Thai Journal of Mathematics
Volume 9 (2011) Number 3 : 531–552

www.math.science.cmu.ac.th/thaijournal
Online ISSN 1686-0209

A New Approximation Method for Equilibrium, Variational Inequality and Fixed Point Problems

Smaila S. Sanni[†], Eugene C. Ukaegbu[†] and Yekini Shehu[‡,1]

[†]Department of Statistics,
University of Nigeria, Nsukka, Nigeria
[‡]Department of Mathematics,
University of Nigeria, Nsukka, Nigeria
e-mail : deltanougt2006@yahoo.com

Abstract : The purpose of this paper is to construct a new iterative scheme and prove strong convergence theorem for approximation of a common fixed point of a countable family of relatively nonexpansive mappings which is also a common solution to an equilibrium and variational inequality problems in a 2-uniformly convex and uniformly smooth real Banach space. We apply our result to convex feasibility problem.

Keywords : Relatively nonexpansive mappings; Generalized projection operator; Equilibrium problem; Variational inequality problem; Banach spaces.
2010 Mathematics Subject Classification : 47H06; 47H09; 47J05; 47J25.

1 Introduction

Let E be a real Banach space with dual E^* and C be nonempty, closed and convex subset of E. A mapping $T : C \to C$ is called *nonexpansive* if

$$||Tx - Ty|| \leq ||x - y||, \quad \forall x, y \in C. \tag{1.1}$$

A point $x \in C$ is called *a fixed point* of T if $Tx = x$. The set of fixed points of T is denoted by $F(T) := \{x \in C : Tx = x\}$.

[1]Corresponding author email: deltanougt2006@yahoo.com (Y. Shehu)

The modulus of smoothness of E is the function $\rho_E : [0,\infty) \to [0,\infty)$ defined by

$$\rho_E(t) := \sup\left\{\frac{1}{2}(||x+y|| + ||x-y||) - 1 : ||x|| \le 1, ||y|| \le t\right\}.$$

E is uniformly smooth if and only if

$$\lim_{t\to 0}\frac{\rho_E(t)}{t} = 0.$$

Let $\dim E \ge 2$. The *modulus of convexity* of E is the function $\delta_E : (0,2] \to [0,1]$ defined by

$$\delta_E(\epsilon) := \inf\left\{1 - \left|\left|\frac{x+y}{2}\right|\right| : ||x|| = ||y|| = 1; \epsilon = ||x-y||\right\}.$$

E is *uniformly convex* if for any $\epsilon \in (0,2]$, there exists a $\delta = \delta(\epsilon) > 0$ such that if $x, y \in E$ with $||x|| \le 1$, $||y|| \le 1$ and $||x-y|| \ge \epsilon$, then $||\frac{1}{2}(x+y)|| \le 1-\delta$. Equivalently, E is uniformly convex if and only if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0,2]$. A normed space E is called *strictly convex* if for all $x, y \in E$, $x \ne y$, $||x|| = ||y|| = 1$, we have $||\lambda x + (1-\lambda)y|| < 1$, $\forall \lambda \in (0,1)$.

We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = ||x||^2 = ||f||^2\}.$$

The following properties of J are well known (The reader can consult [1–3] for more details):

1. If E is uniformly smooth, then J is norm-to-norm uniformly continuous on each bounded subset of E.
2. $J(x) \ne \emptyset$, $x \in E$.
3. If E is reflexive, then J is a mapping from E onto E^*.
4. If E is smooth, then J is single valued.

Let E be a smooth, strictly convex and reflexive real Banach space and let C be a nonempty, closed and convex subset of E. Following Alber [4], the generalized projection Π_C from E onto C is defined by

$$\Pi_C(x) := arg\min_{y\in C}\phi(y,x), \quad \forall x \in E.$$

The existence and uniqueness of Π_C follows from the property of the functional $\phi(x,y)$ and strict monotonicity of the mapping J (see, for example, [3–7]). If E is a Hilbert space, then Π_C is the metric projection of H onto C.

Throughout this paper, we denote by ϕ, the functional on $E \times E$ defined by

$$\phi(x,y) = ||x||^2 - 2\langle x, J(y)\rangle + ||y||^2, \quad \forall x, y \in E. \tag{1.2}$$

It is obvious from (1.2) that

$$(||x|| - ||y||)^2 \leq \phi(x,y) \leq (||x|| + ||y||)^2, \quad \forall x, y \in E \tag{1.3}$$

and

$$\phi\Big(x, J^{-1}\Big(\sum_{i=1}^{n} \lambda_i J x_i\Big)\Big) \leq \sum_{i=1}^{n} \lambda_i \phi(x, x_i) \tag{1.4}$$

for all $\lambda_i \in [0,1]$ and $x, x_i \in E, \quad \forall i = 1, 2, \ldots, n$ such that $\sum_{i=1}^{n} \lambda_i = 1$.

Definition 1.1. Let C be a nonempty subset of E and let $\{T_n\}_{n=0}^{\infty}$ be a countable family of mappings from C into E. A point $p \in C$ is said to be an *asymptotic fixed point* of $\{T_n\}_{n=0}^{\infty}$ if C contains a sequence $\{x_n\}_{n=0}^{\infty}$ which converges weakly to p and $\lim_{n\to\infty} ||x_n - T_n x_n|| = 0$. The set of asymptotic fixed points of $\{T_n\}_{n=0}^{\infty}$ is denoted by $\widehat{F}(\{T_n\}_{n=0}^{\infty})$. We say that $\{T_n\}_{n=0}^{\infty}$ is *countable family of relatively nonexpansive mappings* (see, for example, [8]) if the following conditions are satisfied:

(R1) $F(\{T_n\}_{n=0}^{\infty}) \neq \emptyset$;

(R2) $\phi(p, T_n x) \leq \phi(p, x), \ \forall x \in C, \ p \in F(T_n), \ n \geq 0$;

(R3) $\cap_{n=0}^{\infty} F(T_n) = \widehat{F}(\{T_n\}_{n=0}^{\infty})$.

Definition 1.2. A point $p \in C$ is said to be an *asymptotic fixed point* of T if C contains a sequence $\{x_n\}_{n=0}^{\infty}$ which converges weakly to p and $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$. The set of asymptotic fixed points of T is denoted by $\widehat{F}(T)$. We say that a mapping T is *relatively nonexpansive* (see, for example, [9–14]) if the following conditions are satisfied:

(R1) $F(T) \neq \emptyset$;

(R2) $\phi(p, Tx) \leq \phi(p, x), \ \forall x \in C, \quad p \in F(T)$;

(R3) $F(T) = \widehat{F}(T)$.

Definition 1.2 is a special form of Definition 1.1 as $T_n \equiv T, \quad \forall n \geq 0$. If T satisfies $(R1)$ and $(R2)$, then T is said to be *relatively quasi-nonexpansive.* It is easy to see that the class of relatively quasi-nonexpansive mappings contains the class of relatively nonexpansive mappings. Many authors have studied the methods of approximating the fixed points of relatively quasi-nonexpansive mappings (see, for example, [15, 16] the references contained therein). Clearly, in Hilbert space H, relatively quasi-nonexpansive mappings and quasi-nonexpansive mappings are the same, for $\phi(x,y) = ||x-y||^2, \quad \forall x, y \in H$ and this implies that

$$\phi(p, Tx) \leq \phi(p, x) \Leftrightarrow ||Tx - p|| \leq ||x - p||, \quad \forall x \in C, \quad p \in F(T).$$

It is known that the generalized projection mapping Π_C is relatively quasi-nonexpansive and $F(\Pi_C) = C$ (see, for example, [16]).

