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Weak and Strong Convergence to Common Fixed Points of a Countable Family of Multi-valued Mappings in Banach Spaces

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Abstract : In this paper, we introduce a modified Mann iteration for a countable family of multi-valued mappings. We use the best approximation operator to obtain weak and strong convergence theorems in a Banach space. We apply the main results to the problem of finding a common fixed point of a countable family of nonexpansive multi-valued mappings.

Keywords : Nonexpansive multi-valued mapping; Fixed point; Weak convergence; Strong convergence; Banach space.
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1 Introduction

Let D be a nonempty and convex subset of a Banach spaces E. The set D is called *proximinal* if for each $x \in E$, there exists an element $y \in D$ such that $\|x - y\| = d(x, D)$, where $d(x, D) = \inf\{\|x - z\| : z \in D\}$. Let $CB(D), CCB(D), K(D)$ and $P(D)$ denote the families of nonempty closed bounded subsets, nonempty closed convex bounded subsets, nonempty compact subsets, and nonempty proximinal bounded subsets of D, respectively. The *Hausdorff metric* on $CB(D)$ is defined by

$$H(A, B) = \max\left\{\sup_{x\in A} d(x, B),\ \sup_{y\in B} d(y, A)\right\}$$

for $A, B \in CB(D)$. A single-valued map $T : D \to D$ is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in D$. A multi-valued mapping $T : D \to CB(D)$ is called *nonexpansive* if $H(Tx, Ty) \leq \|x - y\|$ for all $x, y \in D$. An element $p \in D$ is called a *fixed point* of $T : D \to D$ (respectively, $T : D \to CB(D)$) if $p = Tp$ (respectively, $p \in Tp$). The set of fixed points of T is denoted by $F(T)$. The mapping $T : D \to CB(D)$ is called *quasi-nonexpansive* [1] if $F(T) \neq \emptyset$ and $H(Tx, Tp) \leq \|x - p\|$ for all $x \in D$ and all $p \in F(T)$. It is clear that every nonexpansive multi-valued mapping T with $F(T) \neq \emptyset$ is quasi-nonexpansive. But there exist quasi-nonexpansive mappings that are not nonexpansive (see [2]). It is known that if T is a quasi-nonexpansive multi-valued mapping, then $F(T)$ is closed.

Throughout this paper, we denote the weak convergence and the strong convergence by $\rightharpoonup$ and $\to$, respectively. The mapping $T : D \to CB(D)$ is called *hemicompact* if, for any sequence $\{x_n\}$ in D such that $d(x_n, Tx_n) \to 0$ as $n \to \infty$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in D$. We note that if D is compact, then every multi-valued mapping $T : D \to CB(D)$ is *hemicompact*.

A Banach space E is said to satisfy *Opial's condition* [3] if for each $x \in E$ and a sequence $\{x_n\}$ in E such that $x_n \rightharpoonup x$, the following condition holds for all $x \neq y$:

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|.$$

The mapping $T : D \to CB(D)$ is called *demi-closed* if for every sequence $\{x_n\} \subset D$ and any $y_n \in Tx_n$ such that $x_n \rightharpoonup x$ and $y_n \to y$, we have $x \in D$ and $y \in Tx$.

Remark 1.1 ([4]). *If the space E satisfies Opial's condition, then $I - T$ is demi-closed at 0, where $T : D \to K(D)$ is a nonexpansive multi-valued mapping.*

For a single-valued case, in 1953, Mann [5] introduced the following iterative procedure to approximate a fixed point of a nonexpansive mapping T in a real Hilbert space H:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad \forall n \in \mathbb{N}, \tag{1.1}$$

where the initial point x_1 is taken in D arbitrarily and $\{\alpha_n\}$ is a sequence in $(0, 1)$.

However, we note that Mann's iteration process (1.1) has only weak convergence, in general; for instance, see [6–8].

Since 1953, Mann's iteration has extensively been studied by many authors (see, for examples, [9–18]). However, the studying of multivalued nonexpansive mappings is harder than that of single-valued nonexpansive mappings in both Hilbert spaces and Banach spaces. The result of fixed points for multi-valued contractions and nonexpansive mappings by using the Hausdorff metric was initiated by Markin [19]. Later, different iterative processes have been used to approximate fixed points of multi-valued nonexpansive mappings (see also [1, 20–26]).

In 2009, Song and Wang [26] proved strong and weak convergence theorems for Mann's iteration of a multi-valued nonexpansive mapping T in a Banach space. They studied strong convergence of the modified Mann iteration which is independent of the implicit anchor-like continuous path $z_t \in tu + (1-t)Tz_t$.

Let D be a nonempty and closed subset of a Banach space E, $\{\beta_n\} \subset [0,1]$, $\{\alpha_n\} \subset [0,1]$ and $\{\gamma_n\} \subset (0,+\infty)$ such that $\lim_{n\to\infty} \gamma_n = 0$.

(A) Choose $x_0 \in D$,

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n y_n, \ \forall n \geq 0,$$

where $y_n \in Tx_n$ such that $\|y_{n+1} - y_n\| \leq H(Tx_{n+1}, Tx_n) + \gamma_n$.

