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A Note on Kurosh Amitsur Radical and Hoehnke Radical

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Abstract : The notion of Radical classes is introduced in [1]. We prove here some useful equivalent conditions for a subclass of a fixed universal class to be a radical class. We introduce the notion of Hoehnke radical and give some consequences of Hoehnke radical and Kurosh-Amitsur radical.

Keywords : Semirings; Ideal; k-ideals; Radical class.
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1 Introduction

The paper is concerned with generalizing some results in ring theory. In correspondence to the Kurosh-Amitsur radical theory for associative rings, an abstract concept of radical classes and radicals for semirings has been introduced and investigated in a series of publications [2–5] by Olson and several coauthors.

There are some definitions of radical class appearing in the semiring literature. But we were looking for the definition given by Althani [1], who has introduced the definition of radical class in a different way. Using [1], in this paper we give some useful equivalent conditions for a subclass of a fixed universal class to be a radical class. We also introduce the notion of Hoehnke radical and investigate

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some interrelationship between Kurosh-Amitsur radical and Hoehnke radical.

Some of the statements in this paper are more or less known from [6], but we organize and prove them in a somewhat different way more appropriate for our purposes.

2 Preliminaries

Throughout this paper, a semiring will be defined as follows: A semiring is a set S together with two binary operations called addition $(+)$ and multiplication $(\cdot)$ such that $(S,+)$ is a commutative monoid with identity element 0_S; $(S,\cdot)$ is a monoid with identity element 1; multiplication distributes over addition from either side and 0 is multiplicative absorbing element, that is, $r0 = 0r = 0$ for each $r \in S$. A semiring S is commutative if $(S,\cdot)$ is a commutative semigroup.

A subset I of a semiring S will be called an *ideal* of S if I is an additive subsemigroup of $(S,+)$, $IS \subseteq I$ and $SI \subseteq I$. An ideal I of a semiring S is called *proper* if and only if $I \neq S$ and a proper ideal I of S is called *maximal* if and only if there is no ideal J of S satisfying $I \subset J \subset S$. An ideal I of a semiring S will be called *subtractive (k-ideal)* if for $a \in I, a+b \in I, b \in S$ imply $b \in I$.

The k-closure $\overline{I}$ of an ideal I, defined by $\overline{I} = \{s \in S \ / \ s+a \in I \text{ for some } a \in I\}$, is an ideal of S as well. An ideal I of a semiring S is called a k-ideal(closed) if $\overline{I} = I$ is true. We denote the set of all ideals of S by $\mathcal{I}(S)$ and the set of all k-ideals by $\mathcal{K}(S)$. Note that the ideals defined in this way should be called more precisely "semiring ideals". This is of importance if (associative) rings occur in semiring-theoretical investigations, since a ring S, considered as a semiring, may have semiring ideals which are not "ring ideals" in the usual meaning. More precisely, a semiring ideal I of S, is a ring ideal of S if and only if $\overline{I} = I$ holds, i.e. if and only if I is a k-ideal of S. We denote $I \triangleright S$, a semiring ideal in S.

Each homomorphism $\phi\colon S \to T$ of semirings corresponds to a congruence k of S and the homomorphic image $\phi(S)$ is isomorphic to the semiring S/k of congruence classes. In this paper we mainly use congruences that are determined by an ideal I of S according to $sk_Is' \Leftrightarrow$ there are

$$a_i \in I \text{ satisfying } \ s + a_1 = s' + a_2.$$

In this case one usually denotes S/k_I by S/I. Moreover, $k_I = k_{\overline{I}}$ and thus $S/I = S/\overline{I}$ hold for all ideals I of S with the same k-closure $\overline{I}$, S/I has always an absorbing zero, namely the congruence class $\overline{I} = [a]_I = [a]_{\overline{I}}$ determined by each $a \in I$. We also mention that a semiring has in general much more congruences than those determined by its ideals. For a last concept of this kind, let $\phi\colon S \to T$ be a surjective homomorphism for semirings which have a zero. Then ϕ is called a semi-isomorphism and denoted by $\phi\colon S \tilde{\to} T$ if $\phi(0_S) = 0_T$ and $\phi^{-1}(0_T) = 0_S$ are satisfied. We emphasize here that such a semi-isomorphism, despite of misleading name, has in general very little in common with an isomorphism.

Convention: Throughout $R \mapsto S$ is a surjective homomorphism.

