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Strong Convergence Theorems for a Finite Family of Uniformly L-Lipschitzian Mappings in a Banach Space

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Abstract : The purpose of this work is to prove strong convergence theorems for a finite family of uniformly L-Lipschitzian mappings in a Banach space.

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1 Introduction and Preliminaries

Throughout this paper, we assume that E is a real Banach space, E^* is the dual space of E, K is a nonempty closed convex subset of E, T is a self-mapping of K and $J : E \to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2, \|f\| = \|x\|\}, \forall x \in E,$$

where $\langle \cdot, \cdot\rangle$ denotes the duality pairing between E and E^*. The single-valued normalized duality mapping is denoted by j. Recall that a mapping T is said to be *uniformly L-Lipschitzian* if there exists $L > 0$ such that, for any $x, y \in K, \|T^n x - T^n y\| \leq L \|x - y\|, \forall n \geq 1$. A mapping T is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that for any given $x, y \in K, \|T^n x - T^n y\| \leq k_n \|x - y\|, \forall n \geq 1$. A mapping T is said to be *asymptotically pseudo-contractive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ with

$k_n \to 1$ such that for any $x, y \in K$, there exists $j(x-y) \in J(x-y)$ such that $\langle T^n x - T^n y, j(x-y)\rangle \leq k_n \|x-y\|^2, \forall n \geq 1$.

Let C be a nonempty closed convex subset of a real Hilbert space H, and $T: C \to C$ be a mapping. T is said to be a *k-strict pseudo-contraction* if there exists a $k \in [0,1)$ such that $\|Tx - Ty\|^2 \leq \|x-y\|^2 + k\|(I-T)x - (I-T)y\|^2, \forall x, y \in C$. A mapping $T: C \to C$ is said to be an *asymptotically k-strict pseudo-contraction mapping* with sequence $\{k_n\}$ if there exists a constant $k \in [0,1)$ and a sequence $\{k_n\}$ in $[1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that $\|T^n x - T^n y\|^2 \leq k_n \|x-y\|^2 + k\|x - T^n x - (y - T^n y)\|^2$ for all $x, y \in C$ and $\forall n \geq 1$.

Remark 1.1.

(1) It is easy to see that if T is an asymptotically nonexpansive mapping, then T is a uniformly L-Lipschitzian mapping, where $L = \sup_{n\geq 1} k_n$.

(2) (Kim and Xu [1]) Every asymptotically k-strict pseudo-contractive mapping with sequence $\{k_n\}$ is a uniformly L-Lipschitzian mapping with $L = \sup\{\frac{k+\sqrt{(1-k)k_n}}{1+k} : \forall n \geq 1\}$.

The normal Mann's iterative process [2] generates a sequence $\{x_n\}$ in the following manner: for any $x_1 \in K$,

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \forall n \geq 1, \tag{1.1}$$

where $\{\alpha_n\}$ is sequence in $(0,1)$. If T is a nonexpansive mapping with a fixed point and the control sequence $\{\alpha_n\}$ is chosen so that $\Sigma_{n=1}^{\infty}\alpha_n(1-\alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by the normal Mann's iterative process (1.1) converges weakly to a fixed point of T(this is also valid in a uniformly convex Banach space with the Fréchet differentiable norm [3]). In an infinite-dimensional Hilbert space, the normal Mann's iteration algorithm has only weak convergence, in general, even for nonexpansive mapping [4].

Marino and Xu [5] studied the k-strict pseudo-contractions and gave a weak convergence theorem in the framework of Hilbert spaces. That is, they extended the results of Reich [3] from nonexpansive mapping to k-strict pseudo-contractions.