(B) For fixed $u \in D$, the sequence of modified Mann iteration is defined by $x_0 \in D$,

$$x_{n+1} = \beta_n u + \alpha_n x_n + (1-\alpha_n-\beta_n)y_n, \ \forall n \geq 0,$$

where $y_n \in Tx_n$ such that $\|y_{n+1} - y_n\| \leq H(Tx_{n+1}, Tx_n) + \gamma_n$.

Very recently, Shahzad and Zegeye [2] obtained the strong convergence theorems for a quasi-nonexpansive multi-valued mapping. They relaxed the compactness of domain of T and constructed an iterative scheme which removes the restriction of T namely $Tp = \{p\}$ for any $p \in F(T)$. The results provided an affirmative answer to some questions raised in [21]. In fact, they introduced iterations as follows:

Let D be a nonempty and convex subset of a Banach space E, let $T : D \to CB(D)$ and let $\{\alpha_n\}, \{\alpha'_n\} \subset [0,1]$.

(C) The sequence of Ishikawa's iteration is defined by $x_0 \in D$,

$$\begin{aligned} y_n &= \alpha'_n z'_n + (1-\alpha'_n)x_n, \\ x_{n+1} &= \alpha_n z_n + (1-\alpha_n)x_n, \ \forall n \geq 0, \end{aligned}$$

where $z'_n \in Tx_n$ and $z_n \in Ty_n$.

(D) Let $T : D \to P(D)$ and $P_T x = \{y \in Tx : \|x-y\| = d(x,Tx)\}$, where P_T is the best approximation operator. The sequence of Ishikawa's iteration is defined by $x_0 \in D$,

$$\begin{aligned} y_n &= \alpha'_n z'_n + (1-\alpha'_n)x_n, \\ x_{n+1} &= \alpha_n z_n + (1-\alpha_n)x_n, \ \forall n \geq 0, \end{aligned}$$

where $z'_n \in P_T x_n$ and $z_n \in P_T y_n$.

It is remarked that Hussain and Khan [27], in 2003, employed the best approximation operator P_T to study fixed points of *-nonexpansive multi-valued mapping T and strong convergence of its iterates to a fixed point of T defined on a closed and convex subset of a real Hilbert space.

Let D be a nonempty, closed and convex subset of a Banach space E. Let $\{T_n\}_{n=1}^{\infty}$ be a family of multi-valued mappings from D into 2^D and let $P_{T_n}x = \{y_n \in T_n x : \|x - y_n\| = d(x, T_n x)\}$, $n \geq 1$. Let $\{\alpha_n\}$ be a sequence in $(0, 1)$.

(E) The sequence of the modified Mann's iteration is defined by $x_1 \in D$ and

$$x_{n+1} \in \alpha_n x_n + (1 - \alpha_n) P_{T_n} x_n, \ \forall n \geq 1. \tag{1.2}$$

In this paper, we modify Mann's iteration by using the best approximation operator P_{T_n}, $n \geq 1$ to find common fixed points of a countable family of nonexpansive multi-valued mappings $\{T_n\}_{n=1}^{\infty}$, $n \geq 1$. Then we prove weak and strong convergence theorems for a countable family of multi-valued mappings in Banach spaces. Finally, we apply our main result to the problem of finding a common fixed point of a family of nonexpansive multi-valued mappings.

2 Preliminaries

In this section, we give some characterizations and properties of the metric projection in a real Hilbert space.

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let D be a closed and convex subset of H. If, for any point $x \in H$, there exists a unique nearest point in D, denoted by $P_D x$, such that

$$\|x - P_D x\| \leq \|x - y\|, \ \forall y \in D,$$

then P_D is called the *metric projection* of H onto D. We know that P_D is a nonexpansive mapping of H onto D.

Lemma 2.1 ([28]). *Let D be a closed and convex subset of a real Hilbert space H and P_D be the metric projection from H onto D. Then, for any $x \in H$ and $z \in D$, $z = P_D x$ if and only if the following holds:*

$$\langle x - z, y - z \rangle \leq 0, \quad \forall y \in D.$$

Using the proof line in Lemma 3.1.3 of [28], we obtain the following result.

Proposition 2.2. *Let D be a closed and convex subset of a real Hilbert space H. Let $T : D \to CCB(D)$ be a multi-valued mapping and P_T the best approximation operator. Then, for any $x \in D$, $z \in P_T x$ if and only if the following holds:*

$$\langle x - z, y - z \rangle \leq 0, \ \forall y \in Tx.$$

Lemma 2.3 ([28]). *Let H be a real Hilbert space. Then the following equations hold:*

(1) $\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle$ *for all* $x, y \in H$*;*

(2) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2$ *for all* $t \in [0,1]$ *and* $x, y \in H$*.*

We next show that P_T is nonexpansive under some suitable conditions imposed on T.

