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Explicit Iteration Method for Common Fixed Points of a Finite Family of Generalized Asymptotically Nonexpansive Nonself Mappings

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Abstract : Let $I = \{1, 2, \dots, N\}$ and let $\{T_i\}_{i\in I}$ be a finite family of generalized asymptotically nonexpansive nonself mappings with a nonempty set of common fixed points where each T_i is a mapping from a nonempty closed convex subset K of a real uniformly convex Banach space X to X. Let $P : X \to K$ be a nonexpansive retraction of X onto K and let $\{\alpha_n\}$ be a sequence in $[0, 1)$. We prove strong and weak convergence theorems for $\{T_i\}_{i\in I}$ using the iteration generated from arbitrary $x_0 \in K$

$$x_n = P((1-\alpha_n)x_{n-1} + \alpha_n T_n(PT_n)^{m-1}x_{n-1}), \;\; n \geq 1$$

where $n = (m-1)N + i$, $i \in I$ and $T_n = T_{n \bmod N}$.

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1 Introduction

Let K be a nonempty subset of real Banach space X. A self-mapping $T : K \to K$ is said to be *asymptotically nonexpansive* on K if there exists a sequence $\{k_n\}$ in $[0, \infty)$ such that $k_n \to 0$ as $n \to \infty$ and

$$\|T^n x - T^n y\| \leq (1 + k_n)\|x - y\|$$

for each $x, y \in K$ and $n \geq 1$. If $k_n = 0$ for all $n \geq 1$, then T is called a *nonexpansive mapping.*

In 1972, Gobel and Kirk [1] introduced the class of asymptotically nonexpansive self-mappings. They showed that the asymptotically nonexpansive mapping $T : K \to K$ where K is a non-empty closed convex subset of a real uniform convex Banach space has a fixed point. Many authors, by using the Mann and Ishikawa iteration process ([2–5]), studied iterative techniques for approximating fixed points of nonexpansive self-mappings.

Xu and Ori [6] showed that the following implicit iteration process for a finite family of nonexpansive self-mappings $\{T_i\}_{i \in I}$, where $I = \{1, 2, \ldots, N\}$, converges to a commmon fixed-point of the finite family of nonexpansive self-mappings:

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_n x_n \tag{1.1}$$

where $\{\alpha_n\}$ is a real sequence in $(0, 1)$, x_0 is any point in K and $T_n = T_{n \bmod N}$. The implicit iteration method has been used in [7–9] to study the common fixed point of a finite family of strictly pseudocontractive self-mapping, asymptotically quasi-nonexpansive self-mappings and asymptotically nonexpansive self-mappings.

In 1991, a modified Mann iteration process was introduced by Schu [10] to approximate fixed points of an asymptotically nonexpansive self-mapping in a Hilbert space as follows.

Theorem 1.1. *Let H be a Hilbert space, K a nonempty closed convex and bounded subset of H. Let $T : K \to K$ be an asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1, \infty)$ for all $n \geq 1$, $\lim_{n\to\infty} k_n = 1$ and $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ satisfying the condition $0 < a \leq \alpha_n \leq b < 1$, $n \geq 1$, for some constant a, b. Then the sequence $\{x_n\}$ generated from arbitrary $x_1 \in K$ by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 1 \tag{1.2}$$

converges strongly to some fixed point of T.

The iterations (1.1) and (1.2) have been widely used by many authors for self-mappings; however they may not well-defined for nonself-mappings.

Chidume et al. [11] introduced the generalization of asymptotically nonexpansive self-mapping, called *the asymptotically nonexpansive nonself-mapping*; this is a special case of generalized asymptotically nonexpansive nonself-mapping defined as follows.