Let F be a bifunction of $C \times C$ into $\mathbb{R}$. The equilibrium problem (see, for example, [17–29]) is to find $x^* \in C$ such that

$$F(x^*, y) \geq 0, \quad \forall y \in C. \tag{1.5}$$

We shall denote the solutions set of (1.5) by $EP(F)$. Numerous problems in Physics, optimization and economics reduce to find a solution of problem (1.5). The equilibrium problems include fixed point problems, optimization problems and variational inequality problems as special cases (see, for example, [30]).
For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y, \in C$;

(A3) for each $x, y \in C$, $\lim_{n\to\infty} F(tz + (1-t)x, y) \leq F(x, y)$;

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

An operator $B : C \to E^*$ is called *α-inverse-strongly monotone*, if there exists a positive real number α such that

$$\langle x - y, Bx - By\rangle \geq \alpha||Bx - By||^2, \quad \forall x, y \in C, \tag{1.6}$$

and A is said to be *monotone* if

$$\langle x - y, Bx - By\rangle \geq 0, \quad \forall x, y \in C. \tag{1.7}$$

Let B be a monotone operator from C into E^*, the classical variational inequality (see, for example, [31]), denoted by $VI(C, B)$, is to find $x^* \in C$ such that

$$\langle y - x^*, Bx^*\rangle \geq 0, \quad \forall y \in C. \tag{1.8}$$

The variational inequality (1.8) is connected with the convex minimization problem, the complementarity problem, the problem of finding a point $x^* \in E$ such that $Bx^* = 0$ and so on.

It is well known that for a nonexpansive mapping T with $F(T) \neq \emptyset$, the classical *Picard iteration sequence* $x_{n+1} = Tx_n$, $x_1 \in D(T)$ does not always converge. An iterative process commonly used for finding fixed points of nonexpansive mappings is the following: For a convex subset C of a Banach space E and $T : C \to C$, the sequence $\{x_n\}_{n=1}^{\infty}$ is defined iteratively by $x_1 \in C$,

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \quad n \geq 1, \tag{1.9}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $[0, 1]$ satisfying the following conditions:
(i) $\lim_{n\to\infty} \alpha_n = 0$; (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$. The sequence of (1.9) is generally referred to as the *Mann sequence* in the light of [32]. It is generally known that the Mann iterative sequence (1.9) converges weakly to a fixed point of T (see, for example,

[33]). Motivated by (1.9), Matsushita and Takahashi [34] considered the following iterative scheme: $x_0 \in C$,

$$x_{n+1} = \Pi_C J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T x_n)), \quad n \geq 0 \tag{1.10}$$

and proved weak convergence theorems for approximation of a fixed point of relatively nonexpansive mapping T in uniformly convex and uniformly smooth Banach space under appropriate conditions. In order to obtain strong convergence, Matsushita and Takahashi [12] introduced a hybrid iterative scheme for approximation of fixed points of relatively nonexpansive mapping T in a uniformly convex real Banach space which is also uniformly smooth: $x_0 \in C$,

$$\begin{cases} y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T x_n), \\ H_n = \{w \in C : \phi(w, y_n) \leq \phi(w, x_n)\}, \\ W_n = \{w \in C : \langle x_n - w, J x_0 - J x_n \rangle, \\ x_{n+1} = \Pi_{H_n \cap W_n} x_0, \quad n \geq 0. \end{cases} \tag{1.11}$$

They proved that $\{x_n\}_{n=0}^{\infty}$ converges strongly to $\Pi_{F(T)} x_0$, where $F(T) \neq \emptyset$.

One method for solving a point $x^* \in VI(C,B)$ is using the projection algorithm which starts with any $x_1 = x \in C$ and

$$x_{n+1} = P_C(x_n - \lambda_n B x_n), \quad n \geq 1,$$

P_C is the metric projection from real Hilbert H onto C and $\{\lambda_n\}_{n=1}^{\infty}$ is a sequence of positive real numbers. For finding an element of $F(T) \cap VI(C,B)$, Takahashi and Toyoda [35] introduced the following iterative scheme: $x_1 \in C$, and

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) T P_C(x_n - \lambda_n B x_n), \quad n \geq 1$$

and obtained a weak convergence theorem in a Hilbert space. Recently, Iiduka and Takahashi [36] proposed a new iterative scheme: $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x + (1-\alpha_n) T P_C(x_n - \lambda_n B x_n), \quad n \geq 1$$

and obtained a strong convergence theorem in a Hilbert space. In the case when the space is a Banach space E, for finding a unique solution $VI(C,B)$, Alber [4] introduced the following iterative scheme:$x_1 = x \in E$, and

$$x_{n+1} = \Pi_C J^{-1}(J x_n - \lambda_n B x_n), \quad n \geq 1.$$

He proved that $\{x_n\}_{n=1}^{\infty}$ converges strongly to a unique element of z of $VI(C,B)$.

Motivated by Alber [4], Iiduka and Takahashi [37] introduced the following iterative scheme for finding a zero point of an inverse-strongly monotone operator B in a 2-uniformly convex and uniformly smooth Banach space:

$$\begin{cases} x_1 = x \in E, \\ y_n = J^{-1}(J x_n - \lambda_n B x_n), \\ H_n = \{w \in E : \phi(w, y_n) \leq \phi(w, x_n)\}, \\ W_n = \{w \in E : \langle x_n - w, J x - J x_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x_0, \quad n \geq 1. \end{cases}$$

They proved strong convergence theorem of the scheme under the conditions that B is α-inverse-strongly monotone and $A^{-1}0 \neq \emptyset$.

In [34], Matsushita and Takahashi considered the following iterative scheme: $x_0 \in C$,

$$x_{n+1} = \Pi_C J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n)), \quad n \geq 0 \tag{1.12}$$

and proved weak convergence theorems for approximation of a fixed point of relatively nonexpansive mapping T in uniformly convex and uniformly smooth Banach space under appropriate conditions.

In [14], Takahashi and Zembayashi introduced the following hybrid iterative scheme for approximation of fixed point of relatively nonexpansive mapping which is also a solution to an equilibrium problem in a uniformly convex real Banach space which is also uniformly smooth: $x_0 \in C,\ C_1 = C,\ x_1 = \Pi_{C_1}x_0$,

$$\begin{cases} y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ F(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \forall y \in C, \\ C_{n+1} = \{w \in C_n : \phi(w, u_n) \leq \phi(w, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}x_0, \quad n \geq 1, \end{cases}$$

where J is the duality mapping on E. Then, they proved that $\{x_n\}_{n=0}^{\infty}$ converges strongly to $\Pi_\Omega x_0$, where $\Omega = EP(F) \cap F(T) \neq \emptyset$.

Another iteration process which has been found to be successful for approximating fixed points of nonexpansive maps is the Halpern iteration process (see, for example, [38]). Let C be a nonempty, closed and convex subset of a Hilbert space and $T : C \to C$ be a nonexpansive mapping. Assume that $F(T) \neq \emptyset$. For fixed $u \in C$, let the sequence $\{x_n\}_{n=1}^{\infty}$ be generated by $x_1 \in C$,

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \tag{1.13}$$

for all $n \geq 1$. He proved strong convergence of the sequence $\{x_n\}_{n=1}^{\infty}$ to a fixed point of T, where $\alpha_n := n^{-a}, a \in (0, 1)$. He pointed out that the conditions $(C1) : \lim_{n\to\infty} \alpha_n = 0$ and $(C2) : \sum_{n=1}^{\infty} \alpha_n = \infty$ are necessary for the convergence of the Halpern iteration (1.13) to a fixed point of T. The iteration process (1.13) has been proved to be strongly convergent for nonexpansive mapping T both in Hilbert spaces [38–40] and uniformly smooth Banach spaces [41, 42] when the sequence $\{\alpha_n\}$ satisfies the conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$ and

(iii) either $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$.