Remark 2.4. *Let* D *be a closed and convex subset of a real Hilbert space* H*. Let* $T : D \to CCB(D)$ *be a multi-valued mapping. If* $Tx = Ty$*,* $\forall x, y \in D$*, then* P_T *is a nonexpansive multi-valued mapping.*

In fact, let $x, y \in D$. For each $a \in P_T x$, we have

$$d(a, P_T y) \leq \|a-b\|, \ \forall b \in P_T y. \tag{2.1}$$

From Proposition 2.2, we have

$$\langle x-y-(a-b), a-b\rangle = \langle x-a, a-b\rangle + \langle y-b, b-a\rangle \geq 0.$$

It follows that

$$\begin{aligned}\|a-b\|^2 &= \langle x-y, a-b\rangle + \langle a-b-(x-y), a-b\rangle \\ &\leq \langle x-y, a-b\rangle \\ &\leq \|x-y\|\|a-b\|. \end{aligned} \tag{2.2}$$

This implies that

$$\|a-b\| \leq \|x-y\|. \tag{2.3}$$

From (2.1) and (2.3), we obtain

$$d(a, P_T y) \leq \|x-y\|$$

for every $a \in P_T x$. Hence $\sup_{a\in P_T x} d(a, P_T y) \leq \|x-y\|$. Similarly, we can show that $\sup_{b\in P_T y} d(P_T x, b) \leq \|x-y\|$. Therefore $H(P_T x, P_T y) \leq \|x-y\|$.

It is clear that if a nonexpansive multi-valued mapping T satisfies the condition that $Tx = Ty$, $\forall x, y \in D$, then P_T is nonexpansive. The following example shows that if T is a nonexpansive multi-valued mapping satisfying the property that $Tx = Ty$, $\forall x, y \in D$, then Tx is not a singleton for all $x \in D$.

Example 2.5. *Consider* $D = [0,1]$ *with the usual norm. Define* $T : D \to K(D)$ *by*

$$Tx = [0, a], \ a \in (0, 1].$$

For $x, y \in D$, we have $H(Tx, Ty) = 0 \leq \|x-y\|$. Hence T is nonexpansive and $F(T) = [0, a]$.

Next, we show that there exists a nonexpansive multi-valued mapping T which P_T is nonexpansive but $Tx \neq Ty$, $\forall x, y \in D$, $x \neq y$.

Example 2.6. *Consider $D = [0,1]$ with the usual norm. Define $T : D \to K(D)$ by*

$$Tx = [0, x].$$

For $x, y \in D$, $x \neq y$, we have $P_T x = \{x\}$ and $P_T y = \{y\}$. Then $H(Tx, Ty) = \|x - y\| = H(P_T x, P_T y)$. Hence T and P_T are nonexpansive.

Question: Can we remove the assumption that P_T is nonexpansive?

Now, we give an example which T is not nonexpansive, but P_T is nonexpansive.

Example 2.7. *Consider $D = [0,1]$ with the usual norm. Define $T : D \to K(D)$ by*

$$Tx = \begin{cases} [0, x] & , x \in [0, \frac{1}{2}], \\ \{\frac{1}{2}\} & , x \in (\frac{1}{2}, 1]. \end{cases}$$

Since $H(T(\frac{1}{5}), T(\frac{3}{5})) = H([0, \frac{1}{5}], \{\frac{1}{2}\}) = \frac{1}{2} > \frac{2}{5} = \|\frac{1}{5} - \frac{3}{5}\|$, T is not nonexpansive. However, P_T is nonexpansive. In fact,

Case 1: if $x, y \in [0, \frac{1}{2}]$ then $H(P_T x, P_T y) = \|x - y\|$.

Case 2: if $x \in [0, \frac{1}{2}]$ and $y \in (\frac{1}{2}, 1]$ then $H(P_T x, P_T y) = \|x - \frac{1}{2}\| \leq \|x - y\|$.

Case 3: if $x, y \in (\frac{1}{2}, 1]$ then $H(P_T x, P_T y) = 0$.

It would be interesting to study the convergence of a multivalued mapping T by using the best approximation operator P_T.

In order to deal with a family of mappings, we consider the following conditions. Let E be a Banach space and D a subset of E.

(1) Let $\{T_n\}$ and τ be two families of mappings of D into itself with $\emptyset \neq F(\tau) = \bigcap_{n=1}^{\infty} F(T_n)$, where $F(T_n)$ is the set of all fixed points of T_n and $F(\tau)$ is the set of all common fixed points of τ. The family $\{T_n\}$ is said to satisfy the *NST-condition* [29] with respect to τ if, for each bounded sequence $\{z_n\}$ in C,

$$\lim_{n\to\infty} \|z_n - T_n z_n\| = 0 \implies \lim_{n\to\infty} \|z_n - T z_n\| = 0, \quad \forall T \in \tau.$$

(2) Let $\{T_n\}$ and τ be two families of multi-valued mappings of D into 2^D with $\emptyset \neq F(\tau) = \bigcap_{n=1}^{\infty} F(T_n)$, where $F(T_n)$ is the set of all fixed points of T_n and $F(\tau)$ is the set of all common fixed points of τ. The family $\{T_n\}$ is said to satisfy the *SC-condition* with respect to τ if, for each bounded sequence $\{z_n\}$ in D and $s_n \in T_n z_n$,

$$\lim_{n\to\infty} \|z_n - s_n\| = 0 \implies \lim_{n\to\infty} \|z_n - c_n\| = 0, \quad \exists c_n \in T z_n, \quad \forall T \in \tau.$$

It is easy to see that, if the family $\{T_n\}$ of nonexpansive mappings satisfies the $NST-condition$, then $\{T_n\}$ satisfies the $SC-condition$ for single-valued mappings.

