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Some Approximation Theorems by De La Vallée-Pousin Mean

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Abstract : In this paper we apply the generalized de la Vallée-Pousin mean to prove some Korovkin type approximation theorems. We also show by an example that there is a sequence of linear operators for which the Korovkin theorem does not hold but our theorem holds. Finally, we apply regular matrices in proving these theorems.

Keywords : Approximation theorems; Korovkin theorem; De la Vallée-Pousin mean; (V,λ)-summability; Regular matrix.
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1 Introduction and Preliminaries

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ such that

$$\lambda_{n+1} \leq \lambda_n + 1,\ \lambda_1 = 0.$$

The generalized de la Vallée-Pousin mean is defined by

$$t_n(x) =: \frac{1}{\lambda_n} \sum_{j \in I_n} x_j$$

where $I_n = [n - \lambda_n + 1, n]$.

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A sequence $x = (x_j)$ is said to be (V, λ)-*summable* to a number ℓ (see [1]) if

$$t_n(x) \to \ell \text{ as } n \to \infty.$$

In this case ℓ is called the λ-lim x. If $\lambda_n = n$ then (V, λ)-summablity is reduced to the Cesàro summability.

Remark 1.1. *It is observed that if a sequence is convergent to a number ℓ, then it is also (V, λ)-summable to the same number ℓ but the converse need not be true. For example, let the sequence $z = (z_n)$ be defined by*

$$z_n = \begin{cases} (-1)^n, & \text{if } \lambda_n = n \in \mathbb{N}; \\ 0, & \text{otherwise.} \end{cases}$$

Then x is (V, λ)-summable to 0 but not convergent.

In this paper, we generalize the results of Mursaleen [2] by using the concept of (V, λ)-summability. Further we apply regular matrices to get more general results. The classical Korovkin approximation theorem states as follows (see [3–5]): Let (T_n) be a sequence of positive linear operators from $C[a, b]$ into $C[a, b]$. Then $\lim_n \|T_n(f, x) - f(x)\|_\infty = 0$, for all $f \in C[a, b]$ if and only if $\lim_n \|T_n(f_i, x) - f_i(x)\|_\infty = 0$, for $i = 0, 1, 2$, where $f_0(x) = 1$, $f_1(x) = x$ and $f_2(x) = x^2$.

2 Main Results

The following is the (V, λ)-summability version of the classical Korovkin approximation theorem followed by an example to show its importance.

Let $C[a, b]$ be the space of all functions f continuous on $[a, b]$. We know that $C[a, b]$ is a Banach space with norm $||f||_\infty := \sup_{a \le x \le b} |f(x)|$, $f \in C[a, b]$. Suppose that $T_n : C[a, b] \to C[a, b]$. We write $T_n(f, x)$ for $T_n(f(t), x)$ and we say that T is a positive operator if $T(f, x) \ge 0$ for all $f(x) \ge 0$.

Throughout the paper we consider the function f to be continuous on $[a, b]$ and at the end points.

Theorem 2.1. *Let (T_k) be a sequence of positive linear operators from $C[a, b]$ into $C[a, b]$. Then for any function $f \in C[a, b]$ bounded on the whole real line, we have*

$$\lambda - \lim_n ||T_n(f, x) - f(x)||_\infty = 0. \tag{2.1}$$

if and only if

$$\lambda - \lim_n ||T_n(1, x) - 1||_\infty = 0, \tag{2.2}$$

$$\lambda - \lim_n ||T_n(t, x) - x||_\infty = 0, \tag{2.3}$$

$$\lambda - \lim_n ||T_n(t^2, x) - x^2||_\infty = 0. \tag{2.4}$$

Proof. Since each $1, x, x^2$ belongs to $C[a,b]$, conditions (2.2)-(2.4) follow immediately from (2.1). Since $f \in C[a,b]$ and f is bounded on the whole real line, we have

$$|f(x)| \leq M, \quad -\infty < x < \infty.$$

Therefore,

$$|f(t) - f(x)| \leq 2M, \quad -\infty < t, x < \infty. \tag{2.5}$$

Also, since $f \in C[a,b]$ we do have that f is continuous on $[a,b]$, i.e.

$$|f(t) - f(x)| < \epsilon, \quad \forall\ |t - x| < \delta. \tag{2.6}$$

Using (2.5), (2.6) and putting $\psi(t) = (t-x)^2$, we get

$$|f(t) - f(x)| < \epsilon + \frac{2M}{\delta^2}\psi, \quad \forall\ |t - x| < \delta.$$

