\phi Y) + g(X,h\phi Y)] = 0. \tag{3.3}$$

As before, we get from the above equation $k=1$. Consequently, $h=0$. Hence, the manifold is Sasakian. Conversely, if the manifold is Sasakian h is recurrent, trivially. Thus, we can state the following:

Theorem 3.4. *The tensor h of a $(2n+1)$-dimensional $(n>1)$ (k,μ)-contact metric manifold is recurrent if and only if the manifold is Sasakian.*

Definition 3.5. The tensor h of a (k,μ)-contact metric manifold is called *cyclically parallel* if it satisfies

$$g((\nabla_X h)Y,Z) + g((\nabla_Y h)Z,X) + g((\nabla_Z h)X,Y) = 0$$

for all differentiable vector fields on the manifold.

Let us consider a $(2n+1)$-dimensional $(n>1)$ (k,μ)-contact metric manifold satisfying cyclically parallel h. Then by (2.14), we get

$$\begin{aligned} 0 = {} & [(1-k)g(X,\phi Y) + g(X,h\phi Y)]\eta(Z) + [(1-k)g(Y,\phi Z) + g(Y,h\phi Z)]\eta(X) \\ & + [(1-k)g(Z,\phi X) + g(Z,h\phi X)]\eta(Y) + \eta(Y)g(h(\phi X + \phi hX),Z) \\ & - \mu\eta(X)g(\phi hY,Z)\eta(Z)g(h(\phi Y + \phi hY),X) - \mu\eta(Y)g(\phi hZ,X) \\ & + \eta(X)g(h(\phi Z + \phi hZ),Y) - \mu\eta(Z)g(\phi hX,Y). \end{aligned} \tag{3.4}$$

Putting $Z=\xi$ we obtain from (3.4)

$$(1-k)g(X,\phi Y) + 2g(hX,\phi Y) - g(hX,h\phi Y) + \mu g(hX,\phi Y) = 0.$$

In the above equation, replacing ϕY by W, we get

$$(1-k)g(X,W) + 2g(hX,W) - g(hX,hW) + \mu g(hX,W) = 0.$$

The above equation implies $k=1$, for $X=W=\xi$. Consequently, the manifold is Sasakian. The converse is trivial.

The above discussion leads us to state the following:

Theorem 3.6. *The tensor h of a $(2n+1)$-dimensional $(n > 1)$ (k, μ)-contact metric manifold is cyclically parallel if and only if the manifold is Sasakian.*

From Theorem 3.2, Theorem 3.4 and Theorem 3.6, we conclude the following:

Corollary 3.7. *For a $(2n+1)$-dimensional $(n > 1)$ (k, μ)-contact metric manifold the following conditions are equivalent:*

(i) the manifold is Sasakian;

(ii) the tensor h is parallel;

(iii) the tensor h is recurrent;

(iv) the tensor h is cyclically parallel.

Remark 3.8. *It is well known that for a Sasakian manifold $h = 0$. Hence, from the above corollary, it follows that for a $(2n+1)$-dimensional $(n > 1)$ (k, μ)-contact metric manifold, there does not exist proper $(h \neq 0)$ parallel, recurrent and cyclically parallel h.*

To verify the above results we give the following:

Example 3.9. *Let $M = \{(x, y, z, u, v) \in R^5 : (x, y, z, u, v) \neq (0, 0, 0, 0, 0)\}$, where (x, y, z, u, v) are the standard coordinates in R^5. The vector fields*

$$e_1 = 2\left(y\frac{\partial}{\partial z} - \frac{\partial}{\partial x}\right),\ e_2 = 2\frac{\partial}{\partial y},\ e_3 = 2\frac{\partial}{\partial z},\ e_4 = 2\left(v\frac{\partial}{\partial z} - \frac{\partial}{\partial u}\right),\ e_5 = 2\frac{\partial}{\partial v},$$

are linearly independent at each point of M. Let g be the metric defined by $g(e_i, e_j) = 1$ if $i = j$, otherwise 0. Here i and j varies from 1 to 5. Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any Z belongs to $\chi(M)$. Let ϕ be the $(1, 1)$ tensor field defined by $\phi e_1 = -e_2$, $\phi e_2 = e_1$, $\phi e_3 = 0$, $\phi(e_4) = -e_5$, $\phi(e_5) = e_4$. Then using the linearity of ϕ and g we have

$$\eta(e_3) = 1,\ \phi^2 Z = -Z + \eta(Z)e_3,$$

for any $Z, W \in \chi(M)$. Thus for $e_3 = \xi$, $M(\phi, \xi, \eta, g)$ defines an almost contact metric manifold.

