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2-Absorbing and Weakly 2-Absorbing Submodules

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Abstract : Let R be a commutative ring with a nonzero identity and let M be a unitary R-module. We introduce the concepts of 2-absorbing and weakly 2-absorbing submodules of M and give some basic properties of these classes of submodules. Indeed these are generalizations of prime and weak prime submodules. A proper submodule N of M is called a 2-absorbing (resp. weakly 2-absorbing) submodule of M if whenever $a, b \in R$, $m \in M$ and $abm \in N$ (resp. $0 \neq abm \in N$), then $ab \in (N :_R M)$ or $am \in N$ or $bm \in N$. It is shown that the intersection of each distinct pair of prime (resp. weak prime) submodules of M is 2-absorbing (resp. weakly 2-absorbing). We will also show that if R is a commutative ring, M a cyclic R-module and N a 2-absorbing submodule of M, then either (1) $M - radN = P$ is a prime submodule of M such that $P^2 \subseteq N$ or (2) $M - radN = P_1 \cap P_2$, $P_1P_2 \subseteq N$ and $(M - radN)^2 \subseteq N$ where P_1, P_2 are the only distinct minimal prime submodules of N.

Keywords : 2-absorbing submodule; Weakly 2-absorbing submodule; Prime submodule; Weakly prime submodule; Weak prime submodule.
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1 Introduction

Throughout this paper all rings are commutative with a nonzero identity and all modules are considered to be unitary. Prime ideals play an important role in

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commutative ring theory. Let R be a commutative ring. Of course a proper ideal P of R is said to be a prime ideal if $ab \in P$ implies that $a \in P$ or $b \in P$ where $a, b \in R$. Anderson and Smith [1] studied the concept of weakly prime ideals of a commutative ring, where a proper ideal P of R is weakly prime if $a, b \in R$ and $0 \neq ab \in P$ imply that either $a \in P$ or $b \in P$. Another generalization of prime ideals is the concept of almost prime ideals. Let R be an integral domain. A proper ideal I of R is said to be almost prime provided that $a, b \in R$ with $ab \in I - I^2$ imply that $a \in I$ or $b \in I$ (see [2]). This definition can obviously be made for any commutative ring R. A number of generalizations of prime ideals in commutative rings can be found in [3].

Prime submodules play an important role in the module theory over commutative rings. Let M be a module over a commutative ring R. A prime (resp. primary) submodule N of M is a proper submodule N of M with the property that for $a \in R$ and $m \in M$, $am \in N$ implies that $m \in N$ or $a \in (N :_R M)$ (resp. $a^k \in (N :_R M)$ for some positive integer k). In this case $P = (N :_R M)$ (resp. $P = \sqrt{(N :_R M)}$) is a prime ideal of R and we say that N is a P-prime (resp. P-primary) submodule of M. There are several ways to generalize the notion of prime submodules. We could restrict where am lies or we can restrict where a and/or b lie. We begin by mentioning some examples obtained by restricting where ab lies. Weakly prime submodules were introduced by Ebrahimi Atani and Farzalipour in [4]. A proper submodule N of M is weakly prime if for $a \in R$ and $m \in M$ with $0 \neq am \in N$, either $m \in N$ or $a \in (N :_R M)$. Behboodi and Koohi in [5] defined another class of submodules and called it weakly prime. This paper is on the basis of some recent papers devoted to this new class of submodules. Let R be a commutative and M an R-module. A proper submodule N of M is said to be weakly prime when for $a, b \in R$ and $m \in M$, $abm \in N$ implies that $am \in N$ or $bm \in N$. To avoid the ambiguity we call submodules introduced in [4] *weak prime submodules* and those introduced in [5], *weakly prime submodule*.