This means

$$-\epsilon - \frac{2M}{\delta^2}\psi < f(t) - f(x) < \epsilon + \frac{2M}{\delta^2}\psi.$$

Now we could apply $T_n(1,x)$ to this inequality since $T_n(f,x)$ is monotone and linear. Hence,

$$T_n(1,x)\left(-\epsilon - \frac{2M}{\delta^2}\psi\right) < T_n(1,x)\left(f(t) - f(x)\right) < T_n(1,x)\left(\epsilon + \frac{2M}{\delta^2}\psi\right).$$

Note that x is fixed and so $f(x)$ is constant number. Therefore,

$$-\epsilon T_n(1,x) - \frac{2M}{\delta^2}T_n(\psi,x) < T_n(f,x) - f(x)T_n(1,x) < \epsilon T_n(1,x) + \frac{2M}{\delta^2}T_n(\psi,x). \tag{2.7}$$

But,

$$\begin{aligned} T_n(f,x) - f(x) &= T_n(f,x) - f(x)T_n(1,x) + f(x)T_n(1,x) - f(x) \\ &= [T_n(f,x) - f(x)T_n(1,x)] + f(x)[T_n(1,x) - 1]. \end{aligned} \tag{2.8}$$

Using (2.7) and (2.8), we have

$$T_n(f,x) - f(x) < \epsilon T_n(1,x) + \frac{2M}{\delta^2}T_n(\psi,x) + f(x)(T_n(1,x) - 1). \tag{2.9}$$

Now, let us estimate $T_n(\psi,x)$,

$$\begin{aligned} T_n(\psi,x) &= T_n((t-x)^2,x) = T_n(t^2 - 2tx + x^2, x) \\ &= T_n(t^2,x) - 2xT_n(t,x) + x^2T_n(1,x) \\ &= [T_n(t^2,x) - x^2] - 2x[T_n(t,x) - x] + x^2[T_n(1,x) - 1]. \end{aligned}$$

Using (2.9), we get

$$\begin{aligned} T_n(f,x) - f(x) &< \epsilon T_n(1,x) + \frac{2M}{\delta^2}\{[T_n(t^2,x) - x^2] - 2x[T_n(t,x) - x] \\ &\quad + x^2[T_n(1,x) - 1]\} + f(x)(T_n(1,x) - 1) \\ &= \epsilon[T_n(1,x) - 1] + \epsilon + \frac{2M}{\delta^2}\{[T_n(t^2,x) - x^2] - 2x[T_n(t,x) - x] \\ &\quad + x^2[T_n(1,x) - 1]\} + f(x)(T_n(1,x) - 1). \end{aligned}$$

Since ϵ is arbitrary we can write

$$\begin{aligned} ||T_n(f,x) - f(x)||_\infty \leq \left(\epsilon + \frac{2Mb^2}{\delta^2} + M\right)||T_n(1,x) - 1||_\infty + \frac{4Mb}{\delta^2}||T_n(t,x) - x||_\infty \\ + \frac{2M}{\delta^2}||T_n(t^2,x) - x^2||_\infty. \end{aligned} \tag{2.10}$$

Now replacing $T_n(t,x)$ by $B_k(t,x) = \frac{1}{\lambda_k}\sum_{n\in I_k} T_n(t,x)$, taking the limit as $k \to \infty$ on both sides of (2.10) and using conditions (2.2), (2.3) and (2.4), we get

$$\lambda - \lim_n ||T_n(f,x) - f(x)||_\infty = 0.$$

This completes the proof of the theorem. □

In the following we give an example of a sequence of positive linear operators satisfying the conditions of Theorem 2.1 but does not satisfy the conditions of the Korovkin theorem.

The following example shows that our Theorem 2.1 deals even with those sequences which do not satisfy the conditions of the classical Korovkin theorem.

Example 2.2. *Consider the sequence of classical Bernstein polynomials*

$$B_n(f,x) := \sum_{k=0}^{n} f\left(\frac{n}{k}\right)\binom{n}{k} x^k(1-x)^{n-k};\ 0 \leq x \leq 1.$$

Let the sequence (P_n) *be defined by* $P_n : C[0,1] \to C[0,1]$ *with* $P_n(f,x) = (1+ z_n)B_n(f,x)$*, where* z_n *is defined as above. Then*

$$B_n(1,x) = 1,\ B_n(t,x) = x,\ B_n(t^2,x) = x^2 + \frac{x - x^2}{n},$$

and the sequence (P_n) *satisfies the conditions (2.2), (2.3) and (2.4). Hence we have*

$$\lambda - \lim_n ||P_n(f,x) - f(x)||_\infty = 0.$$

On the other hand, we get $P_n(f,0) = (1+ z_n)f(0)$*, since* $B_n(f,0) = f(0)$*, and hence*

$$||P_n(f,x) - f(x)||_\infty \geq | P_n(f,0) - f(0) | = z_n \mid f(0) \mid .$$

We see that (P_n) *does not satisfy the classical Korovkin theorem, since* $\limsup_{n\to\infty} z_n$ *does not exist.*

Next we study a Korovkin type theorem for a sequence of positive linear operators on $L_p[a,b]$ by using the generalized de la Vall ée-Pousin mean.