(3) Let T be a multi-valued mapping from D into 2^D with $F(T) \neq \emptyset$. The mapping T is said to satisfy *Condition I* if there is a nondecreasing function

$f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for $r \in (0, \infty)$ such that $d(x, Tx) \geq f(d(x, F(T)))$ for all $x \in D$.

The following result can be found in [30].

Lemma 2.8 ([30]). *Let D be a bounded and closed subset of a Banach space E. Suppose that a nonexpansive multi-valued mapping $T : D \to P(D)$ has a nonempty fixed point set. If $I - T$ is closed, then T satisfies Condition I on D.*

(4) Let $\{T_n\}$ and τ be two families of multi-valued mappings of D into 2^D with $\emptyset \neq F(\tau) = \bigcap_{n=1}^{\infty} F(T_n)$, where $F(T_n)$ is the set of all fixed points of T_n and $F(\tau)$ is the set of all common fixed points of τ. The family $\{T_n\}$ is said to satisfy *Condition (A)* if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for $r \in (0, \infty)$ such that there exists $T \in \tau$, $d(x, Tx) \geq f(d(x, F(\tau)))$ for all $x \in D$.

We will give examples of a sequence mappings $\{T_n\}$ which satisfy the SC-condition and Condition (A) in the last section.

Now, we need the following lemmas to prove our main results.

Lemma 2.9 ([31]). *Let X be uniformly convex Banach space and $B_r(0)$ be a closed ball of X. Then there exists a continuous, strictly increasing and convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)g(\|x - y\|)$$

for all $x, y \in B_r(0)$ and $\lambda \in [0, 1]$.

Lemma 2.10 ([32]). *Let E be a uniformly convex Banach space and $B_r(0) = \{x \in E : \|x\| \leq r\}$ be a closed ball of E. Then there exists a continuous, strictly increasing and convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that, for any $j \in \{1, 2, ..., m\}$,*

$$\Big\| \sum_{i=1}^{m} \alpha_i x_i \Big\|^2 \leq \sum_{i=1}^{m} \alpha_i \|x_i\|^2 - \frac{\alpha_j}{m-1} \Big(\sum_{i=1}^{m} \alpha_i g(\|x_j - x_i\|) \Big)$$

for all $m \in \mathbb{N}$, $x_i \in B_r(0)$ and $\alpha_i \in [0, 1]$ for all $i = 1, 2, ..., m$ with $\sum_{i=1}^{m} \alpha_i = 1$.

3 Strong and Weak Convergence of the Modified Mann's Iteration in Banach Spaces

In this section, we first prove a strong convergence theorem for a countable family of multi-valued mappings under the SC-condition and Condition (A) and then prove a weak convergence theorem under the SC-condition in Banach spaces.

Theorem 3.1. *Let D be a closed and convex subset of a uniformly convex Banach space E which satisfies Opial's condition. Let $\{T_n\}$ and τ be two families of multi-valued mappings from D into $P(D)$ with $F(\tau) = \cap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $(0, 1)$ such that $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$. Let $\{x_n\}$ be generated by (1.2). Assume that*

(A1) for each $n \in \mathbb{N}$, $H(P_{T_n}x, P_{T_n}p) \leq \|x - p\|$, $\forall x \in D$, $p \in F(\tau)$;

(A2) $I - T$ *is demi-closed at* 0 *for all* $T \in \tau$.

If $\{T_n\}$ *satisfies the SC-condition, then* $\{x_n\}$ *converges weakly to an element in* $F(\tau)$.

Proof. Since $x_{n+1} \in \alpha_n x_n + (1 - \alpha_n)P_{T_n}x_n$, there exists $z_n \in P_{T_n}x_n$ such that $x_{n+1} = \alpha_n x_n + (1 - \alpha_n)z_n$. We note that $P_{T_n}p = \{p\}$ for all $p \in F(\tau)$ and $n \in \mathbb{N}$. It follows from (A1) that

$$\begin{aligned} \|x_{n+1} - p\| &\leq \alpha_n\|x_n - p\| + (1 - \alpha_n)\|z_n - p\| \\ &= \alpha_n\|x_n - p\| + (1 - \alpha_n)d(z_n, P_{T_n}p) \\ &\leq \alpha_n\|x_n - p\| + (1 - \alpha_n)H(P_{T_n}x_n, P_{T_n}p) \\ &\leq \|x_n - p\| \end{aligned} \tag{3.1}$$

for every $p \in F(\tau)$. Then $\{\|x_n - p\|\}$ is a decreasing sequence and hence $\lim_{n\to\infty}\|x_n - p\|$ exists for every $p \in F(\tau)$. For $p \in F(\tau)$, since $\{x_n\}$ and $\{z_n\}$ are bounded, by Lemma 2.9, there exists a continuous, strictly increasing and convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|\alpha_n(x_n - p) + (1 - \alpha_n)(z_n - p)\|^2 \\ &\leq \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|z_n - p\|^2 - \alpha_n(1 - \alpha_n)g(\|x_n - z_n\|) \\ &= \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)d(z_n, P_{T_n}p)^2 - \alpha_n(1 - \alpha_n)g(\|x_n - z_n\|) \\ &\leq \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)H(P_{T_n}x_n, P_{T_n}p)^2 - \alpha_n(1 - \alpha_n)g(\|x_n - z_n\|) \\ &\leq \|x_n - p\|^2 - \alpha_n(1 - \alpha_n)g(\|x_n - z_n\|). \end{aligned}$$