Badawi in [6] generalized the concept of prime ideals in a different way. He defined a nonzero proper ideal I of R to be a 2-absorbing ideal of R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. This definition can obviously be made for any ideal of R. This concept has a generalization, called weakly 2-absorbing ideals, which has studied in [7]. A proper ideal I of R to be a weakly 2-absorbing ideal of R if whenever $a, b, c \in R$ and $0 \neq abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. Anderson and Badawi [8] generalized the concept of 2-absorbing ideals to n-absorbing ideals. According to their definition, a proper ideal I of R is called an n-absorbing (resp., strongly n-absorbing) ideal if whenever $x_1 \cdots x_{n+1} \in I$ for $x_1, ..., x_{n+1} \in R$ (resp, $I_1 \cdots I_{n+1} \subseteq I$ for ideals $I_1, ..., I_{n+1}$ of R), then there are n of the x_i's (resp., n of the I_i's) whose product is in I. In this paper, we generalize the two concepts of 2-absorbing and weakly 2-absorbing ideals to submodules of a module over a commutative ring. Let M ba an R-module and N a proper submodule of M. N is said to be a 2-absorbing submodule (resp. weakly 2-absorbing submodule) of M if whenever $a, b \in R$ and $m \in M$ with $abm \in N$ (resp. $0 \neq abm \in N$), then $ab \in (N :_R M)$ or $am \in N$ or $bm \in N$. A more general concept than 2-absorbing submodules of M is the concept of (n, k)-absorbing submodules. Suppose that k, n are two positive integers with $n > k$. A submodule N of M is called an (n, k)-absorbing submodule if whenever $a_1, a_2, ..., a_{n-1} \in R$ and $m \in M$ with $a_1 a_2 \cdots a_{n-1} m \in N$, then either there are k of a_i's whose product is in $(N :_R M)$ or there are $k - 1$ of a_i's whose product with m lies in N. The concept of weakly (n, k)-absorbing submodules is defined in a similar way.

Here is a brief summary of the paper. In Theorem 2.3 we show that the intersection of each pair of distinct prime (resp. weak prime) submodules of M is 2-absorbing (resp. weakly 2-absorbing). Let R be a commutative ring, M an R-module and N a weakly 2-absorbing submodule of M. In Theorem 2.5 we prove that if N is not 2-absorbing, then $(N :_R M)^2 N = 0$. In Theorem 2.10 we show that if R is a commutative a ring, M a cyclic R-module and N a 2-absorbing submodule of M, then one of the following hold: (1) $M - radN = P$ is a prime submodule of M such that $P^2 \subseteq N$ or (2) $M - radN = P_1 \cap P_2$, $P_1 P_2 \subseteq N$ and $(M - radN)^2 \subseteq N$ where P_1, P_2 are the only distinct minimal prime submodules of N.

2 Results

Proposition 2.1. *Let M be a module over a commutative ring R and N a submodule of M.*

(1) Prime submodules $\Rightarrow$ weak prime submodules $\Rightarrow$ 2-absorbing submodules.

(2) Weakly prime submodules $\Rightarrow$ weakly 2-absorbing submodules.

(3) 2-absorbing submodules $\Rightarrow$ weakly 2-absorbing submodules.

(4) N is (n, k)-absorbing (resp. weakly (n, k)-absorbing) if and only if it is $(k + 1, k)$-absorbing (resp. weakly $(k + 1, k)$-absorbing).

(5) If N is (n, k)-absorbing (resp. weakly (n, k)-absorbing), then it is (n, k')-absorbing (resp. weakly (n, k')-absorbing) for every positive integer $k' > k$.

(6) N is a prime (resp. weak prime) submodule of M if and only if it is a $(2, 1)$-absorbing (resp. weakly $(2, 1)$-absorbing) submodule of M.

(7) N is a 2-absorbing (resp. weakly 2-absorbing) submodule of M if and only if it is a $(3, 2)$-absorbing (resp. weakly $(3, 2)$-absorbing) submodule of M.

Example 2.2. *Let R be a commutative ring and M an R-module. By Proposition 2.1, every 2-absorbing submodule is weakly 2-absorbing but the converse does not necessarily hold. For example consider the case where $R = \mathbb{Z}$, $M = \mathbb{Z}/30\mathbb{Z}$ and $N = 0$. Then $2.3.(5 + 30\mathbb{Z}) = 0 \in N$ while $2.3 \notin (N :_R M)$, $2.(5 + 30\mathbb{Z}) \notin N$ and $3.(5 + 30\mathbb{Z}) \notin N$. Therefore N is not 2-absorbing while it is weakly 2-absorbing.*

Theorem 2.3. *Let R be a commutative ring and let M an R-module.*

(1) The intersection of each pair of distinct prime submodules of M is 2-absorbing.

(2) The intersection of each pair of distinct weak prime submodules of M is weakly 2-absorbing.

Proof. (1) Let N and K be two distinct prime submodules of M. Assume that $a, b \in R$ and $m \in M$ are such that $abm \in N \cap K$ but $am \notin N \cap K$ and $bm \notin N \cap K$. The case $am \notin N$ and $bm \notin N$ leads us to a contradiction since N is prime. A similar argument is true for the case where $am \notin K$ and $bm \notin K$ since K is prime. So assume that $am \notin N$ and $bm \notin K$. Then from $abm \in N$ and $abm \in K$ we have $b \in (N :_R M)$ and $a \in (K :_R M)$. Hence $ab \in (N :_R M) \cap (K :_R M) = (N \cap K :_R M)$. Therefore $N \cap K$ is a 2-absorbing submodule of M.