In [43], Plubtieng and Ungchittrakool introduced the following hybrid projection algorithm for a pair of relatively nonexpansive mappings T and S: $x_0 \in C$,

$$\begin{cases} z_n = J^{-1}(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JTx_n + \beta_n^{(3)} JSx_n) \\ y_n = J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)Jz_n) \\ C_n = \{z \in C : \phi(z, y_n) \leq \phi(z, x_n) + \alpha_n(||x_0||^2 + 2\langle w, Jx_n - Jx_0\rangle)\} \\ Q_n = \{z \in C : \langle x_n - z, Jx_n - Jx_0\rangle \leq 0\} \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \tag{1.14}$$

where $\{\alpha_n\}$, $\{\beta_n^{(1)}\}$, $\{\beta_n^{(2)}\}$ and $\{\beta_n^{(3)}\}$ are sequences in $(0,1)$ satisfying $\beta_n^{(1)}+\beta_n^{(2)}+\beta_n^{(3)} = 1$ and T and S are relatively nonexpansive mappings and J is the single-valued duality mapping on E. They proved under the appropriate conditions on the parameters that the sequence $\{x_n\}$ generated by (1.14) converges strongly to a common fixed point of T and S.

Motivated by (1.13), Kohsaka and Takahashi, [44] introduced and studied the following iterative scheme: $x = x_0 \in E$,

$$x_{n+1} = J^{-1}(\alpha_n Jx + (1-\alpha_n)JJ_{r_n}x_n), \quad n \geq 0 \tag{1.15}$$

where J is the duality mapping and $J_r = (J + rA)^{-1}J$ for all $r > 0$. They proved that if $A^{-1}0 \neq \emptyset, \lim_{n\to\infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} r_n = \infty$, then the sequence generated by (1.15) converges strongly to an element of $A^{-1}0$.

Quite recently, Nilsrakoo and Saejung, [45] proved the following strong convergence theorem for approximation of fixed point of relatively nonexpansive mapping in a uniformly convex and uniformly smooth Banach space.

Theorem 1.3 (Nilsrakoo and Saejung [45])**.** *Let C be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth Banach space E and T be a relatively nonexpansive mapping from C into E. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0,1)$ satisfying: (i) $\lim_{n\to\infty} \alpha_n = 0$, (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$ and (iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Then $\{x_n\}$ defined by $u \in E,\ x_1 \in C$,*

$$x_{n+1} = \Pi_C J^{-1}(\alpha_n Ju + (1-\alpha_n)(\beta_n Jx_n + (1-\beta_n)JTx_n)), \quad n \geq 1 \tag{1.16}$$

converges strongly to $\Pi_{F(T)}u$, where $\Pi_{F(T)}$ is the generalized projection of E onto $F(T)$.

Motivated by the above mentioned results and the on-going research, it is our purpose in this paper to introduce a new iterative scheme and prove strong convergence theorem for a countable family of relatively nonexpansive mappings which is also a common solution to an equilibrium and variational inequality problems in a 2-uniformly convex and uniformly smooth real Banach space. We also apply our result to convex feasibility problem.

2 Preliminaries

We know that the following lemmas hold for generalized projections.

Lemma 2.1 (Alber [4], Kamimura and Takahashi [7]). *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. Then*

$$\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y), \quad \forall x \in C, \ \forall y \in E.$$

Lemma 2.2 (Alber [4], Kamimura and Takahashi [7]). *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. Let $x \in E$ and $z \in C$. Then*

$$z = \Pi_C x \Leftrightarrow \langle y - z, J(x) - J(z)\rangle \leq 0, \quad \forall y \in C.$$

Lemma 2.3 (Matsushita and Takahashi [12]). *Let C be a nonempty, closed and convex subset of a smooth, strictly convex Banach space E. Let T be a relatively nonexpansive mapping of C into itself. Then $F(T)$ is closed and convex.*

Let C be a nonempty, closed and convex subset of a smooth, uniformly convex Banach space E and J be the duality mapping from E into E^*. Then J^{-1} is single-valued, one-one and surjective and it is the duality mapping from E^* into E. We make use of the following function V as studied by Alber [4]:

$$V(x, x^*) = ||x||^2 - 2\langle x, x^*\rangle + ||y||^2 \tag{2.1}$$

for all $x \in E$ and $x^* \in E^*$. Thus, $V(x, x^*) = \phi(x, J^{-1}(x^*))$ for all $x \in E$ and $x^* \in E^*$. We know the following lemma from Alber [4].

Lemma 2.4 (Alber [4]). *Let E be a real reflexive, strictly convex and Banach space and V be as in* (2.1). *Then*

$$V(x, x^*) + 2\langle J^{-1}(x^*) - x, y^*\rangle \leq V(x, x^* + y^*)$$

for all $x \in E$ and $x^*, y^* \in E^*$.

Also, this following lemma holds in a uniformly convex real Banach space.

Lemma 2.5 (Chang et al. [46]). *Let E be a uniformly convex real Banach space. For arbitrary $r > 0$, let $B_r(0) := \{x \in E : ||x|| \leq r\}$. Then, for any given sequence $\{x_n\}_{n=1}^{\infty} \subset B_r(0)$ and for any given sequence $\{\lambda_n\}_{n=1}^{\infty}$ of positive numbers such that $\sum_{i=1}^{\infty} \lambda_i = 1$, there exists a continuous strictly increasing convex function*

$$g : [0, 2r] \to \mathbb{R}, \quad g(0) = 0$$

such that for any positive integers i, j with $i < j$, the following inequality holds:

$$\Big|\Big|\sum_{n=1}^{\infty} \lambda_n x_n\Big|\Big|^2 \leq \sum_{n=1}^{\infty} \lambda_n ||x_n||^2 - \lambda_i \lambda_j g(||x_i - x_j||).$$

The following lemma is an analogue of Lemma 2.5 with respect to ϕ.

Lemma 2.6. *Let E be a uniformly convex real Banach space. For arbitrary $r > 0$, let $B_r(0) := \{x \in E : ||x|| \leq r\}$. Then, for any given sequence $\{x_n\}_{n=1}^{\infty} \subset B_r(0)$ and for any given sequence $\{\lambda_n\}_{n=1}^{\infty}$ of positive numbers such that $\sum_{i=1}^{\infty} \lambda_i = 1$, there exists a continuous strictly increasing convex function*

$$g : [0, 2r] \to \mathbb{R}, \quad g(0) = 0$$

such that for any positive integers i, j with $i < j$, the following inequality holds:

$$\phi\left(x, J^{-1}\left(\sum_{n=1}^{\infty} \lambda_n J x_n\right)\right) \leq \sum_{n=1}^{\infty} \lambda_n \phi(x, x_n) - \lambda_i \lambda_j g(||Jx_i - Jx_j||).$$

It is easy to see that if $\{x_n\}$ and $\{y_n\}$ are bounded sequences of a smooth Banach space E, then $x_n - y_n \to 0,\ n \to \infty$ implies that $\phi(x_n, y_n) \to 0,\ n \to \infty$.

Lemma 2.7 (Blum and Oettli [30]). *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0 \text{ for all } y \in C.$$

Lemma 2.8 (Takahashi and Zembayashi [47]). *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. Assume that $F : C \times C \to \mathbb{R}$ satisfies (A1)-(A4). For $r > 0$ and $x \in E$, define a mapping $T_r : E \to C$ as follows:*

$$T_r(x) = \left\{z \in C : F(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \forall y \in C\right\}$$

for all $z \in E$. Then, the following hold:

1. *T_r is single-valued;*
2. *T_r is firmly nonexpansive-type mapping, i.e., for any $x, y \in E$,*

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle T_r x - T_r y, Jx - Jy\rangle;$$

3. *$F(T_r) = EP(F)$;*
4. *$EP(F)$ is closed and convex.*

Lemma 2.9 (Takahashi and Zembayashi [47]). *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. Assume that $F : C \times C \to \mathbb{R}$ satisfies $(A1) - (A4)$ and let $r > 0$. Then for each $x \in E$ and $q \in F(T_r)$,*

$$\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x).$$

Also, this following lemma will be used in the sequel.

