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Strong Convergence Theorems for Fixed Points of Nonexpansive Semigroups

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Abstract : Let H be a real Hilbert space and $\mathcal{F} = \{T(t) : t \geq 0\}$ be a strongly continuous semigroup of nonexpansive mappings on H with a common fixed point. Let $f : H \to H$ be an α-contraction and $A : H \to H$ be a strongly positive linear bounded self-adjoint operator with the coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. The implicit iterative scheme is given as follows:

$$x_n = \alpha_n \gamma f(x_n) + (I - \alpha_n A)T(t_n)x_n, \quad \forall n \geq 1.$$

It is proved that the sequence $\{x_n\}$ generated in the above iterative process converges strongly to a common fixed point $p \in \bigcap_{t\geq 0} F(T(t))$, where $F(T(t))$ denotes the fixed point of the nonexpansive mapping $T(t)$. The point p also solves the variational inequality $\langle(\gamma f - A)p, x - p\rangle \leq 0$, $\forall x \in \bigcap_{t\geq 0} F(T(t))$.

Keywords : Nonexpansive mapping; Common fixed point; Variational inequality; Iterative method.
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1 Introduction and Preliminaries

Throughout this paper, we denote by $\mathbb{R}^+$ the set of nonnegative real numbers. Let H be a real Hilbert space, C be a nonempty closed convex subset of H and T

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be a nonlinear mapping. Recall that $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

A point $x \in C$ is said to be a *fixed point* of T provided $Tx = x$. Denote by $F(T)$ the set of all fixed points of T, that is, $F(T) = \{x \in C : Tx = x\}$.

Let $\mathcal{F} = \{T(t) : t \geq 0\}$ be a strongly continuous semigroups of nonexpansive mappings on a closed convex subset C of a Hilbert space H, i.e.,

(a) for each $t \in \mathbb{R}^+$, $T(t)$ is a nonexpansive mapping on C;

(b) $T(0)x = x$ for all $x \in C$;

(c) $T(s + t) = T(s)T(t)$ for all $s, t \geq 0$;

(d) for each $x \in H$, the mapping $T(\cdot)s$ from $\mathbb{R}^+$ into C is continuous.

We denote by F the set of common fixed points of $\mathcal{F}$, that is,

$$F := \{x \in C : T(t)x = x, \, t > 0\} = \bigcap_{t>0} F(T(t)).$$

We know that F is nonempty if C is bounded (see [1]). In [2], Shioji and Takahashi proved the following theorem.

Theorem ST. *Let C be a closed convex subset of a Hilbert space H. Let $\{T(t) : t \geq 0\}$ be a strongly continuous semigroup of nonexpansive mappings on C such that $\bigcap_{t\geq 0} F(T(t)) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers satisfying $0 < \alpha_n < 1$, $\lim_{n\to\infty} \alpha_n = 0$, $t_n \geq 0$ and $\lim_{n\to\infty} t_n = \infty$. Fix $u \in C$ and define a sequence $\{x_n\}$ in C by*

$$x_n = \alpha_n u + (1 - \alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds, \quad \forall n \geq 1.$$

Then $\{x_n\}$ converges strongly to the element of F nearest to u.

Recently, Suzuki [3] improved the results of Shinoji and Takahashi [2] and proved the following theorem.

Theorem S. *Let C be a closed convex subset of a Hilbert space H. Let $\{T(t) : t \geq 0\}$ be a strongly continuous semigroup of nonexpansive mappings on C such that $\bigcap_{t\geq 0} F(T(t)) \neq \emptyset$. Let α_n and $\{t_n\}$ be sequences of real numbers satisfying $0 < \alpha_n < 1$, $t_n > 0$ and $\lim_{n\to\infty} t_n = \lim_{n\to\infty} \frac{\alpha_n}{t_n} = 0$. Fix $u \in C$ and define a sequence $\{x_n\}$ in C by*

$$x_n = \alpha_n u + (1 - \alpha_n)T(t_n)x_n, \quad \forall n \geq 1.$$

Then $\{x_n\}$ converges strongly to the element of F nearest to u.

Recall that an operator A is said to be *strongly positive* on H if there exists a constant $\bar{\gamma} > 0$ such that

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2, \quad \forall x \in H.$$

Recall also that a self mapping f is said to be an *α-contraction* on H if there exists a constant $\alpha \in (0,1)$ such that

$$\|f(x) - f(y)\| \leq \alpha\|x - y\|, \quad \forall x, y \in H.$$

Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems (see [4–8] and the references therein). A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping T on a real Hilbert space H:

$$\min_{x\in F(T)} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle, \tag{1.1}$$

where A is a linear bounded self-adjoint operator and b is a given point in H.