It follows that

$$\alpha_n(1 - \alpha_n)g(\|x_n - z_n\|) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2.$$

Since $\lim_{n\to\infty}\|x_n - p\|$ exists and $0 < \liminf_{n\to\infty}\alpha_n \leq \limsup_{n\to\infty}\alpha_n < 1$,

$$\lim_{n\to\infty} g(\|x_n - z_n\|) = 0.$$

By the properties of g, we can conclude that

$$\lim_{n\to\infty}\|x_n - z_n\| = 0.$$

Since $\{T_n\}$ satisfies the SC-condition, there exists $c_n \in Tx_n$ such that

$$\lim_{n\to\infty}\|x_n - c_n\| = 0 \tag{3.2}$$

for every $T \in \tau$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges weakly to some $q_1 \in D$. It follows from (A2) and (3.2) that $q_1 \in Tq_1$ for every $T \in \tau$. Next, we show that $\{x_n\}$ converges weakly to q_1, take another subsequence $\{x_{m_k}\}$ of $\{x_n\}$ converging weakly to some $q_2 \in D$. Again, as above, we can conclude that $q_2 \in Tq_2$ for every $T \in \tau$. Finally, we show that $q_1 = q_2$. Assume $q_1 \neq q_2$. Then by the Opial's condition of E, we have

$$\begin{aligned}\lim_{n\to\infty} \|x_n - q_1\| &= \lim_{k\to\infty} \|x_{n_k} - q_1\| \\ &< \lim_{k\to\infty} \|x_{n_k} - q_2\| \\ &= \lim_{n\to\infty} \|x_n - q_2\| \\ &= \lim_{k\to\infty} \|x_{m_k} - q_2\| \\ &< \lim_{k\to\infty} \|x_{m_k} - q_1\| \\ &= \lim_{n\to\infty} \|x_n - q_1\|,\end{aligned}$$

which is a contradiction. Therefore $q_1 = q_2$. This shows that $\{x_n\}$ converges weakly to a fixed point of T for every $T \in \tau$. This completes the proof. □

Using the above results and Remark 1.1, we obtain the following:

Corollary 3.2. *Let D be a closed and convex subset of a uniformly convex Banach space E which satisfies Opial's condition. Let $\{T_n\}$ and τ be two families of non-expansive multi-valued mappings from D into $K(D)$ with $F(\tau) = \cap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $(0,1)$ such that $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$. Let $\{x_n\}$ be generated by (1.2). Assume that for each $n \in \mathbb{N}$,*

$$H(P_{T_n}x, P_{T_n}p) \leq \|x - p\|,$$

$\forall x \in D$, $p \in F(\tau)$. If $\{T_n\}$ satisfies the SC-condition, then $\{x_n\}$ converges weakly to an element in $F(\tau)$.

Theorem 3.3. *Let D be a closed and convex subset of a uniformly convex Banach space E. Let $\{T_n\}$ and τ be two families of multi-valued mappings from D into $P(D)$ with $F(\tau) = \cap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $(0,1)$ such that $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$. Let $\{x_n\}$ be generated by (1.2). Assume that*

(B1) for each $n \in \mathbb{N}$, $H(P_{T_n}x, P_{T_n}p) \leq \|x - p\|$, $\forall x \in D$, $p \in F(\tau)$;

(B2) the best approximation operator P_T is nonexpansive for every $T \in \tau$;

(B3) $F(\tau)$ is closed.

If $\{T_n\}$ satisfies the SC-condition and Condition (A), then $\{x_n\}$ converges strongly to an element in $F(\tau)$.

Proof. It follows from the proof of Theorem 3.1 that $\lim_{n\to\infty} \|x_n - p\|$ exists for every $p \in F(\tau)$ and $\lim_{n\to\infty} \|x_n - z_n\| = 0$ where $z_n \in P_{T_n}x_n$. Since $\{T_n\}$ satisfies the SC-condition, there exists $c_n \in Tx_n$ such that

$$\lim_{n\to\infty} \|x_n - c_n\| = 0$$

for every $T \in \tau$. This implies that

$$\lim_{n\to\infty} d(x_n, Tx_n) \le \lim_{n\to\infty} d(x_n, P_T x_n) \le \lim_{n\to\infty} \|x_n - c_n\| = 0$$

for every $T \in \tau$. Since that $\{T_n\}$ satisfies Condition (A), we have $\lim_{n\to\infty} d(x_n, F(\tau)) = 0$. It follows from (B3), there is subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $\{p_k\} \subset F(\tau)$ such that