Lemma 2.10 (Kamimura and Takahashi [7]). *Let C be a nonempty closed convex subset of a smooth, uniformly convex Banach space E. Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be sequences in E such that either $\{x_n\}_{n=1}^{\infty}$ or $\{y_n\}_{n=1}^{\infty}$ is bounded. If $\lim_{n\to\infty}\phi(x_n,y_n)=0$, then $\lim_{n\to\infty}||x_n-y_n||=0$.*

Lemma 2.11 (Xu [48]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1}\leq(1-\alpha_n)a_n+\alpha_n\sigma_n+\gamma_n, n\geq 0,$$

where, (i) $\{\alpha_n\}\subset[0,1]$, $\sum\alpha_n=\infty$; (ii) $\limsup\ \sigma_n\leq 0$; (iii) $\gamma_n\geq 0$; $(n\geq 0)$, $\sum\gamma_n<\infty$. Then, $a_n\to 0$ as $n\to\infty$.

Lemma 2.12 (Mainge [49]). *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i}<a_{n_i+1}$ for all $i\in\mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\}\subset\mathbb{N}$ such that $m_k\to\infty$ and the following properties are satisfied by all (sufficiently large) numbers $k\in\mathbb{N}$:*

$$a_{m_k}\leq a_{m_k+1} \text{ and } a_k\leq a_{m_k+1}.$$

In fact, $m_k=\max\{j\leq k: a_j<a_{j+1}\}$.

Lemma 2.13 (Beauzamy [50]). *Let E be a 2-uniformly convex Banach space, then for all x,y from any bounded set of E and $jx\in Jx$, $\ jy\in Jy$, we have*

$$\langle x-y, jx-jy\rangle\geq\frac{c^2}{2}||x-y||^2,$$

where $\frac{1}{c}$ is the 2-uniformly constant of E.

Lemma 2.14 (Rockafellar [51]). *Let C be a nonempty, closed and convex subset of a Banach space E and let B be a monotone and hemicontinuous operator of C into E^* with $C=D(A)$. Let $B\subset E\times E^*$ be an operator defined as follows:*

$$Mv:=\begin{cases} Bv+N_C(v), & v\in C\\ \emptyset, \quad v\notin C.\end{cases}$$

Then M is maximal monotone and $M^{-1}(0)=VI(C,B)$.

In this paper, we shall assume that

(B1) B is α-inverse strongly monotone;

(B2) $||By||\leq||By-Bu||$ for all $y\in C$ and $u\in VI(C,B)$;

(B3) $VI(C,B)\neq\emptyset$.

3 Main Results

Theorem 3.1. *Let E be a 2-uniformly convex real Banach space which is also uniformly smooth. Let C be a nonempty, closed and convex subset of E. Let F be a bifunction from $C \times C \to \mathbb{R}$ satisfying $(A1) - (A4)$, $B : C \to E^*$ an operator satisfying $(B1) - (B3)$ and $\{T_n\}_{n=0}^{\infty}$ a countable family of relatively nonexpansive mappings of C into E such that $\Omega := (\cap_{n=0}^{\infty} F(T_n)) \cap EP(F) \cap VI(C, B) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Suppose $\{x_n\}_{n=0}^{\infty}$ is iteratively generated by u, $u_0 \in E$,*

$$\begin{cases} y_n = \Pi_C J^{-1}(Ju_n - r_n Bu_n), \\ x_n = T_{r_n} y_n, \\ u_{n+1} = \Pi_C J^{-1}(\alpha_n Ju + \beta_n Jx_n + \gamma_n JT_n x_n), \quad n \geq 0, \end{cases} \tag{3.1}$$

with the conditions

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $0 < b \leq \beta_n \gamma_n \leq 1$;

(iii) $0 < a \leq r_n \leq b < \frac{c^2 \alpha}{2}$.

Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to $\Pi_\Omega u$.

Proof. Let $x^* \in \Omega$. Then, we obtain

$$\begin{aligned} \phi(x^*, y_n) &= \phi(x^*, \Pi_C J^{-1}(Ju_n - r_n Bu_n)) \\ &\leq \phi(x^*, J^{-1}(Ju_n - r_n Bu_n)) \\ &= V(x^*, Ju_n - r_n Bu_n) \\ &\leq V(x^*, (Ju_n - r_n Bu_n) + r_n Bu_n) \\ &\quad - 2\langle J^{-1}(Ju_n - r_n Bu_n) - x^*, r_n Bu_n\rangle \\ &= V(x^*, Ju_n) - 2r_n \langle J^{-1}(Ju_n - r_n Bu_n) - x^*, Bu_n\rangle \\ &= \phi(x^*, u_n) - 2r_n \langle u_n - x^*, Bu_n\rangle \\ &\quad + 2\langle J^{-1}(Ju_n - r_n Bu_n) - u_n, -r_n Bu_n\rangle. \end{aligned} \tag{3.2}$$

From condition $(B1)$ and $x^* \in VI(C, B)$, we obtain

$$\begin{aligned} -2r_n \langle u_n - x^*, Bu_n\rangle &= -2r_n \langle u_n - x^*, Bu_n - Bx^*\rangle - 2r_n \langle u_n - x^*, Bx^*\rangle \\ &\leq -2\alpha r_n ||Bu_n - Bx^*||^2. \end{aligned} \tag{3.3}$$

By Lemma 2.13 and condition $(B2)$, we also obtain

$$\begin{aligned}
2\langle J^{-1}(Ju_n - r_nBu_n) - u_n, -r_nBu_n\rangle & \\
&= 2\langle J^{-1}(Ju_n - r_nBu_n) - J^{-1}(Ju_n), -r_nBu_n\rangle \\
&\leq 2||J^{-1}(Ju_n - r_nBu_n) - J^{-1}(Ju_n)||||r_nBu_n|| \\
&\leq \frac{4}{c^2}||(Ju_n - r_nBu_n) - (Ju_n)||||r_nBu_n|| \\
&= \frac{4}{c^2}r_n^2||Bu_n||^2 \\
&\leq \frac{4}{c^2}r_n^2||Bu_n - Bx^*||^2. \qquad (3.4)
\end{aligned}$$

Combining (3.2), (3.3) and (3.4) and $0 < a \leq r_n \leq b < \frac{c^2\alpha}{2}$, we obtain

$$\phi(x^*, y_n) \leq \phi(x^*, u_n) - 2\alpha r_n||Bu_n - Bx^*||^2 + \frac{4}{c^2}r_n^2||Bu_n - Bx^*||^2. \qquad (3.5)$$

From (3.5), we have that

$$\begin{aligned}
\phi(x^*, y_n) &\leq \phi(x^*, u_n) + 2r_n\Big(\frac{2}{c^2}r_n - \alpha\Big)||Bu_n - Bx^*||^2 \\
&\leq \phi(x^*, u_n). \qquad (3.6)
\end{aligned}$$

Using (3.1), (3.6) and the fact that T_{r_n} is relatively quasi-nonexpansive, we have

$$\begin{aligned}
\phi(x^*, x_{n+1}) &= \phi(x^*, T_{r_{n+1}}y_{n+1}) \leq \phi(x^*, y_{n+1}) \leq \phi(x^*, u_{n+1}) \qquad (3.7) \\
&= \phi(x^*, J^{-1}(\alpha_nJu + \beta_nJx_n + \gamma_nJT_nx_n)) \\
&\leq \alpha_n\phi(x^*, u) + \beta_n\phi(x^*, x_n) + \gamma_n\phi(x^*, T_nx_n) \\
&\leq \alpha_n\phi(x^*, u) + \beta_n\phi(x^*, x_n) + \gamma_n\phi(x^*, x_n) \\
&= \alpha_n\phi(x^*, u) + (1 - \alpha_n)\phi(x^*, x_n) \\
&\leq \max\{\phi(x^*, u), \phi(x^*, x_n)\} \\
&\vdots \\
&\leq \max\{\phi(x^*, u), \phi(x^*, x_0)\}.
\end{aligned}$$