In [3], it is proved that the sequence $\{x_n\}$ defined by the following iterative method:

$$x_0 \in H, \quad x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n b, \quad \forall n \geq 0,$$

converges strongly to the unique solution of the minimization problem (1.1) provided the sequence $\{\alpha_n\}$ satisfies certain conditions.

Recently, Marino and Xu [5] studied the following continuous scheme:

$$x_t = t\gamma f(x_t) + (I - tA)Tx_t,$$

where f is an α-contraction on H, A is a strongly positive linear bounded self-adjoint operator and $\gamma > 0$ is a constant. They showed that $\{x_t\}$ converges strongly to a fixed point $\bar{x}$ of T. Also, in [5], they introduced a general iterative scheme by the viscosity approximation method which first was considered by Moudafi [9]:

$$x_0 \in H, \quad x_{n+1} = \alpha_n\gamma f(x_n) + (I - \alpha_n A)Tx_n, \quad \forall n \geq 0$$

and proved that the sequence $\{x_n\}$ converges strongly to the unique solution of the variational inequality:

$$\langle (A - \gamma f)x^*, x - x^*\rangle \geq 0, \quad \forall x \in F(T),$$

which is the optimality condition for the minimization problem:

$$\min_{x\in F(T)} \frac{1}{2}\langle Ax, x\rangle - h(x),$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for any $x \in H$).

In this paper, motivated by the recent work announced in [2, 3, 5, 10–14], we consider the following implicit iterative scheme:

$$x_n = \alpha_n \gamma f(x_n) + (I - \alpha_n A)T(t_n)x_n, \quad \forall n \geq 1,$$

where $\gamma > 0$ is a constant, f is an α-contraction on H, A is a strongly positive linear bounded self-adjoint operator on H and prove that the sequence $\{x_n\}$ generated in the above iterative process converge strongly to a common fixed point $p \in F$. Also, we show that the point p solves the variational inequality:

$$\langle (\gamma f - A)p, x - p\rangle \leq 0, \quad \forall x \in F.$$

The results presented in this paper mainly improve and extend the corresponding results announced in Shioji and Takahashi [2], Suzuki [3] and Xu [14].

In order to prove our main result, we need the following concepts and lemmas.

Recall that a space X satisfies *Opial's condition* ([15]) if, for each sequence $\{x_n\}$ in X which converges weakly to point $x \in X$,

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|, \quad \forall y \in X\ (y \neq x).$$

Lemma 1.1 ([5]). *Assume that A is a strongly positive linear bounded self-adjoint operator on a Hilbert space H with the coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.*

Lemma 1.2. *Let H be a Hilbert space, C be a closed convex subset of H, $f : H \to H$ be an α-contraction and A be a strongly positive linear bounded operator with the coefficient $\bar{\gamma} > 0$. Then, for any $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$, we see that*

$$\langle x - y, (A - \gamma f)x - (A - \gamma f)y\rangle \geq (\bar{\gamma} - \gamma\alpha)\|x - y\|^2, \quad \forall x, y \in H. \tag{1.2}$$

That is, $A - \gamma f$ is strongly monotone with the coefficient $\bar{\gamma} - \alpha\gamma$.

Proof. From the definition of strongly positive linear bounded operators, we see that

$$\langle x - y, A(x - y)\rangle \geq \bar{\gamma}\|x - y\|^2.$$

On the other hand, we have

$$\langle x - y, \gamma f x - \gamma f y\rangle \leq \gamma\alpha\|x - y\|^2.$$

It follows that

$$\begin{aligned}\langle x - y, (A - \gamma f)x - (A - \gamma f)y\rangle &= \langle x - y, A(x - y)\rangle - \langle x - y, \gamma f x - \gamma f y\rangle \\ &\geq (\bar{\gamma} - \gamma\alpha)\|x - y\|^2, \quad \forall x, y \in H.\end{aligned}$$

This completes the proof. □

Remark 1.3. Taking $\gamma = 1$ and $A = I$ (: the identity mapping), we have the following inequality:

$$\langle x - y, (I - f)x - (I - f)y\rangle \geq (1 - \alpha)\|x - y\|^2, \quad \forall x, y \in H.$$

Furthermore, if f is a nonexpansive mapping in (1.2), then we have

$$\langle x - y, (I - f)x - (I - f)y\rangle \geq 0, \quad \forall x, y \in H. \tag{1.3}$$

2 Main Results

Now, we are ready to give our main results.