Let $P : X \to K$ be a nonexpansive retraction of X onto K; that is, $P : X \to K$ is a nonexpansive continuous map such that $P(x) = x$ for all $x \in K$. A nonself mapping $T : K \to X$ is said to be *generalized asymptotically nonexpansive* if there exist sequences $\{k_n\}$ and $\{r_n\}$ in $[0, \infty)$ such that $k_n \to 0$ as $n \to \infty$ and

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq (1 + k_n)\|x - y\| + r_n$$

for all $x, y \in K$ and $n \geq 1$. If $r_n = 0$ for all $n \geq 1$, then T is said to be *asymptotically nonexpansive.*

They proved results on strong and weak convergence for asymptotically nonexpansive nonself-mapping in uniformly convex Banach spaces where their iteration process is

$$x_{n+1} = P((1 - \alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n) \text{ and } x_1 \in K.$$

In 2006, Wang [12] proved results on strong and weak convergence for common fixed points of two nonself asymptotically nonexpansive mappings in uniformly convex Banach spaces. Recently, he construct an explicit iteration scheme: for $n \geq 1$ and arbitrary $x_0 \in K$

$$x_n = P((1 - \alpha_n)x_{n-1} + \alpha_n T_n(PT_n)^{m-1}x_{n-1}) \tag{1.3}$$

where $n = (m - 1)N + i$, $T_n = T_{n \bmod N} = T_i$, $i \in I$ and $\{\alpha_n\}$ is a sequence in $[0, 1)$, to approximate a common fixed point of a finite family of nonself asymptotically nonexpansive mappings $\{T_i\}_{i \in I}$. He showed results on weak and strong convergence for such mappings [13].

Motivated by Wang's work, we prove strong and weak convergence theorems for a finite family of generalized asymptotically nonexpansive nonself-mappings using iteration (1.3).

2 Preliminaries

In this section, we recall some concepts and results which are needed to prove our main results.

Definition 2.1. Let X and Y be Banach spaces. A mapping $T : X \to Y$ is said to be *completely continuous* if, for any sequence $\{x_n\}$ in X such that $x_n \to x$ weakly, we have $\|Tx_n - Tx\| \to 0$.

Definition 2.2. Let X be a Banach spaces. A mapping T with domain D and range R in X is said to be *demiclosed* at 0 if, for any sequence $\{x_n\}$ in D such that x_n converges weakly to $x \in D$ and Tx_n converges strongly to 0, we have $Tx = 0$.

Definition 2.3. Let X and Y be Banach spaces. A mapping $T : X \to Y$ is said to be *demicompact* if, for any sequence $\{x_n\}$ in X such that $\|x_n - Tx_n\| \to 0$, there exist a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $x \in X$ such that $\|x_{n_j} - x\| \to 0$.

Definition 2.4. A Banach space X is said to satisfy *Opial's property* if for any distinct elements x and y in X and for each sequence $\{x_n\}$ weakly convergent to x,

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|.$$

Definition 2.5. Let X be a Banach spaces and let K be a subset of X. Let $\{T_i\}_{i\in I}$ be a family of nonself mappings from K to X with a nonempty set F of common fixed points, where $I = \{1, 2, \dots, N\}$. We say that $\{T_i\}_{i\in I}$ satisfies condition $(\bar{A})$ if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0, \infty)$ such that

$$\frac{1}{N}\sum_{i=1}^{N} \|x - T_i x\| \geq f(d(x, F))$$

for all $x \in K$, where $d(x, F) = \inf\{\|x - p\| : p \in F\}$.

Lemma 2.6 ([14])**.** *Let $\{x_n\}$ and $\{y_n\}$ be nonnegative sequences such that $x_{n+1} \leq x_n + y_n$ for all $n \geq 1$. If $\sum_{n=1}^{\infty} y_n < \infty$, then $\lim_{n\to\infty} x_n$ exists.*

Lemma 2.7 ([10])**.** *Let X be a real uniformly convex Banach space. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X. Let $\{\alpha_n\}$ be a sequence such that $0 < p \leq \alpha_n \leq q < 1$ for all integer $n \geq 1$. If $\limsup_{n\to\infty} \|x_n\| \leq r$, $\limsup_{n\to\infty} \|y_n\| \leq r$, and $\lim_{n\to\infty} \|\alpha_n x_n + (1 - \alpha_n) y_n\| = r$ for some $r \geq 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.8 ([15])**.** *Let X be a Banach space which satisfies Opial's condition and let $\{x_n\}$ be a sequence in X. Let $u, v \in X$ be such that $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exist. If $\{x_{n_k}\}$ and $\{x_{n_j}\}$ are subsequences of $\{x_n\}$ which converges weakly to u and v, respectively, then $u = v$.*

3 Main Results

In this section, we let X be a real Banach space, and let K be a nonempty closed convex subset of X which is also a nonexpansive retract of X with nonexpansive retraction P.