Hence, $\{x_n\}_{n=0}^{\infty}$ is bounded and also is $\{T_nx_n\}_{n=0}^{\infty}$. Since E is uniformly smooth, E^* is uniformly convex. Then from Lemma 2.6, we have for some $M > 0$ that

$$\begin{aligned}
\phi(x^*, x_{n+1}) \leq \phi(x^*, u_{n+1}) &\leq \alpha_n\phi(x^*, u) + \beta_n\phi(x^*, x_n) + \gamma_n\phi(x^*, T_nx_n) \\
&\quad - \beta_n\gamma_ng(||Jx_n - JT_nx_n||) \\
&\leq \alpha_n\phi(x^*, u) + (1 - \alpha_n)\phi(x^*, x_n) - \beta_n\gamma_ng(||Jx_n - JT_nx_n||) \\
&\leq \alpha_nM + \phi(x^*, x_n) - \beta_n\gamma_ng(||Jx_n - JT_nx_n||). \qquad (3.8)
\end{aligned}$$

This implies that

$$\begin{aligned}
0 < bg(||Jx_n - JT_nx_n||) &\leq \beta_n\gamma_ng(||Jx_n - JT_nx_n||) \\
&\leq \alpha_nM + \phi(x^*, x_n) - \phi(x^*, x_{n+1}). \qquad (3.9)
\end{aligned}$$

Now put $z_n := J^{-1}(\alpha_n Ju + \beta_n Jx_n + \gamma_n JT_n x_n),\ n \geq 0$. Then, we show that

$$\limsup_{n\to\infty}\langle z_n - z, Ju - Jz\rangle \leq 0,$$

where $z := \Pi_\Omega u$. To do this inequality, we choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle x_n - z, Ju - Jz\rangle = \lim_{j\to\infty}\langle x_{n_j} - z, Ju - Jz\rangle.$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ that converges weakly to p. The rest of the proof will be divided into two parts.

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\phi(x^*, x_n)\}_{n=n_0}^\infty$ is nonincreasing. Then $\{\phi(x^*, x_n)\}_{n=0}^\infty$ converges and $\phi(x^*, x_n) - \phi(x^*, x_{n+1}) \to 0,\ n \to \infty$. This implies from (3.9) and condition (i) that

$$g(||Jx_n - JT_n x_n||) \to 0, \quad n \to \infty.$$

By property of g, we have

$$||Jx_n - JT_n x_n|| \to 0, \quad n \to \infty.$$

Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we have

$$||x_n - T_n x_n|| \to 0, \quad n \to \infty. \tag{3.10}$$

This implies that

$$\phi(x_n, T_n x_n) \to 0, \quad n \to \infty.$$

Since $x_{n_j} \rightharpoonup p$ and $\{T_n\}_{n=0}^\infty$ are uniformly closed, we have $p \in (\cap_{n=0}^\infty F(T_n))$.

Next, we show that $p \in VI(C, B)$. From (3.5) and (3.7), we obtain

$$\begin{aligned}
\phi(x^*, x_n) &\leq \phi(x^*, u_n) - 2\alpha r_n ||Bu_n - Bx^*||^2 + \frac{4}{c^2} r_n^2 ||Bu_n - Bx^*||^2 \\
&= \phi(x^*, u_n) + 2r_n\Big(\frac{2}{c^2} r_n - \alpha\Big)||Bu_n - Bx^*||^2 \\
&\leq \alpha_{n-1}\phi(x^*, u) + (1 - \alpha_{n-1})\phi(x^*, x_{n-1}) \\
&\quad + 2r_n\Big(\frac{2}{c^2} r_n - \alpha\Big)||Bu_n - Bx^*||^2 \qquad (3.11)\\
&\leq \alpha_{n-1}\phi(x^*, u) + \phi(x^*, x_{n-1}) + 2r_n\Big(\frac{2}{c^2} r_n - \alpha\Big)||Bu_n - Bx^*||^2.
\end{aligned}$$

Hence, we obtain

$$-2r_n\Big(\frac{2}{c^2} r_n - \alpha\Big)||Bu_n - Bx^*||^2 \leq \alpha_{n-1}\phi(x^*, u) + \phi(x^*, x_{n-1}) - \phi(x^*, x_n) \to 0,$$

as $n \to \infty$. Since $0 < a \leq r_n \leq b < \frac{c^2\alpha}{2}$, we obtain from the last inequality that

$$\lim_{n\to\infty}||Bu_n - Bx^*|| = 0.$$

By Lemma 2.4 and (3.4), we have

$$\begin{aligned}\phi(u_n, y_n) &= \phi(u_n, \Pi_C J^{-1}(Ju_n - r_n Bu_n)) \le \phi(u_n, J^{-1}(Ju_n - r_n Bu_n))\\ &= V(u_n, Ju_n - r_n Bu_n)\\ &\le V(u_n, (Ju_n - r_n Bu_n) + r_n Bu_n)\\ &\quad - 2\Big\langle J^{-1}(Ju_n - r_n Bu_n) - u_n, r_n Bu_n\Big\rangle\\ &= \phi(u_n, u_n) + 2\langle J^{-1}(Ju_n - r_n Bu_n) - u_n, -r_n Bu_n\rangle\\ &= 2\langle J^{-1}(Ju_n - r_n Bu_n) - u_n, -r_n Bu_n\rangle\\ &\le \frac{4}{c^2} b^2 \|Bu_n - Bx^*\|^2 \to 0, \quad n \to \infty. \end{aligned} \tag{3.12}$$

It then follows from Lemma 2.10 that $\lim_{n\to\infty} \|y_n - u_n\| = 0$. Since J is uniformly norm-to-norm continuous on bounded sets and $\lim_{n\to\infty} \|y_n - u_n\| = 0$, we obtain

$$\lim_{n\to\infty} \|Jy_n - Ju_n\| = 0.$$

Now, let $B \subset E \times E^*$ be an operator as follows:

$$Mv := \begin{cases} Bv + N_C(v), & v \in C \\ \emptyset, & v \notin C. \end{cases}$$

By Lemma 2.14, M is maximal monotone and $M^{-1}(0) = VI(C, B)$. Let $(v, w) \in G(M)$. Since $w \in Mv = Bv + N_C(v)$, we have $w - Bv \in N_C(v)$. Since $y_n \in C$, we get

$$\langle v - y_n, w - Bv\rangle \ge 0. \tag{3.13}$$

On the other hand, from $y_n = \Pi_C J^{-1}(Ju_n - r_n Bu_n)$ and Lemma 2.2 we obtain

$$\langle v - y_n, Jy_n - (Ju_n - r_n Bu_n)\rangle \ge 0,$$

and hence

$$\Big\langle v - y_n, \frac{Ju_n - Jy_n}{r_n} - Bu_n\Big\rangle \le 0. \tag{3.14}$$

Then, by (3.13), (3.14) and replacing n by n_j, we obtain that

$$\begin{aligned}\langle v - y_{n_j}, w\rangle &\ge \langle v - y_{n_j}, Bv\rangle\\ &\ge \langle v - y_{n_j}, Bv\rangle + \Big\langle v - y_{n_j}, \frac{Ju_{n_j} - Jy_{n_j}}{r_{n_j}} - Bu_{n_j}\Big\rangle\\ &= \Big\langle v - y_{n_j}, Bv - Bu_{n_j} + \frac{Ju_{n_j} - Jy_{n_j}}{r_{n_j}}\Big\rangle\\ &= \langle v - y_{n_j}, Bv - By_{n_j}\rangle + \langle v - y_{n_j}, By_{n_j} - Bu_{n_j}\rangle\\ &\quad + \Big\langle v - y_{n_j}, \frac{Ju_{n_j} - Jy_{n_j}}{r_{n_j}}\Big\rangle \qquad (3.15)\\ &\ge -\|v - y_{n_j}\|\|By_{n_j} - Bu_{n_j}\| - \|v - y_{n_j}\|\Big\|\frac{Ju_{n_j} - Jy_{n_j}}{r_{n_j}}\Big\|.\end{aligned}$$

Hence, we have $\langle v-p, w\rangle \geq 0$ as $j \to \infty$, since the uniform continuity of J and B implies that the right side of (3.15) goes to 0 as $j \to \infty$. Thus, since M is maximal monotone, we have $p \in M^{-1}(0)$ and hence $p \in VI(C,B)$.