Theorem 2.1. *Let H be a real Hilbert space and $\{T(t) : t \geq 0\}$ be a strongly continuous semigroup of nonexpansive mappings on H such that $F \neq \emptyset$. Let $f : H \to H$ be an α-contraction and $A : H \to H$ a strongly positive linear bounded self-adjoint operator with the coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers satisfying $0 < \alpha_n < 1$, $t_n > 0$ and $\lim_{n\to\infty} t_n = \lim_{n\to\infty} \frac{\alpha_n}{t_n} = 0$. Define a sequence $\{x_n\}$ in the following manner:*

$$x_n = \alpha_n \gamma f(x_n) + (I - \alpha_n A)T(t_n)x_n, \quad \forall n \geq 1. \tag{2.1}$$

Then $\{x_n\}$ converges strongly to $p \in F$ which solves the following variational inequality:

$$\langle (A - \gamma f)p, p - x\rangle \leq 0, \quad \forall x \in F. \tag{2.2}$$

Proof. First, we show that the fixed point equation (2.1) is well defined. For any $n \geq 1$, define a mapping T_n as follows:

$$T_n x = \alpha_n \gamma f(x) + (I - \alpha_n A)T(t_n)x.$$

It follows that

$$\begin{aligned}\|T_n x - T_n y\| &= \|\alpha_n \gamma (f(x) - f(y)) + (I - \alpha_n A)(T(t_n)x - T(t_n)y)\| \\ &\leq \alpha_n \alpha \gamma \|x - y\| + (1 - \alpha_n \bar{\gamma})\|x - y\| \\ &= (1 - \alpha_n(\bar{\gamma} - \alpha\gamma))\|x - y\|, \quad \forall x, y \in H.\end{aligned}$$

Hence T_n has a unique fixed point x_n, which uniquely solves the fixed point equation

$$x_n = \alpha_n \gamma f(x_n) + (I - \alpha_n A)T(t_n)x_n.$$

The uniqueness of the solution of the variational inequality (2.2) is a consequence of the strong monotonicity of $A - \gamma f$. Suppose that $p, q \in F$ are solutions to (2.2). It follows that

$$\langle (A - \gamma f)p, p - q\rangle \leq 0 \tag{2.3}$$

and

$$\langle (A - \gamma f)q, q - p\rangle \leq 0. \tag{2.4}$$

Adding up (2.3) and (2.4), one obtains that

$$\langle (A - \gamma f)p - (A - \gamma f)q, p - q\rangle \leq 0, \quad \forall x \in F.$$

From Lemma 1.2, one sees that $p = q$. Next, we use p to denote the unique solution of the variational inequality (2.2). Observing $\lim_{n\to\infty} t_n = \lim_{n\to\infty} \frac{\alpha_n}{t_n} = 0$, we may assume, without loss of generality, that $\alpha_n < \|A\|^{-1}$ for all $n \geq 1$. From Lemma 1.1, we know that, if $0 < \alpha_n \leq \|A\|^{-1}$, then $\|I - \alpha_n A\| \leq 1 - \alpha_n \bar{\gamma}$.

Next, we show that $\{x_n\}$ is bounded. Fixing $x \in F$, we have

$$\begin{aligned}\|x_n - x\|^2 &= \langle \alpha_n(\gamma f(x_n) - Ax) + (I - \alpha_n A)(T(t_n)x_n - x), x_n - x\rangle \\ &= \alpha_n \gamma \langle f(x_n) - f(x), x_n - x\rangle + \alpha_n \langle \gamma f(x) - Ax, x_n - x\rangle \\ &\quad + \langle (I - \alpha_n A)(T(t_n)x_n - x), x_n - x\rangle,\end{aligned}$$

from which it follows that

$$\|x_n - x\|^2 \leq \frac{1}{\bar{\gamma} - \gamma\alpha}\langle \gamma f(x) - Ax, x_n - x\rangle. \tag{2.5}$$

That is,

$$\|x_n - x\| \leq \frac{1}{\bar{\gamma} - \gamma\alpha}\|\gamma f(x) - Ax\|.$$

This implies that $\{x_n\}$ is bounded. Let $\{x_{n_i}\}$ be an arbitrary subsequence of $\{x_n\}$. Then there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to a point p.