In order to get a strong convergence result, one has to modify the normal Mann iteration algorithm. Some attempts have been made and several important results have been reported (see,e.g., [5–9]). Recently, Acedo and Xu [10] studied the following cyclic algorithm. Let $x_0 \in C$ and $\{\alpha_n\}$ be a sequence in $(0,1)$, $\{x_n\}$ is generated in the following way:

$$\begin{aligned} x_1 &= \alpha_0 x_0 + (1-\alpha_0)T_0 x_0, \\ x_2 &= \alpha_1 x_1 + (1-\alpha_1)T_1 x_1, \\ &\vdots \\ x_N &= \alpha_{N-1} x_{N-1} + (1-\alpha_{N-1})T_{N-1} x_{N-1}, \\ x_{N+1} &= \alpha_N x_N + (1-\alpha_N)T_0 x_N, \\ &\vdots \end{aligned}$$

In general, x_{n+1} is defined by

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)T_{[n]}x_n, \forall n \geq 0, \tag{1.2}$$

where $T_{[n]} = T_i$, with $i = n(modN), 0 \leq i \leq N-1$. They also proved weak and strong convergence theorems for k-strictly pseudo-contractive mappings in Hilbert spaces by cyclic algorithm (1.2).

Very recently, Qin and Cho [11] studied the following cyclic algorithm. Let $x_0 \in C$ and $\{\alpha_n\}$ be a sequence in $(0,1)$, $\{x_n\}$ is generated by the following way:

$$\begin{aligned}
x_1 &= \alpha_0 x_0 + (1-\alpha_0)T_1 x_0, \\
x_2 &= \alpha_1 x_1 + (1-\alpha_1)T_2 x_1, \\
&\vdots \\
x_N &= \alpha_{N-1} x_{N-1} + (1-\alpha_{N-1})T_N x_{N-1}, \\
x_{N+1} &= \alpha_N x_N + (1-\alpha_N)T_1^2 x_N, \\
&\vdots \\
x_{2N} &= \alpha_{2N-1} x_{2N-1} + (1-\alpha_{2N-1})T_N^2 x_{2N-1}, \\
x_{2N+1} &= \alpha_{2N} x_{2N} + (1-\alpha_{2N})T_1^3 x_{2N}, \\
&\vdots
\end{aligned}$$

We can rewrite the above table in the following compact form:

$$x_n = \alpha_{n-1} x_{n-1} + (1-\alpha_{n-1})T_{i(n)}^{h(n)} x_{n-1}, \forall n \geq 1, \tag{1.3}$$

where $i = i(n) \in \{1, 2, \ldots, N\}, h = h(n) \geq 1$ is a positive integer and $h(n) \to \infty$ as $n \to \infty$,$n = (h-1)N + i$. They proved weak and strong convergence theorems for asymptotically k-strictly pseudo-contractive mappings in Hilbert spaces by the cyclic algorithm (1.3).

In this paper, motivated and inspired by Acedo and Xu [10], Qin and Cho [11], we introduce the following algorithm for uniformly L-Lipschitzian mapping. Let $x_1 \in K$ and $\{\alpha_n\}, \{\beta_n\}$ be sequences in $(0,1)$. The sequence $\{x_n\}_{n=1}^{\infty}$ is generated in the following way:

$$\begin{aligned}
x_2 &= (1-\alpha_1)x_1 + \alpha_1 T_1 y_1, \\
y_1 &= (1-\beta_1)x_1 + \beta_1 T_1 x_1, \\
x_3 &= (1-\alpha_2)x_2 + \alpha_2 T_2 y_2, \\
y_2 &= (1-\beta_2)x_2 + \beta_2 T_2 x_2, \\
&\vdots \\
x_{N+1} &= (1-\alpha_N)x_N + \alpha_N T_N y_N, \\
y_N &= (1-\beta_N)x_N + \beta_N T_N x_N,
\end{aligned}$$

$$x_{N+2} = (1-\alpha_{N+1})x_{N+1} + \alpha_{N+1}T_1^2 y_{N+1},$$
$$y_{N+1} = (1-\beta_{N+1})x_{N+1} + \beta_{N+1}T_1^2 x_{N+1},$$
$$\vdots$$

We can rewrite the above table in the following compact form:

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_{i(n)}^{h(n)} y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T_{i(n)}^{h(n)} x_n \end{cases} \tag{1.4}$$

where $i = i(n) \in \{1, 2, \ldots, N\}, h = h(n) \geq 1$ is a positive integer and $h(n) \to \infty$ as $n \to \infty$,$n = (h-1)N + i$.