Finally, we show that $p \in EP(F)$. Now, by Lemma 2.9, (3.8) and condition (i), we obtain

$$\begin{aligned}\phi(x_n, y_n) &= \phi(T_{r_n} y_n, y_n)\\ &\leq \phi(x^*, y_n) - \phi(x^*, x_n)\\ &\leq \phi(x^*, u_n) - \phi(x^*, x_n)\\ &\leq \alpha_{n-1} M + \phi(x^*, x_{n-1}) - \phi(x^*, x_n) \to 0, \quad n \to \infty.\end{aligned}$$

Using Lemma 2.10, we have $\lim_{n\to\infty} ||x_n - y_n|| = 0$. Now, since $x_{n_j} \rightharpoonup p$ and $\lim_{n\to\infty} ||x_n - y_n|| = 0$, we obtain that $y_{n_j} \rightharpoonup p$. Also, since J is uniformly norm-to-norm continuous on bounded sets and $\lim_{n\to\infty} ||x_n - y_n|| = 0$, we obtain

$$\lim_{n\to\infty} ||Jx_n - Jy_n|| = 0.$$

Since $\liminf_{n\to\infty} r_n > 0$,

$$\lim_{n\to\infty} \frac{||Jx_n - Jy_n||}{r_n} = 0. \tag{3.16}$$

Since $x_n = T_{r_n} u_n,\ n \geq 0$, by Lemma 2.8, we have

$$F(x_n, y) + \frac{1}{r_n}\langle y - x_n, Jx_n - Jy_n\rangle \geq 0, \quad \forall y \in C.$$

Furthermore, replacing n by n_j in the last inequality and using $(A2)$, we obtain

$$\frac{1}{r_{n_j}}\langle y - x_{n_j}, Jx_{n_j} - Jy_{n_j}\rangle \geq F(y, x_{n_j}). \tag{3.17}$$

By $(A4)$, (3.16) and $x_{n_j} \rightharpoonup p$, we have

$$F(y, p) \leq 0, \quad \forall y \in C.$$

For fixed $y \in C$, let $z_{t,y} := ty + (1-t)p$ for all $t \in (0,1]$. This implies that $z_t \in C$. This yields that $F(z_t, p) \leq 0$. It follows from $(A1)$ and $(A4)$ that

$$\begin{aligned}0 = F(z_t, z_t) &\leq tF(z_t, y) + (1-t)F(z_t, p)\\ &\leq tF(z_t, y)\end{aligned}$$

and hence $0 \leq F(z_t, y)$. From condition $(A3)$, we obtain

$$F(p, y) \geq 0, \quad \forall y \in C.$$

This implies that $p \in EP(F)$. Hence, we have $p \in (\cap_{n=0}^{\infty} F(T_n)) \cap EP(F) \cap VI(C,B) = \Omega$.

Let $w_n := J^{-1}\Big(\frac{\beta_n}{1-\alpha_n}Jx_n + \frac{\gamma_n}{1-\alpha_n}JT_nx_n\Big), \quad n \geq 0$, then

$$\phi(x_n, w_n) \leq \frac{\beta_n}{1-\alpha_n}\phi(x_n, x_n) + \frac{\gamma_n}{1-\alpha_n}\phi(x_n, T_nx_n) \to 0, \quad n \to \infty. \tag{3.18}$$

By Lemma 2.10, it follows that $||x_n - w_n|| \to 0,\ n \to \infty$. Furthermore,

$$\begin{aligned}\phi(w_n, z_n) &= \phi(w_n, J^{-1}(\alpha_n Ju + (1-\alpha_n)Jw_n)) \\ &\leq \alpha_n\phi(w_n, u) + (1-\alpha_n)\phi(w_n, w_n) \\ &= \alpha_n\phi(w_n, u) \to 0, \quad n \to \infty.\end{aligned} \tag{3.19}$$

Again, by Lemma 2.10, it follows that $||w_n - z_n|| \to 0,\ n \to \infty$. Then

$$||x_n - z_n|| \leq ||w_n - z_n|| + ||x_n - w_n|| \to 0, \quad n \to \infty. \tag{3.20}$$

By (3.20), and Lemma 2.2, we obtain

$$\begin{aligned}\limsup_{n\to\infty}\langle z_n - z, Ju - Jz\rangle &= \limsup_{n\to\infty}\langle x_n - z, Ju - Jz\rangle \\ &= \lim_{j\to\infty}\langle x_{n_j} - z, Ju - Jz\rangle \\ &= \langle p - z, Ju - Jz\rangle \leq 0.\end{aligned} \tag{3.21}$$

Therefore,

$$\begin{aligned}\phi(z, x_{n+1}) &\leq \phi(z, J^{-1}(\alpha_n Ju + \beta_n Jx_n + \gamma_n JT_nx_n)) \\ &= V(z, \alpha_n Ju + \beta_n Jx_n + \gamma_n JT_nx_n) \\ &\leq V(z, \alpha_n Ju + \beta_n Jx_n + \gamma_n JT_nx_n - \alpha_n(Ju - Jz)) \\ &\quad - 2\langle J^{-1}(\alpha_n Ju + \beta_n Jx_n + \gamma_n JT_nx_n) - z, -\alpha_n(Ju - Jz)\rangle \\ &= V(z, \alpha_n Jz + \beta_n Jx_n + \gamma_n JT_nx_n) \\ &\quad + 2\alpha_n\langle z_n - z, Ju - Jz\rangle \\ &= \phi(z, J^{-1}(\alpha_n Jz + \beta_n Jx_n + \gamma_n JT_nx_n)) \\ &\quad + 2\alpha_n\langle z_n - z, Ju - Jz\rangle \\ &\leq \alpha_n\phi(z, z) + \beta_n\phi(z, x_n) + \gamma_n\phi(z, T_nx_n) \\ &\quad + 2\alpha_n\langle z_n - p, Ju - Jz\rangle \\ &\leq (1-\alpha_n)\phi(z, x_n) + 2\alpha_n\langle z_n - z, Ju - Jz\rangle.\end{aligned} \tag{3.22}$$

Now, using (3.21), (3.22) and Lemma 2.11, we obtain $\phi(z, x_n) \to 0,\ n \to \infty$. Hence, $x_n \to z,\ n \to \infty$.