For each $i \in I$, we let T_i be a generalized asymptotically nonexpansive nonself mapping from K to X with respect to $\{k_n^{(i)}\}$ and $\{r_n^{(i)}\}$ such that $\sum_{n=1}^{\infty} k_n^{(i)} < \infty$ and $\sum_{n=1}^{\infty} r_n^{(i)} < \infty$.

Let F denote the set of common fixed points of $\{T_i\}_{i\in I}$; here we assume that $F \neq \varnothing$.

Let $\{\alpha_n\}$ be a sequence in $[0, 1)$ and let x_0 be an arbitrary element in K. For $n = (m - 1)N + i \geq 1$, $i \in I$, we let $T_n = T_i$ and

$$x_n = P[(1 - \alpha_n)x_{n-1} + \alpha_n T_n (PT_n)^{m-1} x_{n-1}]. \tag{3.1}$$

Lemma 3.1. *For each $q \in F$, $\lim_{n\to\infty} \|x_n - q\|$ exists.*

Proof. Let $k_n = \max\{k_n^{(1)}, k_n^{(2)}, \ldots, k_n^{(N)}\}$ for each integer n. Then $0 \le k_n \le k_n^{(1)} + k_n^{(2)} + \cdots + k_n^{(N)}$. Since for each $i \in I$, $\sum_{n=1}^{\infty} k_n^{(i)} < \infty$, we have $\sum_{n=1}^{\infty} k_n < \infty$. Similarly, we let $r_n = \max\{r_n^{(1)}, r_n^{(2)}, \ldots, r_n^{(N)}\}$ for each integer n; thus $\sum_{n=1}^{\infty} r_n < \infty$.

From (3.1), we have, for any $q \in F$,

$$\begin{aligned}\|x_n - q\| &= \|P[(1-\alpha_n)x_{n-1} + \alpha_n T_n(PT_n)^{m-1}x_{n-1}] - Pq\| \\ &\le \|(1-\alpha_n)x_{n-1} + \alpha_n T_n(PT_n)^{m-1}x_{n-1} - q\| \\ &= \|(1-\alpha_n)(x_{n-1} - q) + \alpha_n[T_n(PT_n)^{m-1}x_{n-1} - T_n(PT_n)^{m-1}q]\| \\ &\le (1-\alpha_n)\|x_{n-1} - q\| + \alpha_n[(1+k_m)\|x_{n-1} - q\| + r_m] \\ &= (1+\alpha_n k_m)\|x_{n-1} - q\| + \alpha_n r_m \\ &\le (1+k_m)\|x_{n-1} - q\| + r_m. \end{aligned} \tag{3.2}$$

Let $s_n = (1+k_n)^{N-1} + \cdots + (1+k_n) + 1$. Then

$$\begin{aligned}\|x_N - q\| &\le (1+k_1)\|x_{N-1} - q\| + r_1 \\ &\le (1+k_1)^2\|x_{N-2} - q\| + (1+k_1)r_1 + r_1 \\ &\le (1+k_1)^3\|x_{N-3} - q\| + (1+k_1)^2 r_1 + (1+k_1)r_1 + r_1 \\ &\;\vdots \\ &\le (1+k_1)^N\|x_0 - q\| + s_1 r_1.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\|x_{2N} - q\| &\le (1+k_2)^N\|x_N - q\| + s_2 r_2 \\ &\le (1+k_1)^N(1+k_2)^N\|x_0 - q\| + (1+k_2)^N s_1 r_1 + s_2 r_2.\end{aligned}$$