Let ∇ be the Levi-Civita connection with respect to the metric g. Then we have $[e_1, e_2] = -2e_3$, $[e_4, e_5] = -2e_3$ and $[e_i, e_j] = 0$, for all other i, j.

Taking $e_3 = \xi$ and using Koszul formula for the metric g, it can be easily calculated that

$$\nabla_{e_1} e_3 = e_2,\ \nabla_{e_2} e_3 = -e_1,\ \nabla_{e_4} e_3 = -e_1,$$

$$\nabla_{e_5} e_3 = -e_4,\ \nabla_{e_1} e_2 = -e_3,\ \nabla_{e_3} e_2 = -e_1,$$

$$\nabla_{e_3} e_1 = e_2,\ \nabla_{e_3} e_4 = -e_1,\ \nabla_{e_5} e_4 = -e_3,$$

$$\nabla_{e_3} e_5 = -e_4,\ \nabla_{e_4} e_5 = e_3.$$

and the remaining $\nabla_{e_i} e_j = 0$.

From the above results it is easy to verify that M *is a* (k,μ)*-contact metric manifold with* $k = 1$ *and* $\mu = 0$. *For a contact metric manifold we get*

$$\nabla_X \xi = -\phi X - \phi h X.$$

Hence,

$$\nabla_{e_1} e_3 = -\phi e_1 - \phi h e_1.$$

The above equation gives $e_2 = e_2 + h\phi e_1$. *Therefore,* $h\phi e_1 = 0$, *that is,* $he_2 = 0$. *Similarly we can prove that* $he_1 = he_3 = he_4 = he_5 = 0$. *Consequently,* $h = 0$ *and it is parallel, recurrent and cyclically parallel.*

It is to be noted that the above example is compatible with Corollary 3.7 and Remark 3.8.

4 (k, μ)-Contact Metric Manifolds of Dimension Three with η-Recurrent Ricci Tensor

Definition 4.1. The Ricci tensor of a three-dimensional (k,μ)-contact metric manifold M^3 is called *η-recurrent* if there exists an $1-$form A such that

$$(\nabla_Z S)(\phi X, \phi Y) = A(Z) S(\phi X, \phi Y), \tag{4.1}$$

where A is defined by $g(Z,\rho) = A(Z)$, ρ is a unit vector field and X, Y, Z are arbitrary differentiable vector fields on the manifold. If the 1-form vanishes identically on the manifold, then the Ricci tensor is called η-parallel.

The notion of η-parallel Ricci tensor was introduced by Kon [8] in the context of Sasakian manifold. From the definition, it follows that if the Ricci tensor is η-parallel, then it is η-recurrent with $A(Z) = 0$, but the converse is not true, in general. From (2.11), using (2.14) we get

$$\begin{aligned}(\nabla_Z S)(X,Y) = {} & [2(n-1)+\mu]\{[(1-k)g(Z,\phi X) + g(Z, h\phi X)]\eta(Y) \\ & + \eta(X) g(h(\phi Z + \phi h Z), Y) - \mu \eta(Z) g(\phi h X, Y)\} \\ & + [2(1-n) + n(2k+\mu)]((\nabla_Z \eta)(X)\eta(Y) + \eta(X)(\nabla_Z \eta)(Y)).\end{aligned} \tag{4.2}$$

From (4.2) we have

$$(\nabla_Z S)(\phi X, \phi Y) = -\mu[2(n-1)+\mu]\eta(Z) g(\phi h \phi X, \phi Y). \tag{4.3}$$

Let the Ricci tensor of M^3 is η-recurrent. Then by (2.11), (4.1) and (4.3), we get

$$-\mu[2(n-1)+\mu]\eta(Z) g(\phi h \phi X, \phi Y) = A(Z)[-\mu g(\phi X, \phi Y) + \mu g(h\phi X, \phi Y)]. \tag{4.4}$$

In (4.4), taking Z orthogonal to $\xi(\neq \rho)$ we get

$$\mu A(Z) g(\phi X - h\phi X, \phi Y) = 0.$$

The above equation yields $\mu = 0$. Hence, for $n = 1$ (2.11) yields

$$S(X,Y) = 2k\eta(X)\eta(Y).$$

Let X, Y be orthogonal to ξ, then the above equation reduces to $S(X,Y) = 0$. Putting $X = Y = e_i$, $i = 1,2,3$, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold, we get, from the above equation $r = 0$. Therefore, from (2.13), $k = 0$.