<u>Case 2</u>. Suppose there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\phi(x^*, x_{n_i}) < \phi(x^*, x_{n_i+1})$$

for all $i \in \mathbb{N}$. Then, by Lemma 2.12, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$,

$$\phi(x^*, x_{m_k}) \leq \phi(x^*, x_{m_k+1}) \quad \text{and} \quad \phi(x^*, x_k) \leq \phi(x^*, x_{m_k+1})$$

for all $k \in \mathbb{N}$. This together with (3.9) gives

$$0 < bg(||Jx_{m_k} - JT_{m_k}x_{m_k}||) \leq \alpha_{m_k}M + \phi(x^*, x_{m_k}) - \phi(x^*, x_{m_k+1}) \leq \alpha_{m_k}M$$

for all $k \in \mathbb{N}$. It then follows that

$$g(||Jx_{m_k} - JT_{m_k}x_{m_k}||) \to 0, \quad k \to \infty.$$

By the same arguments as in Case 1, we can show that

$$\limsup_{k\to\infty}\langle z_{m_k} - z, Ju - Jz\rangle \leq 0. \tag{3.23}$$

From (3.22), we have

$$\phi(z, x_{m_k+1}) \leq (1 - \alpha_{m_k})\phi(z, x_{m_k}) + 2\alpha_{m_k}\langle z_{m_k} - z, Ju - Jz\rangle. \tag{3.24}$$

Since $\phi(z, x_{m_k}) \leq \phi(z, x_{m_k+1})$, we have

$$\begin{aligned} \alpha_{m_k}\phi(z, x_{m_k}) &\leq \phi(z, x_{m_k}) - \phi(z, x_{m_k+1}) + 2\alpha_{m_k}\langle z_{m_k} - z, Ju - Jz\rangle \\ &\leq 2\alpha_{m_k}\langle z_{m_k} - z, Ju - Jz\rangle. \end{aligned}$$

In particular, since $\alpha_{m_k} > 0$, we get

$$\phi(z, x_{m_k}) \leq 2\langle z_{m_k} - z, Ju - Jz\rangle. \tag{3.25}$$

It then follows from (3.23) that $\phi(z, x_{m_k}) \to 0,\ k \to \infty$. From (3.25) and (3.24), we have

$$\phi(z, x_{m_k+1}) \to 0,\ k \to \infty.$$

Since $\phi(z, x_k) \leq \phi(z, x_{m_k+1})$ for all $k \in \mathbb{N}$, we conclude that $x_k \to z,\ k \to \infty$. This implies that $x_n \to z,\ n \to \infty$ and this completes the proof. □

Corollary 3.2. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C \to \mathbb{R}$ satisfying (A1)−(A4), $B : C \to H$ is α-inverse strongly monotone and T a nonexpansive mapping of C into H such that $\Omega := F(T) \cap EP(F) \cap VI(C, B) \neq \emptyset$. Let $\{\alpha_n\},\ \{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Suppose $\{x_n\}_{n=0}^{\infty}$ is iteratively generated by $u, u_0 \in E$,*

$$\begin{cases} y_n = P_C(u_n - r_nBu_n), \\ x_n = T_{r_n}y_n, \\ u_{n+1} = P_C(\alpha_n u + \beta_n x_n + \gamma_n Tx_n), \quad n \geq 0, \end{cases}$$

with the conditions

- *(i)* $\lim_{n\to\infty}\alpha_n = 0,\ \sum_{n=0}^{\infty}\alpha_n = \infty$;
- *(ii)* $0 < b \leq \beta_n\gamma_n \leq 1$;
- *(iii)* $0 < a \leq r_n \leq b < 2\alpha$.

Then, $\{x_n\}_{n=0}^{\infty}$ *converges strongly to* $P_\Omega u$.

Next, we apply our Theorem 3.1 to convex feasibility problem. First, we introduce the following lemma which was proved by Reich [52].

Lemma 3.3 (Reich [52]). *Let E be a uniformly convex Banach space with uniformly Gâteaux differentiable norm, let $\{C_i\}_{i=1}^{m}$ be a finite family of closed and convex subsets of E and let Π_i be the generalized projection from E onto C_i for each $i = 1, 2, \ldots, m$. Then*

$$\phi(p, \Pi_m\Pi_{m-1}\ldots\Pi_2\Pi_1 x) \leq \phi(p, x)$$

for each $p \in \widehat{F}(\Pi_m\Pi_{m-1}\ldots\Pi_2\Pi_1)$, $x \in E$ and $\widehat{F}(\Pi_m\Pi_{m-1}\ldots\Pi_2\Pi_1) = \cap_{i=1}^{m} C_i$.

As direct consequence of Theorem 3.1 and Lemma 3.3, we can prove the following result.

Theorem 3.4. *Let E be a 2-uniformly convex real Banach space which is also uniformly smooth. Let C be a nonempty, closed and convex subset of E. Let F be a bifunction from $C \times C \to \mathbb{R}$ satisfying $(A1) - (A4)$, $B : C \to E^*$ an operator satisfying $(B1) - (B3)$ and let $\{C_i\}_{i=1}^{m}$ be a finite family of closed and convex subsets of E such that $\Omega := (\cap_{i=1}^{m} C_i) \cap EP(F) \cap VI(C, B) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Suppose $\{x_n\}_{n=0}^{\infty}$ is iteratively generated by u, $u_0 \in E$,*

$$\begin{cases} y_n = \Pi_C J^{-1}(Ju_n - r_n Bu_n), \\ x_n = T_{r_n} y_n, \\ u_{n+1} = \Pi_C J^{-1}(\alpha_n Ju + \beta_n Jx_n + \gamma_n J\Pi_m\Pi_{m-1}\ldots\Pi_2\Pi_1 x_n), \quad n \geq 0, \end{cases}$$

with the conditions

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $0 < b \leq \beta_n\gamma_n \leq 1$;

(iii) $0 < a \leq r_n \leq b < \frac{c^2\alpha}{2}$.

Then, $\{x_n\}_{n=0}^{\infty}$ *converges strongly to* $\Pi_\Omega u$.

Proof. Put $T := \Pi_m\Pi_{m-1}\ldots\Pi_2\Pi_1$. It is clear that $F(T) \subset \widehat{F}(T)$ and $\cap_{i=1}^{m} C_i \subset F(T)$. By Lemma 3.3, we have that T is a relatively nonexpansive mapping and $F(T) = \cap_{i=1}^{m} C_i$. Applying Theorem 3.1, we obtain the desired result. □

Acknowledgements : We would like to thank the referee for his comments and suggestions on the manuscript.

References

[1] C.E. Chidume, Geometric properties of Banach spaces and nonlinear iterations, Springer Verlag Series: Lecture Notes in Mathematics Vol. 1965 (2009), XVII, 326p, ISBN 978-1-84882-189-7.

[2] W. Takahashi, Nonlinear Functional Analysis-Fixed Point Theory and Applications, Yokohama Publishers Inc., Yokohama, 2000.

[3] W. Takahashi, Nonlinear functional analysis, Yokohama Publishers Inc., Yokohama, 2000.

[4] Y.I. Alber, Metric and generalized projection operator in Banach spaces: properties and applications, in Theory and Applications of Nonlinear Operators of Accretive and Monotone Type vol 178 of Lecture Notes in Pure and Applied Mathematics, pp 15-50, Dekker, New York, NY, USA, 1996.

[5] Y.I. Alber, S. Reich, An iterative method for solving a class of nonlinear operator equations in Banach spaces, Panamer. Math. J. 4 (1994) 39–54.

[6] I. Cioranescu, Geometry of Banach spaces, duality mappings and nonlinear problems, Kluwer Academic, Dordrecht, 1990.

[7] S. Kamimura, W. Takahashi, Strong convergence of a proximal-type algorithm in a Banach space, SIAM J. Optim. 13 (2002) 938–945.

[8] Y. Su, H.K. Xu, X. Zhang, Strong convergence theorems for two countable families of weak relatively nonexpansive mappings and applications, Nonlinear Anal. 73 (2010) 3890–3906.

[9] D. Butnariu, S. Reich, A.J. Zaslavski, Asymptotic behaviour of relatively nonexpansive operators in Banach spaces, J. Appl. Anal. 7 (2001) 151–174.