Theorem 2.3. *Let (T_n) be the sequence of positive linear operators $T_n : L_p[a,b] \to L_p[a,b]$ and let the sequence $\{||T_n||\}$ be uniformly bounded. If*

$$\lambda - \lim_n ||T_n(1,x) - 1||_{L_p} = 0,$$

$$\lambda - \lim_n ||T_n(t,x) - x||_{L_p} = 0,$$

and

$$\lambda - \lim_n ||T_n(t^2,x) - x^2||_{L_p} = 0.$$

Then for any function $f \in L_p[a,b]$, we have

$$\lambda - \lim_n ||T_n(f,x) - f(x)||_{L_p} = 0.$$

Remark 2.4. *We can reformulate the above theorem under the same hypothesis as follows, that is, if*

$$\lim_n ||D_n(1,x) - 1||_{L_p} = 0,$$

$$\lim_n ||D_n(t,x) - x||_{L_p} = 0,$$

and

$$\lim_n ||D_n(t^2,x) - x^2||_{L_p} = 0,$$

hold. Then for any function $f \in L_p[a,b]$, we have

$$\lim_n ||D_n(f,x) - f(x)||_{L_p} = 0,$$

where $D_n = \frac{1}{\lambda_n}\sum_{k\in I_n} T_k$.

Now we present a slight general result.

Theorem 2.1. *Let (T_n) be a sequence of positive linear operators on $L_p[a,b]$ such that*

$$\lim_n \frac{1}{\lambda_n} \sum_{k\in I_n} ||T_n - T_k|| = 0.$$

If

$$\lambda - \lim_n ||T_n(t^\nu,x) - x^\nu||_{L_p} = 0 \quad (\nu = 0,1,2). \tag{2.11}$$

Then for any function $f \in L_p[a,b]$, we have

$$\lim_n ||T_n(f,x) - f(x)||_{L_p} = 0. \tag{2.12}$$

Proof. From Theorem 2.3, we have that if (2.11) holds then

$$\lambda - \lim_n ||T_n(f,x) - f(x)||_{L_p} = 0,$$

which is equivalent to

$$\lim_n ||D_n(f,x) - f(x)||_{L_p} = 0,$$

that $(D_n(f,x))$ is convergent to $f(x)$ in L_p-norm. Now

$$\begin{aligned} T_n - D_n &= T_n - \frac{1}{\lambda_n}\sum_{k\in I_n} T_k \\ &= \frac{1}{\lambda_n}\sum_{k\in I_n}(T_n - T_k). \end{aligned}$$

Hence using the hepothesis we get

$$\lim_n ||T_n(f,x) - f(x)||_{L_p} = \lim_n ||D_n(f,x) - f(x)||_{L_p} = 0,$$

that is (2.12) holds. □

3 λ-Convergence with Order

In this section we define the order of λ-convergence of a sequence of positive linear operators and give an analogue of Theorem 2.1.

Definition 3.1. The number sequence $x = (x_k)$ is λ-*convergent to the number* L *with order* $0 < \beta < 1$ if

$$\lim_n \frac{1}{(\lambda_n)^{1-\beta}} \sum_{j\in I_n} x_j = L.$$

In this case, we write

$$x_n - L = o((\lambda_n)^{-\beta}), \text{ as } n \to \infty.$$

Theorem 3.2. *Suppose that* $T_n : C[a,b] \to C[a,b]$ *is a sequence of positive linear operator satisfying the following conditions*

$$||T_n(1,x) - 1||_\infty = o((\lambda_n)^{-\beta_1}),$$

$$||T_n(t,x) - x||_\infty = o((\lambda_n)^{-\beta_2}),$$

$$||T_n(t^2,x) - x^2||_\infty = o((\lambda_n)^{-\beta_3}).$$

Then for any function $f \in C[a,b]$, *we have*

$$||T_n(f,x) - f(x)||_\infty = o((\lambda_n)^{-\beta}), \quad \textit{as } n \to \infty,$$

where $\beta = min\{\beta_1, \beta_2, \beta_3\}$.