Thus, we have for a (k,μ)-contact metric manifold of dimension three with η-recurrent Ricci tensor $R(X,Y)\xi = 0$. Hence, by Lemma 2.1 we have the manifold is flat.

Conversely, if the manifold is flat, then $R(X,Y)Z = 0$ and $S(X,Y) = 0$. Hence, the Ricci tensor is trivially η-recurrent.

Now, we are in a position to state the following:

Theorem 4.2. *A three-dimensional (k,μ)-contact metric manifold has η-recurrent Ricci tensor if and only if the manifold is flat.*

For $n = 1$ and $\mu = 0$, from (2.9) we obtain a three-dimensional (k,μ)-contact metric manifold with η-recurrent Ricci tensor satisfies $\phi Q = Q\phi$.

Again, if a contact metric manifold of dimension three satisfies $\phi Q = Q\phi$, then from Lemma 2.3 it follows that the manifold is an η-Einstein manifold. Hence, from (2.15) and (2.2) we have

$$(\nabla_W S)(\phi X, \phi Y) = 0.$$

Hence, the Ricci tensor of the manifold is η-parallel. Consequently, it is η-recurrent. The above discussion helps us to state the following:

Theorem 4.3. *A three-dimensional (k,μ)-contact metric manifold satisfies $\phi Q = Q\phi$ if and only if its Ricci tensor is $\eta-$recurrent.*

From Theorem 4.2 and Theorem 4.3, we obtain the following:

Corollary 4.4. *For a three-dimensional (k,μ)-contact metric manifold the following conditions are equivalent:*

(i) the manifold has η-recurrent Ricci tensor;

(ii) the manifold satisfies $\phi Q = Q\phi$;

(iii) the manifold is flat.

Suppose the manifold satisfies $\phi Q = Q\phi$. Then by above corollary it is flat. If a manifold is flat, then it is obviously locally ϕ-symmetric. Hence, it is clear that every three-dimensional (k,μ)-contact metric manifold with $\phi Q = Q\phi$ is locally ϕ-symmetric.

Conversely, suppose that the manifold is locally ϕ-symmetric. Then

$$\phi^2((\nabla_W R)(X,Y)Z) = 0$$

for X, Y, Z, W orthogonal to ξ. The above equation yields

$$g((\nabla_W R)(X, Y)Z, U) = 0.$$

Putting $X = U = e_i$, $i = 1, 2, 3$, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold, we get from the above equation

$$(\nabla_W S)(Y, Z) = 0.$$

Replacing Y by ϕY and Z by ϕZ, we get from the above equation $(\nabla_W S)(\phi Y, \phi Z) = 0$. Therefore, the Ricci tensor is η-parallel, consequently, it is $\eta-$recurrent and by Corollary 4.4 it satisfies $\phi Q = Q\phi$.

Now, we can state the following:

Corollary 4.5. *A three-dimensional (k, μ)-contact metric manifold is locally ϕ-symmetric if and only if it satisfies $\phi Q = Q\phi$.*

Remark 4.6. *In the paper [13], the authors proved that a three-dimensional contact metric manifold with $\phi Q = Q\phi$ is locally ϕ-symmetric if and only if the scalar curvature of the manifold is constant. From Corollary 4.5 it is seen that the condition scalar curvature is constant is not required for (k, μ)-contact metric manifolds. Hence, we observe that our result improves the result of the paper [13] regarding (k, μ)-contact metric manifolds.*

In the following we give an example.