[10] D. Butnariu, S. Reich, A.J. Zaslavski, Weak convergence of orbits of nonlinear operator in reflexive Banach spaces, Numer. Funct. Anal. Optim. 24 (2003) 489–508.

[11] Y. Censor, S. Reich, Iterations of paracontractions and firmly nonexpansive operators with applications to feasibility and optimization, Optimization 37 (1996) 323–339.

[12] S. Matsushita, W. Takahashi, A strong convergence theorem for relatively nonexpansive mappings in Banach spaces, J. Approx. Theory 134 (2005) 257–266.

[13] X. Qin, Y. Su, Strong convergence theorems for relatively nonexpansive mappings in a Banach space, Nonlinear Anal. 67 (2007) 1958–1965.

[14] W. Takahashi, K. Zembayashi, Strong convergence theorem by a new hybrid method for equilibrium problems and relatively nonexpansive mappings, Fixed Point Theory Applications, Vol. 2008, Article ID 528476, 11 pages.

[15] W. Nilsrakoo, S. Saejung, Strong convergence to common fixed points of countable relatively quasi-nonexpansive mappings, Fixed Point Theory and Applications, Vol. 2008, Article ID 312454, 19 pages.

[16] X. Qin, Y.J. Cho, S.M. Kang, Convergence theorems of common elements for equilibrium problems and fixed point problems in Banach spaces, J. Comput. Appl. Math. 225 (2009) 20–30.

[17] A. Chinchuluun, P.M. Pardalos, A. Migdalas, L. Pitsoulis, Pareto Optimality, Game Theory and Equilibria Springer-Verlag Series: Optimization and Its Applications, Vol. 17, 2008.

[18] P.L. Combettes, S.A. Hirstoaga, Equilibrium programming in Hilbert spaces, J. Nonlinear Convex Anal. 6 (2005) 117–136.

[19] C.A. Floudas, A. Christodoulos, P.M. Pardalos, Encyclopedia of Optimization - 2ND EDITION, Springer, (2009), XXXIV, 4626 p. 613 illus., ISBN 978-0-387-74760-6.

[20] F. Giannessi, A. Maugeri, P.M. Pardalos, Equilibrium Problems: Nonsmooth Optimization and Variational Inequality Models, Springer Vol. 58 (2002), ISBN 978-1-4020-0161-1.

[21] L. Lin, System of generalized vector quasi-equilibrium problems with applications to fixed point theorems for a family of nonexpansive multivalued mappings, J. Global Optim. 34 (1) (2006) 15–32.

[22] Y. Liu, A general iterative method for equilibrium problems and strict pseudo-contractions in Hilbert spaces, Nonlinear Anal. 71 (2009) 4852–4861.

[23] A. Moudafi, Weak convergence theorems for nonexpansive mappings and equilibrium problems, J. Nonlinear Convex Anal. 9 (2008) 37–43.

[24] P.M. Pardalos, T.M. Rassias, A.A. Khan, Nonlinear Analysis and Variational Problems, Springer, 2010.

[25] S. Plubtieng, R. Punpaeng, A new iterative method for equilibrium problems and fixed point problems of nonexpansive mappings and monotone mappings, Appl. Math. Comput. 197 (2008) 548–558.

[26] X. Qin, M. Shang, Y. Su, Strong convergence of a general iterative algorithm for equilibrium problems and variational inequality problems, Math. Com. Model. 48 (2008) 1033–1046.

[27] Y. Su, M. Shang, X. Qin, An iterative method of solution for equilibrium and optimization problems, Nonlinear Anal. 69 (2008) 2709–2719.

[28] S. Takahashi, W. Takahashi, Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces, J. Math. Anal. Appl. 331 (2007) 506–518.

[29] R. Wangkeeree, An extragradient approximation method for equilibrium problems and fixed point problems of a countable family of nonexpansive mappings, Fixed Point Theory and Applications, Vol. 2008, Article ID 134148, 17 pages.

[30] E. Blum, W. Oettli, From optimization and variational inequalities to equilibrium problems, Math. Student 63 (1994) 123–145.

[31] D. Kinderlehrer, G. Stampaccia, An introduction to variational inequalities and their applications, Academic Press, New York, 1984.

[32] W.R. Mann, Mean value methods in iteration, Proc. Amer. Math. Soc. 4 (1953) 506–510.

[33] A. Genel, J. Lindenstrauss, An example concerning fixed points, Isreal J. Math. 22 (1975) 81–86.

[34] S. Matsushita, W. Takahashi, Weak and strong convergence theorems for relatively nonexpansive mappings in a Banach space, Fixed Point Theory and Applications, Vol. 2004, 37–47.

[35] W. Takahashi, M. Toyoda, Weak convergence theorems for nonexpansive mappings and monotone mappings, J. Optim. Theory Appl. 118 (2003) 417–428.

[36] H. Iiduka, W. Takahashi, Strong convergence theorems for nonexpansive mappings and inverse-strongly monotone mappings, Nonlinear Anal. 61 (2005) 341–350.

[37] H. Iiduka, W. Takahashi Strong convergence studied by a hybrid type method for monotone operators in a Banach space, Nonlinear Anal. 68 (12) 3679–3688.

[38] B. Halpern, Fixed points of nonexpansive maps, Bull. Amer. Math. Soc. 3 (1967) 957–961.

[39] P.L. Lion, Approximation de points fixes de contractions, C. R. Acad. Sci. sér. A-B Paris 284 (1977) 1357–1359.

[40] R. Wittmann, Approximation of fixed points of nonexpasive mappings, Arch. Math. Soc. 59 (1992) 486–491.

[41] S. Reich, Strong convergence theorems for resolvents of accretive operators in Banach spaces, J. Math. Anal. Appl. 75 (1980) 287–292.

[42] S. Shioji, W. Takahashi, Strong convergence of approximated sequences for nonexpansive mappings in Banach spaces, Proc. Amer. Math. Soc. 125 (1997) 3641–3645.

[43] S. Plubtieng, K. Ungchittrakool, Strong convergence theorems for a common fixed point of two relatively nonexpansive mappings in a Banach space, J. Approx. Theory 149 (2007) 103–115.

[44] F. Kohsaka, W. Takahashi, Strong convergence of an iterative sequence for maximal monotone operators in a Banach space, Abstract and Applied Analysis, Vol. 2004, 239–249.

[45] W. Nilsrakoo, S. Saejung, Strong convergence theorems by Halpern-Mann iterations for relatively nonexpansive mappings in Banach spaces, Appl. Math. Comput. 217 (2011) 6577–6586.

[46] S. Chang, J.K. Kim, X.R. Wang, Modified block iterative algorithm for solving convex feasibility problems in Banach spaces, Journal of Inequalities and Applications, Vol. 2010, Article ID 869684, 14 pages.

[47] W. Takahashi, K. Zembayashi, Strong and weak convergence theorems for equilibrium problems and relatively nonexpansive mappings in Banach spaces, Nonlinear Anal. 70 (2009) 45–57.

[48] H.K. Xu, Iterative algorithms for nonlinear operators, J. London Math. Soc. 66 (2) (2002) 240–256.

[49] P.E. Maingé, Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization, Set-Valued Anal. 16 (2008) 899–912.

[50] B. Beauzamy, Introduction to Banach spaces and their geometry, Second Ed. North-Holland, 1985.

[51] R.T. Rockafellar, Monotone operators and proximal point algorithm, SIAM J. Control Optim. 14 (1976) 877–898.

[52] S. Reich, Constructive techniques for accretive and monotone operators, in Applied Nonlinear Analysis (Proc. Third Internat. Conf., Univ. Texas, Arlington, Tex., 1978), pp. 335345, Academic Press, New York, NY, USA, 1979.

(Received 3 May 2011)
(Accepted 20 May 2011)