(2) The proof is completely similar to that of (1). □

Lemma 2.4. *Let R be a ring, M an R-module and N, K submodules of M with $K \subseteq N$. Then N is a 2-absorbing submodule of M if and only if N/K is a 2-absorbing R-submodule of M/K.*

Proof. Suppose first that N is a 2-absorbing submodule of M and let $a, b \in R$ and $m \in M$ be such that $ab(m + K) \in N/K$. Then $abm \in N$ and N 2-absorbing gives $ab \in (N :_R M)$ or $am \in N$ or $bm \in N$. Therefore $ab \in (N/K :_R M/K)$ or $a(m + k) \in N/K$ or $b(m + K) \in N/K$, that is N/K is a 2-absorbing submodule of M/K. Conversely, assume that N/K is a 2-absorbing submodule of M/K. Suppose that $a, b \in R$ and $m \in M$ are such that $abm \in N$. Then we have $ab(m + K) \in N/K$. Therefore $ab \in (N/K :_R M/K)$ or $a(m + K) \in N/K$ or $b(m+K) \in N/K$ since N/K is 2-absorbing in M/K. Therefore $ab \in (N :_R M)$ or $am \in N$ or $bm \in N$. This implies that N is a 2-absorbing submodule of M. □

Let R be a commutative ring and M an R-module. By Proposition 2.1(2), every 2-absorbing submodule of M is weakly 2-absorbing. But the converse does not necessarily hold. For example consider the case where $R = \mathbb{Z}$, $M = \mathbb{Z}/60\mathbb{Z}$ and $N = \{0\}$. As N is weakly prime in M, it is weakly 2-absorbing in M, but N is not 2-absorbing. The following result provides some condition under which a weakly 2-absorbing submodule is 2-absorbing.

Theorem 2.5. *Let R be a commutative ring, M an R-module and N a weakly 2-absorbing submodule of M. If N is not 2-absorbing, then $(N :_R M)^2N = 0$.*

Proof. Assume that $(N :_R M)^2N \neq 0$. We will prove that N is 2-absorbing. Let $a, b \in R$ and $m \in M$ be such that $abm \in N$. If $abc \neq 0$, then $ab \in (N :_R M)$ or $am \in N$ or $bm \in N$ since N is weakly 2-absorbing. So assume that $abm = 0$. Assume first that $abN \neq 0$, say $abn_0 \neq 0$ for some $n_0 \in N$. Then $0 \neq abn_0 = ab(m + n_0) \in N$. Since N is weakly 2-absorbing we get $ab \in (N :_R M)$ or $a(m + n_0) \in N$ or $b(m + n_0) \in N$. Hence $ab \in (N :_R M)$ or $am \in N$ or $bm \in N$. Hence we may assume that $abN = 0$. If $am(N :_R M) \neq 0$, then there exists $r_0 \in (N :_R M)$ such that $ar_0m \neq 0$. Then $0 \neq ar_0m = a(b + r_0)m \in N$. Since N is weakly 2-absorbing we have $a(b + r_0) \in (N :_R M)$ or $am \in N$ or $(b + r_0)m \in N$. Therefore $ab \in (N :_R M)$ or $am \in N$ or $bm \in N$. So we can assume that $am(N :_R M) = 0$. Likewise we can assume that $bm(N :_R M) = 0$. Since $(N :_R M)^2N \neq 0$, there exist $a_0, b_0 \in (N :_R M)$ and $x_0 \in N$ with $a_0b_0x_0 \neq 0$. If $ab_0x_0 \neq 0$, then $0 \neq ab_0x_0 = a(b+b_0)(m+x_0) \in N$ implies that $a(b+b_0) \in (N :_R M)$ or $a(m+x_0) \in N$ or $(b+b_0)(m+x_0) \in N$. Hence $ab \in (N :_R M)$ or $am \in N$ or $bm \in N$. So we can assume that $ab_0x_0 = 0$. Likewise we can assume that $a_0b_0m = 0$ and $a_0bx_0 = 0$. Then from $0 \neq a_0b_0x_0 = (a+a_0)(b+b_0)(m+x_0) \in N$ we get $(a+a_0)(b+b_0) \in (N :_R M)$ or $(a+a_0)(m+x_0) \in N$ or $(b+b_0)(m+x_0) \in N$. Therefore $ab \in (N :_R M)$ or $am \in N$ or $bm \in N$, and so N is 2-absorbing. □

Let R be a commutative ring. An R-module M is called a multiplication module if for each submodule N of M, $N = IM$ for some ideal I of R. In this case we can take $I = (N : M)$ (see [9]).