Then, for $n = (m-1)N + i$, $i \in I$,

$$\begin{aligned}\|x_n - q\| \le &(1+k_1)^N \cdots (1+k_{m-1})^N(1+k_m)^i\|x_0 - q\| \\ &+ (1+k_2)^N \cdots (1+k_{m-1})^N(1+k_m)^i s_1 r_1 \\ &+ (1+k_3)^N \cdots (1+k_{m-1})^N(1+k_m)^i s_2 r_2 \\ &+ \cdots + (1+k_m)^i s_{m-1} r_{m-1} \\ &+ [(1+k_m)^i + \cdots + (1+k_m) + 1]r_m.\end{aligned}$$

Since $1 + x \le e^x$ as $x \ge 0$, we have, for each $i \in I$,

$$\begin{aligned}(1+k_i)^N \cdots (1+k_{m-1})^N(1+k_m)^i &\le [(1+k_1)\cdots(1+k_m)]^N \\ &\le (e^{k_1} \cdots e^{k_m})^N \\ &= e^{(k_1 + \cdots + k_m)N}.\end{aligned}$$

Since $\sum_{n=0}^{\infty} k_n < \infty$, the nonnegative sequence $\{k_n\}$ converges to 0 and hence there exists a constant $n_0 > 0$ such that $0 \leq k_n \leq 1$ for all $n \geq n_0$. Then, for any $n \geq n_0$,

$$\begin{aligned} s_n &= (1+k_n)^{N-1} + \cdots + (1+k_n) + 1 \\ &= [(1+k_n)^N - 1]/k_n \\ &= \tbinom{N}{1} + \tbinom{N}{2}k_n + \tbinom{N}{3}k_n^2 + \cdots + \tbinom{N}{N}k_n^{N-1} \\ &\leq \tbinom{N}{1} + \tbinom{N}{2} + \tbinom{N}{3} + \cdots + \tbinom{N}{N} \\ &= 2^N - 1. \end{aligned}$$

Then there exists a positive constant C such that $s_n \leq C$ for all $n \geq 1$. Therefore

$$\begin{aligned} \|x_n - q\| &\leq e^{(k_1+\cdots+k_m)N}(\|x_0 - q\| + s_1 r_1 + s_2 r_2 + \cdots + s_m r_m) \\ &\leq e^{(k_1+\cdots+k_m)N}[\|x_0 - q\| + C(r_1 + r_2 + \cdots + r_m)]. \end{aligned}$$

Since $\sum_{n=1}^{\infty} k_n < \infty$ and $\sum_{n=1}^{\infty} r_n < \infty$, we conclude that $\|x_n - q\|$ is bounded; that is, there exists a constant $M > 0$ such that $\|x_n - q\| \leq M$ for all $n \geq 0$. From (3.2), we have, for $n \geq 1$,

$$\begin{aligned} \|x_n - q\| &\leq \|x_{n-1} - q\| + k_m\|x_{n-1} - q\| + r_m \\ &\leq \|x_{n-1} - q\| + k_m M + r_m. \end{aligned}$$

By Lemma 2.6, we have that $\lim_{n\to\infty} \|x_n - q\|$ exists. □

Lemma 3.2. *If X is uniformly convex and $\{\alpha_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0,1)$, then $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ for each $i \in I$.*

Proof. Let $q \in F$. By Lemma 3.1, $\lim_{n\to\infty} \|x_n - q\|$ exists; let $c = \lim_{n\to\infty} \|x_n - q\|$. By (3.2), assuming $n+1 = (m-1)N + i$ for $i \in I$, we have

$$\begin{aligned} \|x_{n+1} - q\| &\leq \|(1-\alpha_{n+1})(x_n - q) + \alpha_{n+1}[T_{n+1}(PT_{n+1})^{m-1}x_n - q]\| \\ &\leq (1+k_m)\|x_n - q\| + r_m \end{aligned}$$

where $k_n = \max\{k_n^{(1)}, k_n^{(2)}, \ldots, k_n^{(N)}\}$ and $r_n = \max\{r_n^{(1)}, r_n^{(2)}, \ldots, r_n^{(N)}\}$ for each integer n. Since both $\|x_{n+1} - q\|$ and $(1+k_m)\|x_n - q\| + r_m$ converge to c, we have

$$\lim_{n\to\infty} \|(1-\alpha_{n+1})(x_n - q) + \alpha_{n+1}[T_{n+1}(PT_{n+1})^{m-1}x_n - q]\| = c.$$