Example 4.7. *In the paper [1], the authors gave examples of (k, μ)-contact metric manifolds. In the similar way we construct the following example. Consider $M = \{(x, y, z) \in R^3, (x, y, z) \neq (0, 0, 0)\}$, where (x, y, z) are the standard coordinates in R^3. Let M be generated by three linearly independent vector fields e_1, e_2 and e_3 satisfying*

$$[e_2, e_3] = 2e_1,\ [e_3, e_1] = c_2 e_2,\ [e_1, e_2] = c_3 e_3. \tag{4.5}$$

We take c_2, c_3 as real numbers. Let $\{\omega_i\}$ be the dual $1-$form to the vector field $\{e_i\}$. Using (4.5) we get

$$d\omega(e_1, e_2) = -dw_1(e_3, e_2) = 1 \text{ and } dw_1(e_i, e_j) = 0$$

for others i, j. We take $e_1 = \xi$. Define the Riemannian metric by $g(e_i, e_j) = \delta_{ij}$. Let $\phi e_3 = -e_2, \phi e_2 = e_3$. For g as an associated metric, we have $\phi^2 = -I + \omega_1 \otimes e_1$. Hence $M(\phi, e_1, \omega_1, g)$ is a contact metric manifold. By Koszul formula we can calculate the following:

$$\nabla_{e_1} e_1 = 0, \quad \nabla_{e_2} e_2 = 0, \quad \nabla_{e_3} e_3 = 0,$$
$$\nabla_{e_1} e_2 = \tfrac{1}{2}(c_2 + c_3 - 2)e_3, \quad \nabla_{e_2} e_1 = \tfrac{1}{2}(c_3 - c_2 - 2)e_3, \quad \nabla_{e_1} e_3 = -\tfrac{1}{2}e_2,$$
$$\nabla_{e_3} e_1 = \tfrac{1}{2}(2 + c_2 - c_3)e_2.$$

The non-vanishing components of the curvature tensor of the manifold can be calculated as

$$R(e_2,e_1)e_1 = [1 - \frac{(c_3-c_2)^2}{4}]e_2 + (2-c_2-c_3)he_2,$$

$$R(e_3,e_1)e_1 = [1 - \frac{(c_3-c_2)^2}{4}]e_3 + (2-c_2-c_3)he_3.$$

Here $k = 1 - \frac{(c_3-c_2)^2}{4}$, $\mu = 2 - c_2 - c_3$. *In this example, if we choose* $c_2 = -1$ *and* $c_3 = 3$, *then* $k = \mu = 0$ *and the manifold becomes flat. Consequently, by Corollary 4.4 the manifold satisfies* $\phi Q = Q\phi$ *and has* η*-recurrent Ricci tensor. The manifold is locally* ϕ*-symmetric also.*

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References

[1] D.E. Blair, T. Koufogiorgos, B.J. Papantoniou, Contact metric manifolds satisfying a nullity condition, Israel J. Math. 19 (1995) 189–214.

[2] E. Boeckx, A full classification of contact metric (k,μ)-spaces, Illinois J. of Math. 44 (2000) 212–219.

[3] D. Perrone, Homogeneous contact Riemannian three-manifolds, Illinois J. Math. 42 (1998) 243–258.

[4] R. Sharma, Certain results on K-contact and (k,μ)-contact metric manifolds, J. Geom. 89 (2008) 138–147.

[5] A. Ghosh, R. Sharma, J.T. Cho, Contact metric manifolds with η-parallel torsion tensor, Ann. Glob. Anal. Geom. 34 (2008) 287–299.

[6] S.S. Chern, R.S. Hamilton, On Riemannian metrics adapted to three-dimensional contact manifolds, Lecture Note in Mathematics, Springer-Verlag, Berlin and New York, Vol 1111, 1985.

[7] E. Boeckx, J.T. Cho, η-parallel contact metric spaces, Diff. Geom. Appl. 22 (2005) 275–285.

[8] M. Kon, Invariant submanifolds in Sasakian manifolds, Math. Ann. 219 (1976) 277–290.

[9] D.E. Blair, Contact manifolds in Riemannian geometry, Lecture notes in Math., 509, Springer Verlag, New York, 1973.

[10] S. Tanno, Ricci curvatures of contact Riemannian manifolds, Tohoku Math. J. 40 (1988) 441–448.

[11] D.E. Blair, Two remarks on contact metric structures, Tohoku Math. J. 29 (1977) 319–324.

[12] B.J. Papantoniou, Contact manifolds, harmonic curvature tensor and (k, μ)-nullity distribution, Comment. Math. Univ. Caroliane 34 (1993) 323–334.

[13] D.E. Blair, T. Koufogiorgos, R. Sharma, A classification of three-dimensional contact metric manifolds with $Q\phi = \phi Q$, Kodai Math. J. 13 (1990) 391–401.

[14] T. Takahashi, Sasakian ϕ-symmetric spaces, Tohoku Math. J. 29 (1977) 91–113.

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