$$\|x_{n_k} - p_k\| < \frac{1}{2^k} \tag{3.3}$$

for all k. From (3.1), we obtain

$$\begin{aligned}\|x_{n_{k+1}} - p\| &\le \|x_{n_{k+1}-1} - p\| \\ &\le \|x_{n_{k+1}-2} - p\| \\ &\vdots \\ &\le \|x_{n_k} - p\|\end{aligned}$$

for all $p \in F(\tau)$. This implies that

$$\|x_{n_{k+1}} - p_k\| \le \|x_{n_k} - p_k\| < \frac{1}{2^k}. \tag{3.4}$$

Next, we show that $\{p_k\}$ is a Cauchy sequence in D. From (3.3) and (3.4), we have

$$\begin{aligned}\|p_{k+1} - p_k\| &\le \|p_{k+1} - x_{n_{k+1}}\| + \|x_{n_{k+1}} - p_k\| \\ &< \frac{1}{2^{k-1}}.\end{aligned} \tag{3.5}$$

This implies that $\{p_k\}$ is a Cauchy sequence in D and thus converges to $q \in D$. Since P_T is nonexpansive for every $T \in \tau$,

$$d(p_k, Tq) \le d(p_k, P_T q) \le H(P_T p_k, P_T q) \le \|p_k - q\| \tag{3.6}$$

for every $T \in \tau$. It follows that $d(q, Tq) = 0$ for every $T \in \tau$ and thus $q \in F(\tau)$. It implies by (3.3) that $\{x_{n_k}\}$ converges strongly to q. Since $\lim_{n\to\infty} \|x_n - q\|$ exists, it follows that $\{x_n\}$ converges strongly to q. This completes the proof. □

We know that if T is a quasi-nonexpansive multi-valued mapping, then $F(T)$ is closed. So we have the following result:

Corollary 3.4. *Let D be a closed and convex subset of a uniformly convex Banach space E. Let $\{T_n\}$ and τ be two families of nonexpansive multi-valued mappings from D into $P(D)$ with $F(\tau) = \cap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $(0,1)$ such that $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$. Let $\{x_n\}$ be generated by (1.2). Assume that for each $n \in \mathbb{N}$, $H(P_{T_n}x, P_{T_n}p) \le \|x - p\|$, $\forall x \in D$, $p \in F(\tau)$ and the best approximation operator P_T is nonexpansive for every $T \in \tau$.*

If $\{T_n\}$ satisfies the SC-condition and Condition (A), then $\{x_n\}$ converges strongly to an element in $F(\tau)$.

4 Application

Let E be a Banach space and D a nonempty, closed and convex subset of E. Let $\{T_i\}_{i=0}^N$ be a family of nonexpansive multi-valued mappings of D into $CB(D)$ and let $\{\beta_{i,n}\} \subset [0,1]$ be such that $\sum_{i=0}^N \beta_{i,n} = 1$ for all $n \in \mathbb{N}$. We define the mapping $S_n : D \to 2^D$ as follows:

$$S_n = \sum_{i=0}^{N} \beta_{i,n} P_{T_i}, \tag{4.1}$$

where $T_0 = I$ the identity mapping. We also show that the mapping S_n defined by (4.1) satisfies the condition imposed on our main theorem.

Lemma 4.1. *Let D be a closed and convex subset of a uniformly convex Banach space E. Let $\{T_i\}_{i=1}^N$ be a family of nonexpansive multi-valued mappings of D into $P(D)$ and let $\{\beta_{i,n}\}_{i=0}^N$ be sequences in $(0,1)$ such that $0 < \liminf_{n\to\infty} \beta_{i,n} \leq \limsup_{n\to\infty} \beta_{i,n} < 1$ for all $i \in \{0,1,...,N\}$ and $\sum_{i=0}^N \beta_{i,n} = 1$ for all $n \in \mathbb{N}$. For all $n \in \mathbb{N}$, let S_n be the mapping defined by (4.1). Assume that the best approximation operator P_{T_i} is nonexpansive for all $i \in \{1,2,...,N\}$. Then the followings hold:*

(1) $\cap_{n=1}^{\infty} F(S_n) = \cap_{i=0}^{N} F(T_i)$;

(2) $\{S_n\}$ satisfies the SC-condition;

(3) for each $n \in \mathbb{N}$, $H(P_{S_n}x, P_{S_n}p) \leq \|x-p\|$ for all $x \in D$ and $p \in \cap_{i=0}^N F(T_i)$.