Let R be a commutative ring and M an R-module. For a submodule N of M, if $N = IM$ for some ideal I of R, then we say that I is a presentation ideal of N. Note that it is possible that for a submodule N, no such presentation ideal exist. For example, assume that M is a vector space over an arbitrary field F with $dim_F M \geq 2$ and let N be a proper subspace of M such that $N \neq 0$. Then M is finite length (so M is both Noetherian, Artinian and injective), but M is not multiplication and N has not any presentation. Clearly, every submodule of M has a presentation ideal if and only if M is a multiplication module. Let N and K be submodules of a multiplication R-module M with $N = I_1M$ and $K = I_2M$ for some ideals I_1 and I_2 of R. The product N and K denoted by NK is defined by $NK = I_1I_2M$. Then by [10, Theorem 3.4], the product of N and K is independent of presentations of N and K. Moreover, for $a, b \in M$, by ab, we mean the product of Ra and Rb. Clearly, NK is a submodule of M and $NK \subseteq N \cap K$ (see [10]).

A submodule N of an R-module M is called a nilpotent submodule if $(N :_R M)^k N = 0$ for some positive integer k (see [11]).

Corollary 2.6. *Let R be a commutative ring and let M be an R-module. Assume that N is a weakly 2-absorbing submodule of M that is not 2-absorbing, then*

(1) N is nilpotent.

(2) If M is a multiplication module, then $N^3 = 0$.

Proposition 2.7. *Let R be a ring and M a faithful multiplication R-module. Let N be a weakly 2-absorbing submodule of M. If N is not 2-absorbing, then $N \subseteq M - rad0$.*

Proof. Suppose that N is not 2-absorbing. By Theorem 2.5, $(N :_R M)^2N = 0$. In this case we have $(N :_R M)^3 \subseteq ((N :_R M)^2N :_R M) = (0 :_R M) = 0$ since M is faithful, so that $(N :_R M)^3 = 0$. If $a \in (N :_R M)$, then $a^3 = 0$ and hence $a \in sqrt0$. So that $(N :_R M) \subseteq \sqrt{0}$ and therefore $N = (N :_R M)M \subseteq \sqrt{0}M = M - rad0$. □

Theorem 2.8. *Let R be a commutative ring and M a finitely generated, faithful multiplication R-module. If N a nonzero weakly 2-absorbing projective submodule of M. Then N is never 2-absorbing.*

Proof. We first prove that $(0:_R (N:_R M)^2N) = (0:_R N)$. Clearly $(0:_R N) \subseteq (0:_R (N:_R M)^2N)$. For the reverse containment, pick an element $a \in (0:_R (N:_R M)^2N)$. Since M is a multiplication module, $N = (N:_R M)M$. So it follows from $a(N:_R M)^2N = 0$ that $a(N:_R M)^3 \subseteq (0:_R M) = 0$. So $a(N:_R M)^3 = 0$. Therefore $(0:_R M)\cap(N:_R M) \supseteq a(N:_R M)^2$. Let $Tr(N) = \sum_{\phi\in Hom_R(N,R)} \phi(N)$ be the trace ideal of N. Since N is projective, $Tr(N)$ is a pure ideal of R, $(0:_R N) = (0:_R Tr(N))$ and $N = Tr(N)N$ by [12, Proposition 3.30]. Hence

$$(N:_R M) = (Tr(N)N:_R M) = Tr(N)(N:_R M) \subseteq Tr(N).$$

Consequently $a(N:_R M)^2 \subseteq Tr(N) \cap (0:_R Tr(N)$. Since $Tr(N)$ is a pure ideal of R, from [13], we get $Tr(N) \cap (0:_R Tr(N)) = 0$. Therefore $a(N:_R M)^2 = 0$. By repeating this method, we will have $a(N:_R M) = 0$, and hence therefore $aN = 0$, that is $a \in (0:_R N)$. Thus $(0:_R (N:_R M)^2N) \subseteq (0:_R N)$. Therefore $(0:_R (N:_R M)^2N) = (0:_R N)$. Now if N is not 2-absorbing, then $N^3 = 0$. Then $(0:_R N) = (0:_R N^3) = R$ which is a contradiction. □

Proposition 2.9. *Let R be a commutative a ring, M a cyclic R-module and N a submodule of M.*

(1) N is a 2-absorbing submodule of M if and only if $(N:_R M)$ is a 2-absorbing ideal of R.