Theorem 2.1 ([6])**.** *Let S be a semiring, T a semiring with an absorbing zero 0_T, and $\phi\colon S \to T$ a surjective homomorphism. Then $K = \phi^{-1}(0_T)$ is a k-ideal of S (also called the kernel of ϕ) and $\phi([s]_K) = \phi(s)$ for all $s \in S$ defines a semi-isomorphism $\phi\colon S/K \tilde{\to} T$ which satisfies $\phi \circ {k_K}^{\#} = \phi$, where ${k_K}^{\#}$ denotes the natural homomorphism of S onto $S/K = S/k_K$.*

Theorem 2.2 ([6])**.** *For a semiring S with an absorbing zero 0, let S be a subsemiring which contains 0 and B an ideal of S. Then $\phi([a]_{A\cap\overline{B}}) = [a]_B$ for all $a \in A \subseteq A + B$ defines a semi-isomorphism*

$$\phi\colon A/A \cap \overline{B} \tilde{\to} A + B/B.$$

Theorem 2.3 ([6])**.** *Let A, B be ideals of a semiring S with the additional condition $A \subseteq B$. Then $\overline{\phi}([s]_B) = [[s]_A]_{\overline{B}/A}$ for all $s \in S$ defines an isomorphism*

$$\overline{\phi}\colon S/B \to (S/A)/(\overline{B}/A).$$

Definition 2.4 ([1])**.** Let $\mathcal{R}$ be a class of semirings. A semiring (ideal) belonging to the class $\mathcal{R}$, will be called a $\mathcal{R}$-*semiring* ($\mathcal{R}$-ideal).

Definition 2.5 ([1])**.** A class $\mathcal{R}$ of semirings is called a *radical class* whenever the following three conditions are satisfied:

(a) $\mathcal{R}$ is homomorphically closed; i.e. if S is a homomorphic image of a $\mathcal{R}$-semiring R then S is also a $\mathcal{R}$-semiring.

(b) Every semiring R contains a $\mathcal{R}$-ideal $\mathcal{R}(R)$ which in turn contains every other $\mathcal{R}$-ideal of R.

(c) The factor semiring $R/\mathcal{R}(R)$ does not contain any nonzero $\mathcal{R}$-ideal; i.e. $\mathcal{R}(R/\mathcal{R}(R)) = 0$.

Proposition 2.6. *Assuming conditions (a) and (b) on a class $\mathcal{R}$ of semirings, condition (c) is equivalent to*
(c′) If I is an ideal of the semiring R and if both I and R/I are in $\mathcal{R}$, then R itself is in $\mathcal{R}$.

Proof. (c) $\Longrightarrow$ (c′) follows by [6, Theorem 1.3]. Conversely, suppose that (c') holds. If $\mathcal{R}(R/\mathcal{R}(R)) = K/\mathcal{R}(R) \neq 0$ for some ideal K of R, then $K \in \mathcal{R}$ by (c'). Thus $K \subset \mathcal{R}(R)$ and $K/\mathcal{R}(R) = 0$, a contradiction to the fact that $\mathcal{R}(R/\mathcal{R}(R)) \neq 0$. Hence (c). □

$\mathcal{R}$ is closed under extensions if $\mathcal{R}$ satisfies (c').

Proposition 2.7. *Assuming conditions (a) and (c′) on a class $\mathcal{R}$ of semirings, condition (b) is equivalent to*
(b′) If $I_1 \subset I_2 \subset \cdots \subset I_\lambda \subset \ldots$ is an ascending chain of ideals of a semiring R and if each I_λ is in $\mathcal{R}$, then $\bigcup I_\lambda$ is in $\mathcal{R}$.

Proof. Suppose that (b) holds and $B = \bigcup I_\lambda$.Therefore $B = \mathcal{R}(B)$ is in $\mathcal{R}$ and hence the if part. Conversely, assume that (b') holds. Then we can apply Zorn's lemma and obtain a maximal $\mathcal{R}$-ideal B of R. If K is an $\mathcal{R}$ ideal of R, then $B+K/K$ is in $\mathcal{R}$ by (a). Thus both K and $B+K/K$ are in $\mathcal{R}$ and by (c') $B+K$ is in $\mathcal{R}$. Since B is maximal with respect to this property, K must be in B and thus $\mathcal{R}(R) = B$ which is in $\mathcal{R}$. □

$\mathcal{R}$ has inductive property if $\mathcal{R}$ satisfies (b').

Theorem 2.8. *A non-empty sub class $\mathcal{R}$ of a universal class $\mathbb{U}$ is a radical class if and only if*

a) $\mathcal{R}$ is homomorphically closed.

b′) $\mathcal{R}$ has the inductive property.

c′) $\mathcal{R}$ is closed under extensions.