Proof. (1) It is easy to see that $\cap_{i=0}^N F(T_i) \subset \cap_{n=1}^{\infty} F(S_n)$. Next, we show that $\cap_{n=1}^{\infty} F(S_n) \subset \cap_{i=1}^N F(T_i)$. Let $p \in \cap_{n=1}^{\infty} F(S_n)$ and $x^* \in \cap_{i=0}^N F(T_i)$. Then there exists $z_i \in P_{T_i}p$ such that $p = \beta_{0,n}p + \sum_{i=1}^N \beta_{i,n} z_i$ for all $n \in \mathbb{N}$. From Lemma 2.10, there exists a continuous, strictly increasing and convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that

$$\begin{aligned}
\|p - x^*\|^2 &= \|\beta_{0,n}(p - x^*) + \sum_{i=1}^{N} \beta_{i,n}(z_i - x^*)\|^2 \\
&\leq \beta_{0,n}\|p - x^*\|^2 + \sum_{i=1}^{N} \beta_{i,n}\|z_i - x^*\|^2 - \frac{\beta_{0,n}}{N}\big(\sum_{i=1}^{N} \beta_{i,n} g(\|z_i - p\|)\big) \\
&= \beta_{0,n}\|p - x^*\|^2 + \sum_{i=1}^{N} \beta_{i,n} d(z_i, P_{T_i}x^*)^2 - \frac{\beta_{0,n}}{N}\big(\sum_{i=1}^{N} \beta_{i,n} g(\|z_i - p\|)\big) \\
&\leq \beta_{0,n}\|p - x^*\|^2 + \sum_{i=1}^{N} \beta_{i,n} H(P_{T_i}p, P_{T_i}x^*)^2 - \frac{\beta_{0,n}}{N}\big(\sum_{i=1}^{N} \beta_{i,n} g(\|z_i - p\|)\big) \\
&\leq \|p - x^*\|^2 - \frac{\beta_{0,n}}{N}\big(\sum_{i=1}^{N} \beta_{i,n} g(\|z_i - p\|)\big).
\end{aligned}$$

By the properties of g, we can conclude that

$$z_i = p, \quad \forall i = 1, 2, ..., N.$$

Hence $p \in \cap_{i=1}^N F(T_i)$. This completes the proof.

(2) Let $\{v_n\} \subset D$ be a bounded sequence and $a_n \in S_n v_n$ be such that $\lim_{n\to\infty} \|a_n - v_n\| = 0$. Then there exists $z_{i,n} \in P_{T_i} v_n,\ i = 1, 2, ..., N$ such that $a_n = \beta_{0,n} v_n + \sum_{i=1}^N \beta_{i,n} z_{i,n}$. Since $\{v_n\}$ and $\{z_{i,n}\}$ are bounded, by Lemma 2.10, there exists a continuous, strictly increasing and convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that

$$\begin{aligned}
\|a_n - p\|^2 &= \|\beta_{0,n}(v_n - p) + \sum_{i=1}^N \beta_{i,n}(z_{i,n} - p)\|^2 \\
&\le \beta_{0,n}\|v_n - p\|^2 + \sum_{i=1}^N \beta_{i,n}\|z_{i,n} - p\|^2 - \frac{\beta_{0,n}}{N}\big(\sum_{i=1}^N \beta_{i,n} g(\|z_{i,n} - v_n\|)\big) \\
&= \beta_{0,n}\|v_n - p\|^2 + \sum_{i=1}^N \beta_{i,n} d(z_{i,n}, P_{T_i}p)^2 - \frac{\beta_{0,n}}{N}\big(\sum_{i=1}^N \beta_{i,n} g(\|z_{i,n} - v_n\|)\big) \\
&\le \beta_{0,n}\|v_n - p\|^2 + \sum_{i=1}^N \beta_{i,n} H(P_{T_i}v_n, P_{T_i}p)^2 - \frac{\beta_{0,n}}{N}\big(\sum_{i=1}^N \beta_{i,n} g(\|z_{i,n} - v_n\|)\big) \\
&\le \|v_n - p\|^2 - \frac{\beta_{0,n}}{N}\big(\sum_{i=1}^N \beta_{i,n} g(\|z_{i,n} - v_n\|)\big), \quad \forall p \in \cap_{i=1}^N F(T_i).
\end{aligned}$$

This implies that

$$\frac{\beta_{0,n}}{N}\big(\sum_{i=1}^N \beta_{i,n} g(\|z_{i,n} - v_n\|)\big) \le \|v_n - a_n\|\big(\|v_n - p\| + \|a_n - p\|\big), \quad \forall p \in \cap_{i=1}^N F(T_i).$$

By assumptions, we get $\lim_{n\to\infty} g(\|z_{i,n} - v_n\|) = 0$ for all $i = 1, 2, ..., N$. By the properties of g, we can conclude that

$$\lim_{n\to\infty} \|z_{i,n} - v_n\| = 0, \quad \forall i = 1, 2, ..., N.$$

Hence $\{S_n\}$ satisfies the SC-condition.