Proof. We can rewrite the inequality (2.10) as follows:

$$\frac{||T_n(f,x)-f(x)||_\infty}{(\lambda_k)^{1-\beta}} \leq \left(\epsilon + \frac{2Mb^2}{\delta^2} + M\right) \frac{||T_n(1,x)-1||_\infty}{(\lambda_k)^{1-\beta_1}} \left(\frac{(\lambda_k)^{1-\beta_1}}{(\lambda_k)^{1-\beta}}\right) + \frac{4Mb}{\delta^2} \frac{||T_n(t,x)-x||_\infty}{(\lambda_k)^{1-\beta_2}} \left(\frac{(\lambda_k)^{1-\beta_2}}{(\lambda_k)^{1-\beta}}\right) + \frac{2M}{\delta^2} \frac{||T_n(t^2,x)-x^2||_\infty}{(\lambda_k)^{1-\beta_3}} \left(\frac{(\lambda_k)^{1-\beta_3}}{(\lambda_k)^{1-\beta}}\right).$$

Hence,

$$||T_n(f,x)-f(x)||_\infty = o((\lambda_n)^{-\beta}), \quad \text{as } n \to \infty,$$

where β =min$\{\beta_1, \beta_2, \beta_3\}$. □

4 Approximation by Regular Matrices

In this section, we discass the applications of regular matrices in proving above approximation theorems. Let c be the Banach space of all convergent sequences of real numbers with the usual supremum norm. If $x = (x_k)$ is a number sequence and $Ax = (A_n(x))$ is an infinite matrix, then Ax is the sequence whose nth term is given by $A_n(x) = \sum_k a_{nk}x_k$. We say that x is A-*summable* to L if $\lim_n A_n(x) = L$, provided that the series $\sum_k a_{nk}x_k$ converges for each n; L is called the A-limit of x. Let X and Y be any two sequence spaces. If $x \in X$ implies $Ax \in Y$, then we say that the matrix A maps X into Y. We denote by (X, Y), the class of all matrices A which map X into Y.

A matrix $A = (a_{nk})$ is said to be *regular* [6] if $Ax \in c$ for all $x \in c$ with A-lim $x = \lim x$.

The following are the well-known Silverman-Toeplitz conditions for the regularity of A.

Lemma 4.1. *A matrix $A = (a_{nk})$ is regular if and only if*

(i) $||A|| = \sup_n \sum_k |a_{nk}| < \infty$,

(ii) $\lim_n a_{nk} = 0$, *for each k*,

(iii) $\lim_n \sum_k a_{nk} = 1$.

Remark 4.2. *Let us define $A = (a_{nk})$ by*

$$a_{nk} = \begin{cases} \frac{1}{\lambda_n}, & \text{if } n - \lambda_n + 1 \leq k \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

Then the A-summability is reduced to the (V, λ)-summability.

Results of Section 2 can be easily proved on the similar lines, by taking the A-transform on both sides of (2.10) and using the conditions of regularity. Hence we have the following:

Theorem 4.3. *Let $A = (a_{nk})$ be a regular matrix. Suppose that $T_k : C[a,b] \to C[a,b]$ is a sequence of positive linear operator and $G_n(\cdot,\cdot) = \sum_k a_{nk} T_k(\cdot,\cdot)$ satisfying the following conditions*

$$\lim_n ||G_n(1,x) - 1||_\infty = 0, \quad \lim_n ||G_n(t,x) - x||_\infty = 0,$$

$$\lim_n ||G_n(t^2,x) - x^2||_\infty = 0.$$

Then for any function $f \in C[a,b]$ bounded on the whole real line, we have

$$\lim_n ||G_n(f,x) - f(x)||_\infty = 0.$$

Theorem 4.4. *Let $A = (a_{nk})$ be a regular matrix and (T_k) be the sequence of positive linear operators $T_k : L_p[a,b] \to L_p[a,b]$ and let the sequence $\{||T_k||\}$ be uniformly bounded. If*

$$\lim_n ||G_n(1,x) - 1||_{L_p} = 0, \quad \lim_n ||G_n(t,x) - x||_{L_p} = 0,$$

and

$$\lim_n ||G_n(t^2,x) - x^2||_{L_p} = 0.$$

Then for any function $f \in L_p[a,b]$, we have

$$\lim_n ||G_n(f,x) - f(x)||_{L_p} = 0.$$

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