Since $\|x_{n+1} - q\|$ converges to c, we have

$$\limsup_{n\to\infty} \|T_{n+1}(PT_{n+1})^{m-1}x_n - q\| \leq \lim_{n\to\infty} \|x_{n+1} - q\| = c.$$

It follows from Lemma 2.7 that

$$\lim_{n\to\infty} \|x_n - T_{n+1}(PT_{n+1})^{m-1}x_n\| = 0.$$

Since

$$\begin{aligned}\|x_{n+1}-x_n\| &= \|P[(1-\alpha_{n+1})x_n+\alpha_{n+1}T_{n+1}(PT_{n+1})^{m-1}x_n]-Px_n\|\\ &\le \|(1-\alpha_{n+1})x_n+\alpha_{n+1}T_{n+1}(PT_{n+1})^{m-1}x_n-x_n\|\\ &= \alpha_{n+1}\|[T_{n+1}(PT_{n+1})^{m-1}x_n-x_n]\|,\end{aligned}$$

it follows that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. By induction, we have

$$\lim_{n\to\infty}\|x_{n+j}-x_n\|=0$$

for any positive integer j. Since, for any $x,y\in K$ and $i\in I$,

$$\|T_ix-T_iy\|\le k_1^{(i)}\|x-y\|+r_1^{(i)}\le k_1\|x-y\|+r_1,$$

we have, for $n>N$,

$$\begin{aligned}\|x_n-T_{n+1}x_n\| &\le \|x_n-T_{n+1}(PT_{n+1})^{m-1}x_n\|+\|T_{n+1}(PT_{n+1})^{m-1}x_n-T_{n+1}Px_n\|\\ &\le \|x_n-T_{n+1}(PT_{n+1})^{m-1}x_n\|+k_1\|(PT_{n+1})^{m-1}x_n-Px_n\|+r_1\\ &\le \|x_n-T_{n+1}(PT_{n+1})^{m-1}x_n\|+k_1\|T_{n+1}(PT_{n+1})^{m-2}x_n-x_n\|+r_1\\ &\le \|x_n-T_{n+1}(PT_{n+1})^{m-1}x_n\|\\ &\quad +k_1[\|T_{n+1-N}(PT_{n+1-N})^{m-2}x_n-T_{n+1-N}(PT_{n+1-N})^{m-2}x_{n-N}\|\\ &\quad +\|T_{n+1-N}(PT_{n+1-N})^{m-2}x_{n-N}-x_{n-N}\|+\|x_{n-N}-x_n\|]+r_1.\end{aligned}$$

Then $\lim_{n\to\infty}\|x_n-T_{n+1}x_n\|=0$. Since, for each $i\in I$,

$$\begin{aligned}\|x_n-T_{n+i}x_n\| &\le \|x_n-x_{n+i-1}\|+\|x_{n+i-1}-T_{n+i}x_{n+i-1}\|+\|T_{n+i}x_{n+i-1}-T_{n+i}x_n\|\\ &\le (1+k_1)\|x_n-x_{n+i-1}\|+\|x_{n+i-1}-T_{n+i}x_{n+i-1}\|+r_1,\end{aligned}$$

we have $\lim_{n\to\infty}\|x_n-T_{n+i}x_n\|=0$ which completes the proof. □

Theorem 3.3. *Suppose that X is uniformly convex and $\{\alpha_n\}\subset[\delta,1-\delta]$ for some $\delta\in(0,1)$. If $\{T_i\}_{i\in I}$ satisfies condition (A′), then $\{x_n\}$ converges strongly to a common fixed point in F.*

Proof. From the proof of Lemma 3.1, we can show that, for any positive integers n, m and N_0 with $n\ge N_0+mN$,

$$\|x_n-p\|\le M\|x_{N_0}-p\|+C\sum_{i=m}^{\infty}r_i$$

where $M=e^{N\sum_{i=1}^{\infty}k_i}$ and $C\ge(1+k_n)^{N-1}+\cdots+(1+k_n)+1$ for all positive integers n.