(3) For $p \in \cap_{i=1}^N F(T_i)$, we know that $P_{T_i}p = \{p\}$ for every $i \in \{1, 2, ..., N\}$. Let $n \in \mathbb{N}$. Then for each $x \in D$ and $a_n \in P_{S_n}x$, there exists $z_{i,n} \in P_{T_i}x$ such that $a_n = \beta_{0,n}x + \sum_{i=1}^N \beta_{i,n} z_{i,n}$. It follows that

$$\begin{aligned}
\|a_n - p\| &\le \beta_{0,n}\|x-p\| + \sum_{i=1}^{N}\beta_{i,n}\|z_{i,n}-p\| \\
&= \beta_{0,n}\|x-p\| + \sum_{i=1}^{N}\beta_{i,n} d(z_{i,n}, P_{T_i}p) \\
&\le \beta_{0,n}\|x-p\| + \sum_{i=1}^{N}\beta_{i,n} H(P_{T_i}x, P_{T_i}p) \\
&\le \|x-p\|. \qquad (4.2)
\end{aligned}$$

This implies that $d(P_{S_n}x, p) \le \|a_n - p\| \le \|x-p\|$. It follows from (4.2) that $\sup_{a_n \in P_{S_n}x} d(a_n, P_{S_n}p) \le \|x-p\|$. Since $P_{S_n}p = \{p\}$, $\sup_{p \in P_{S_n}p} d(P_{S_n}x, p) \le \|x-p\|$. This shows that $H(P_{S_n}x, P_{S_n}p) \le \|x-p\|$ for all $x \in D$ and $p \in \cap_{i=1}^N F(T_i)$. This completes the proof. □

Lemma 4.2. *Let D be a closed and convex subset of a Banach space E. Let $\{T_i\}_{i=1}^N$ be a family of nonexpansive multi-valued mappings of D into $P(D)$ and, let $\{\beta_{i,n}\}_{i=0}^N$ be sequences in $(0,1)$ such that $0 < \liminf_{n\to\infty}\beta_{i,n} \le \limsup_{n\to\infty}\beta_{i,n} < 1$ for all $i \in \{0,1,...,N\}$ and $\sum_{i=0}^N \beta_{i,n} = 1$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let S_n be the mapping defined by (4.1). Assume that there exists $i_0 \in \{1,2,...,N\}$ such that $F(T_{i_0}) = \cap_{i=1}^N F(T_i) \neq \emptyset$ and $I - T_{i_0}$ is closed. Then $\{S_n\}$ satisfies Condition (A).*

Proof. Since $I - T_{i_0}$ is closed, it follows by Lemma 2.8 that T_{i_0} satisfies Condition I. Then there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for $r \in (0,\infty)$ such that

$$d(x, T_{i_0}x) \ge f(d(x, F(T_{i_0}))) = f(d(x, \cap_{i=1}^N F(T_i))$$

for all $x \in D$. This completes the proof. □

Using Lemma 4.1, we obtain the following:

Corollary 4.3. *Let D be a closed and convex subset of a uniformly convex Banach space E which satisfies Opial's condition. Let $\{T_i\}_{i=1}^N$ be a family of nonexpansive multi-valued mappings of D into $P(D)$ with $\cap_{i=1}^N F(T_i) \neq \emptyset$ and, let $\{\beta_{i,n}\}_{i=0}^N$ be sequences in $(0,1)$ such that $0 < \liminf_{n\to\infty}\beta_{i,n} \le \limsup_{n\to\infty}\beta_{i,n} < 1$ for all $i \in \{0,1,...,N\}$ and $\sum_{i=0}^N \beta_{i,n} = 1$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let S_n be the mapping defined by (4.1). Assume that the best approximation operator P_{T_i} is nonexpansive for all $i \in \{1,2,...,N\}$. Let $\{x_n\}$ be generated by*

$$x_{n+1} \in \alpha_n x_n + (1-\alpha_n)P_{S_n}x_n, \quad n \ge 1. \qquad (4.3)$$

If $0 < \liminf_{n\to\infty}\alpha_n \le \limsup_{n\to\infty}\alpha_n < 1$, then $\{x_n\}$ converges weakly to an element in $\cap_{i=1}^N F(T_i)$.

Proof. Putting $T_n = S_n$ for all $n \geq 1$ in the Theorem 3.1, we obtain the desired result. $\square$

Now, using Lemma 4.1 and Lemma 4.2, we obtain the following:

Corollary 4.4. *Let D be a closed and convex subset of a uniformly convex Banach space E. Let $\{T_i\}_{i=1}^N$ be a family of nonexpansive multi-valued mappings of D into $P(D)$ with $\cap_{i=1}^N F(T_i) \neq \emptyset$ and, let $\{\beta_{i,n}\}_{i=0}^N$ be sequences in $(0,1)$ such that $0 < \liminf_{n\to\infty} \beta_{i,n} \leq \limsup_{n\to\infty} \beta_{i,n} < 1$ for all $i \in \{0,1,...,N\}$ and $\sum_{i=0}^N \beta_{i,n} = 1$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let S_n be the mapping defined by (4.1). Assume that the best approximation operator P_{T_i} is nonexpansive for all $i \in \{1,2,...,N\}$ and there exists $i_0 \in \{1,2,...,N\}$ such that $F(T_{i_0}) = \cap_{i=1}^N F(T_i)$ with $I - T_{i_0}$ is closed. Let $\{x_n\}$ be generated by*

$$x_{n+1} \in \alpha_n x_n + (1-\alpha_n) P_{S_n} x_n, \quad n \geq 1. \tag{4.4}$$

If $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$, then $\{x_n\}$ converges strongly to an element in $\cap_{i=1}^N F(T_i)$.

Proof. Putting $T_n = S_n$ for all $n \geq 1$ in the Theorem 3.3, we obtain the desired result. $\square$

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