Theorem 2.9. *For any subclass $\mathcal{R}$ of a fixed universal class $\mathbb{U}$, the following conditions are equivalent*

I. *$\mathcal{R}$ is a radical class.*

II. *(R1) If $R \in \mathcal{R}$ then every $R \mapsto S \neq 0$ there is a $I \lhd S$ such that $0 \neq I \in \mathcal{R}$.*

   (R2) If R is a semiring of a universal class $\mathbb{U}$ and for every $R \mapsto S \neq 0$ there is a $I \lhd S$ such that $0 \neq I \in \mathcal{R}$, then $R \in \mathcal{R}$.

III. *$\mathcal{R}$ satisfies condition (R1), has the induction property and closed under extensions.*

3 Examples

3.1 Köthes Nil Radical

$$\mathcal{N} = \{A \;:\; \forall a \in A\ \exists n \geq 1,\ \ n \text{ depending on } a \text{ such that } a^n = 0\},$$

i.e. $\mathcal{N}$ is a class of all nil semirings. Then $\mathcal{N}$ is a radical class called the Nil radical class, usually denoted by $\mathcal{N}(A)$.

3.2 Von-Neumann Radical

A semiring is said to be Von - Neumann regular if for every $a \in A$ $a = aba\ \forall\ b \in A$ or $a \in aAa$. The class

$$\mathcal{V} = \{A : A \text{ is Von-Neumann regular}\} = \{a \in A : a = aba\ \forall b \in A\}$$

is a radical class.

4 Hoehnke Radical

From an axiomatic point of view a radical $\mathcal{R}$ may be defined as an assignment $\mathcal{R} : R \longrightarrow \mathcal{R}(R)$ designating a certain ideal $\mathcal{R}(R)$ to each semiring R. Such an assignment $\mathcal{R}$ is called Hoehnke radical if

i) $f(\mathcal{R}(R)) \subseteq \mathcal{R}(f(R))$ for every homomorphism $f : R \mapsto \mathcal{R}(R)$.

ii) $\mathcal{R}(R/\mathcal{R}(R)) = 0$, for every semiring R.

A Hoehnke radical $\mathcal{R}$ may also satisfy the following conditions:

iii) $\mathcal{R}$ is complete: If $I \rhd R$ and $\mathcal{R}(I) = I$ then $I \subseteq \mathcal{R}(I)$.

iv) $\mathcal{R}$ is idempotent: $\mathcal{R}(\mathcal{R}(R)) = \mathcal{R}(R)$, for every semiring R.

Theorem 4.1. *If $\mathcal{R}$ is a Kurosh-Amitsur radical then the assignment $A \to \mathcal{R}(R)$ is a complete, idempotent, Hoehnke radical. Conversely, if $\mathcal{R}$ is a complete, idempotent, Hoehnke radical, then there is a Kurosh-Amitsur radical $\wp$ such that $\mathcal{R}(R) = \wp(R)$ for every semiring R. Moreover $\wp = \{R : \mathcal{R}(R) = R\}$.*

Proof. If part is immediate. Conversely, assume that $\mathcal{R}$ is a complete, idempotent, Hoehnke radical and define the class $\wp$ by $\wp = \{R : \mathcal{R}(R) = R\}$. Claim that $\wp$ is a Kurosh-Amitsur radical class such that $\mathcal{R}(R) = \wp(R)$ for every semiring R.

Let R be in $\wp$ and $\phi\colon R \to S$ be a surjective homomorphism. Then by (i) $S = \phi(R) = \phi(\mathcal{R}(R)) \subseteq \mathcal{R}(\phi(R)) = \mathcal{R}(S)$. Therefore $\mathcal{R}(S) = S \in \wp$. This gives us condition (a) for $\wp$. For every semiring R, $\wp(R) = \{I \rhd R : I \in \wp\} = \{I \rhd R : R(I) = I\}$. If $I \rhd R$ and $\mathcal{R}(I) = I$, then $I \subseteq \wp(R)$ and by (iii) $I \subseteq \mathcal{R}(\wp(R))$. Therefore $\wp(R) \subseteq \mathcal{R}(R)) \subseteq \wp(R)$. Shows that $\wp(R) = \mathcal{R}(R)$ and hence $\wp(R) \in \wp$. Hence condition (b) for $\wp$.

Also $\wp(R) = \mathcal{R}(\wp(R))$ and by completeness property (iii), $\wp(R) \subseteq \mathcal{R}(R)$. But by idempotentness property (iv), $\mathcal{R}(\mathcal{R}(R)) = \mathcal{R}(R)$,implies that $\mathcal{R}(R) = \wp(R)$ for all semirings R. Moreover $\wp(A/\wp(R)) = (R/\mathcal{R}(R)) = 0$. Hence the theorem. ☐

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