By Lemma 3.2, we have $\lim_{n\to\infty}\|x_n-T_ix_n\|=0$ for each $i\in I$. Since $\{T_i\}_{i\in I}$ satisfies condition (A′), $\lim_{n\to\infty}d(x_n,F)=0$. Let $\epsilon>0$ be given. There exists a positive integer N_0 such that $d(x_n, F) < \frac{\epsilon}{5M}$ for all $n \geq N_0$. Then there exists $p \in F$ such that

$$\|x_{N_0} - p\| < \frac{\epsilon}{4M}.$$

In addition, since $\sum_{i=1}^{\infty} r_i < \infty$, there exists a positive integer m such that

$$\sum_{i=m}^{\infty} r_i < \frac{\epsilon}{4C}.$$

Thus, for $n, k \geq N_0 + mN$,

$$\begin{aligned}\|x_n - x_k\| &\leq \|x_n - p\| + \|x_k - p\| \\ &\leq 2M\|x_{N_0} - p\| + 2C\sum_{i=m}^{\infty} r_i < \epsilon.\end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in K. We assume that $x_n \to q \in K$ as $n \to \infty$. By Lemma 3.2, we have $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ for each $i \in I$; by the continuity of T_i, we conclude that q is a common fixed point of $\{T_i\}_{i\in I}$. □

Theorem 3.4. *Suppose that X is uniformly convex and $\{\alpha_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0, 1)$. If T_k is completely continuous for some $k \in I$ and $I - T_i$ is demiclosed at zero for all $i \in I$, then $\{x_n\}$ converges strongly to a common fixed point in F.*

Proof. From Lemma 3.1 and Lemma 3.2, $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ for each $i \in I$. Then $\{T_i x_n\}$ is bounded for each $i \in I$. Assume without loss of generality that T_1 is completely continuous. Then there exist an element $p \in K$ and a subsequence $\{T_1 x_{n_j}\}$ such that $\|T_1 x_{n_j} - p\| \to 0$ as $j \to \infty$. Since

$$\|x_{n_j} - p\| \leq \|x_{n_j} - T_1 x_{n_j}\| + \|T_1 x_{n_j} - p\|,$$

we have $\lim_{j\to\infty} \|x_{n_j} - p\| = 0$. Since each $I - T_i$ is demiclosed, we have that $p \in F$. By Lemma 3.1, $\lim_{n\to\infty} \|x_n - p\|$ exists and hence equals zero. Then $\{x_n\}$ converges strongly to a common fixed point in F. □

Theorem 3.5. *Suppose that X is uniformly convex and $\{\alpha_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0, 1)$. If T_i is demicompact for some $i \in I$ and $I - T_i$ is demiclosed at zero for each i, then $\{x_n\}$ converges strongly to a common fixed point in F.*

Proof. Without lost of generality, we suppose that T_1 is demicompact; by Lemma 3.1 and Lemma 3.2 we have that a sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - T_1 x_n\| = 0$. Since T_1 is demicompact, there exist $q \in K$ and a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to q$ strongly. Moreover we have $q \in F$, by Lemma 3.2 together with the assumption that $I - T_i$ is demiclosed at zero for all $i \in I$. By Lemma 3.1, $\{x_n\}$ converges strongly to q, a common fixed point of $\{T_i\}_{i\in I}$. □

Theorem 3.6. *Suppose that X is uniformly convex and $\{\alpha_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0, 1)$. If X satisfies Opial's property and $I - T_i$ is demiclosed at zero for each i, then $\{x_n\}$ converges weakly to a common fixed point in F.*

Proof. It follows from Lemma 3.1 that $\lim_{n\to\infty}\|x_n - q\|$ exists for all $q \in F$. To complete the proof, we have to show that a sequence $\{x_n\}$ has a unique weak subsequential limit in F. Let q_1 and q_2 be weak limits of subsequences $\{x_{n_k}\}$ and $\{x_{n_j}\}$, respectively. By Lemma 3.2 and the assumption that each $I - T_i$ is demiclosed at zero, we have $q_1, q_2 \in F$. By Lemma 2.8, $q_1 = q_2$. Therefore $\{x_n\}$ converges weakly to a common fixed point in F. □

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