(2) If in addition M is faithful, then N is a weakly 2-absorbing submodule of M if and only if $(N:_R M)$ is a weakly 2-absorbing ideal of R.

Proof. (1) Let N ba a 2-absorbing submodule of M. Assume that $M = Rm$ for some $m \in M$ and let $abc \in (N:_R M)$ for some $a, b, c \in R$, but $ab \notin (N:_R M)$ and $bc \notin (N:_R M)$. There exist $r, s \in R$ such that $ab(rm) \notin N$ and $bc(sm) \notin N$. Now it follows from $abcm \in N$, $abm \notin N$ and $bcm \notin N$ that $ac \in (N:_R M)$ since N is 2-absorbing. Conversely, assume that $(N:_R M)$ is a 2-absorbing ideal of R and let $abx \in N$ for some $a, b \in R$ and $m \in N$. There exists $c \in R$ with $x = rm$. Then $abc \in (N:_R m) = (N:_R M)$ and $(N:_R M)$ weakly 2-absorbing gives $ab \in (N:_R M)$ or $ac \in (N:_R M)$ or $bc \in (N:_R M)$. Therefore $ab \in (N:_R M)$ or $ax \in N$ or $bx \in N$. Hence N is 2-absorbing.

(2) Let $0 \neq abc \in (N:_R M)$ for some $a, b, c \in R$, but $ab \notin (N:_R M)$ and $bc \notin (N:_R M)$. Then $0 \neq abcm \in N$ for otherwise $abcm = 0$ implies that $abc \in (0:_R M) = 0$, a contradiction. Now a similar argument as in the proof of part (1) will complete the proof. □

Let N be a proper submodule of a nonzero R-module M. Then the M-radical of N, denoted by M-rad N, is defined in [14] to be the intersection of all prime submodules of M containing N. It is shown in [9, Theorem 2.12] that if N is a proper submodule of a multiplication R-module M, then M-rad $N = Rad((N:_R M))M$.

Theorem 2.10. *Let R be a commutative a ring, M a cyclic R-module and N a 2-absorbing submodule of M. Then one of the following hold:*

(1) $M - radN = P$ *is a prime submodule of* M *such that* $P^2 \subseteq N$*.*

(2) $M - radN = P_1 \cap P_2$*,* $P_1P_2 \subseteq N$ *and* $(M - radN)^2 \subseteq N$ *where* P_1, P_2 *are the only distinct minimal prime submodules of* N*.*

Proof. By Proposition 2.9, $(N :_R M)$ is a 2-absorbing submodule of M. It follows from [6, Theorem 2.4] that either $Rad((N :_R M)) = p$ is a prime ideal of R such that $p^2 \subseteq (N :_R M)$ or $Rad((N :_R M)) = p_1 \cap p_2$, $p_1p_2 \subseteq (N :_R M)$, and $Rad((N :_R M))^2 \subseteq (N :_R M)$ where p_1 and p_2 are minimal prime ideals of $(N :_R M)$. If the former case holds, then since M is multiplication, we have $M - radN = Rad((N :_R M))M = pM$ is a prime submodule of M by [9, Corollary 2.11, Theorem 2.12] and $(pM)^2 = p^2M \subseteq (N :_R M)M = N$. Now assume that the latter case holds. Then, by [9, Corollary 2.11], p_1M and p_2M are minimal prime submodules of N and $M - radN = Rad((N :_R M))M = (p_1 \cap p_2)M = (p_1 + ann(M))M \cap (p_2 + ann(M))M = p_1M \cap p_2M$ by [9, Theorem 2.12]. Moreover $(p_1M)(p_2M) = (p_1p_2)M \subseteq (N :_R M)M = N$ and $(M - radN)^2 = (Rad((N :_R M))M)^2 = (Rad((N :_R M)))^2M \subseteq (N :_R M)M = N$. □

Theorem 2.11. *Let* N *be a* p*-primary submodule of a cyclic* R*-module. Then* N *is* 2*-absorbing if and only if* $(pM)^2 \subseteq N$*.*

Proof. Assume first that N is 2-absorbing. Since N is p-primary, it follows from [9, Theorem 2.12] that $M - radN = pM$. Hence $(pM)^2 \subseteq N$ by Theorem 2.10. Conversely, assume that $(pM)^2 \subseteq N$ and let $abm \in N$ for some $a, b \in R$ and $m \in M$, $am \notin N$ and $bm \notin N$. Since N is p-primary, it follows that $a, b \in \sqrt{(N :_R M)} = p$. Thus $ab \in p^2 \subseteq (N :_R M)$. Therefore N is 2-absorbing. □

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