Next, we show that $p \in F$. In fact, put $z_j = x_{n_{i_j}}$, $\gamma_j = \alpha_{n_{i_j}}$ and $s_j = t_{n_{i_j}}$ for all $j \geq 1$. Fix $t > 0$. Noticing that

$$\begin{aligned}\|z_j - T(t)p\| &\leq \sum_{k=0}^{[\frac{t}{s_j}]-1} \|T((k+1)s_j)z_j - T(ks_j)z_j\| \\ &\quad + \left\|T\left(\left[\frac{t}{s_j}\right]s_j\right)z_j - T\left(\left[\frac{t}{s_j}\right]s_j\right)p\right\| + \left\|T\left(\left[\frac{t}{s_j}\right]s_j\right)p - T(t)p\right\| \\ &\leq \left[\frac{t}{s_j}\right]\|T(s_j)z_j - z_j\| + \|z_j - p\| + \left\|T\left(t - \left[\frac{t}{s_j}\right]s_j\right)p - p\right\| \\ &\leq \frac{\gamma_j}{s_j}t\|AT(s_j)z_j - \gamma f(z_j)\| + \|z_j - p\| \\ &\quad + \max\{\|T(s)p - p\| : 0 \leq s \leq s_j\}, \quad \forall j \geq 1,\end{aligned}$$

we have

$$\liminf_{j\to\infty} \|z_j - T(t)p\| \leq \liminf_{j\to\infty} \|z_j - p\|.$$

From Opial's condition, we have $T(t)p = p$. It follows that $p \in F$. In the inequality (2.5), replacing p with x, we have

$$\|z_j - p\|^2 \leq \frac{1}{\bar{\gamma} - \alpha\gamma}\langle \gamma f(p) - Ap, z_j - p\rangle. \tag{2.6}$$

Taking the limit as $j \to \infty$ in (2.6), we obtain that $\lim_{j\to\infty} \|z_j - p\| = 0$. Since the subsequence $\{x_{n_i}\}$ is arbitrary, it follows that $\{x_n\}$ converges strongly to p.

Finally, we prove that $p \in F$ is a solution of the variational inequality (2.2). From (2.1), we see that

$$(A - \gamma f)x_n = -\frac{1}{\alpha_n}(I - \alpha_n A)(x_n - T(t_n)x_n).$$

In view of (1.3), we see that

$$\begin{aligned}\langle (A-\gamma f)x_n, x_n - x\rangle &= -\frac{1}{\alpha_n}\langle (I-\alpha_n A)(x_n - T(t_n)x_n), x_n - x\rangle \\ &= -\frac{1}{\alpha_n}\langle (I-T(t_n))x_n - (I-T(t_n))x, x_n - x\rangle \\ &\quad + \langle A(I-T(t_n))x_n, x_n - x\rangle \\ &\le \langle A(I-T(t_n))x_n, x_n - x\rangle \\ &= \langle A(\alpha_n \gamma f(x_n) - \alpha_n AT(t_n)x_n), x_n - x\rangle \\ &= \alpha_n \langle A(\gamma f(x_n) - AT(t_n)x_n), x_n - x\rangle.\end{aligned}$$

It follows that

$$\langle (A-\gamma f)p, p - x\rangle \le 0, \quad \forall x \in F.$$

That is, $p \in F$ is the unique solution to the variational inequality (2.2). This completes the proof. □

Taking $\gamma = 1$ and $A = I$ in Theorem 2.1, we have the following result.

Corollary 2.2. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $\{T(t) : t \ge 0\}$ be a strongly continuous semigroup of nonexpansive mappings of C into itself such that $F \neq \emptyset$ and $f : C \to C$ be an α-contraction. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers satisfying $0 < \alpha_n < 1$, $t_n > 0$ and $\lim_{n\to\infty} t_n = \lim_{n\to\infty} \frac{\alpha_n}{t_n} = 0$. Define a sequence $\{x_n\}$ in the following manner:*

$$x_n = \alpha_n f(x_n) + (1-\alpha_n)T(t_n)x_n, \quad \forall n \ge 1. \tag{2.7}$$

Then $\{x_n\}$ converges strongly to $p \in F$ which solves the following variational inequality:

$$\langle (f-I)p, x-p\rangle \le 0, \quad \forall x \in F. \tag{2.8}$$

Remark 2.3. If $f(x) = u \in C$, a fixed point, for all $x \in C$, then Corollary 2.2 is reduced to Suzuki's results [3]. Corollary 2.2 also can be viewed as an improvement of the corresponding results in Shioji and Takahashi [2].

Remark 2.4. It is of interest to improve Theorem 2.1 to some Banach space.

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