In this paper, algorithm (1.4) in the framework of Banach spaces, we prove two strong convergence theorems for the finite family of uniformly L-Lipschitzian mappings.

In order to prove our main results, we need the following lemmas.

Lemma 1.2 (Chang [12]). *Let E be a real Banach space and $J : E \to 2^{E^*}$ be the normalized duality mapping. Then, for any $x, y \in E$,*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \forall j(x+y) \in J(x+y).$$

Lemma 1.3 (Moore and Nnoli [13]). *Let $\{\theta_n\}$ be a sequence of nonnegative real numbers and $\{\lambda_n\}$ be a real sequence satisfying the following conditions:*

$$0 \leq \lambda_n \leq 1, \sum_{n=0}^{\infty} \lambda_n = \infty.$$

If there exists a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ such that $\theta_{n+1}^2 \leq \theta_n^2 - \lambda_n \phi(\theta_{n+1}) + \sigma_n, \forall n \geq n_0$, where n_0 is some nonnegative integer and $\{\sigma_n\}$ is a sequence of nonnegative numbers such that $\sigma_n = \circ(\lambda_n)$, then $\theta_n \to 0$ as $n \to \infty$.

Lemma 1.4 ([14]). *Let $\{a_n\}$ and $\{b_n\}$ be two nonnegative real sequences satisfying the following condition:*

$$a_{n+1} \leq (1+\lambda_n)a_n + b_n, \forall n \geq n_0,$$

where $\{\lambda_n\}$ is a sequence in $(0,1)$ with $\sum_{n=0}^{\infty} \lambda_n < \infty$. If $\sum_{n=0}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

2 Main Results

Theorem 2.1. *Let K be a nonempty closed convex subset of a real Banach space E. Let $N \geq 1$ be an integer,for each $1 \leq i \leq N, T_i : K \to K$ be a uniformly L_i-Lipschitzian for $L_i > 0$ and $L = \max\{L_i : 1 \leq i \leq N\}$. Assume that the* *common fixed point set* $\bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ *and* x^* *be a point in* $\bigcap_{i=1}^{N} F(T_i)$. *Let* $\{k_{h(n)}\} \subset [1, \infty)$ *be a sequence with* $k_{h(n)} \to 1$. *For any* $x_1 \in K$, *let* $\{x_n\}$ *be the sequence generated by the cyclic algorithm* (1.4). *Let* $\{\alpha_n\}$ *be a sequence in* $[0, \frac{1}{2}]$ *and* $\{\beta_n\}$ *be a sequence in* $[0, 1]$ *satisfying the following conditions:*

(a1) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(a2) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$;

(a3) $\sum_{n=1}^{\infty} \beta_n < \infty$;

(a4) $\sum_{n=1}^{\infty} \alpha_n (k_{h(n)} - 1) < \infty$.

If there exists a strict increasing function $\phi : [0, \infty) \to [0, \infty)$ *with* $\phi(0) = 0$ *such that*

$$\langle T_{i(n)}^{h(n)} - x^*, j(x - x^*) \rangle \leq k_{h(n)} \|x - x^*\|^2 - \phi(\|x - x^*\|)$$

for all $j(x - x^*) \in J(x - x^*)$ *and* $x \in K, i = i(n) = 1, 2, \ldots, N$, *then* $\{x_n\}$ *converges strongly to* x^*.

Proof. First, we prove that the sequence $\{x_n\}$ defined by (1.4) is bounded. In fact, it follows from (1.4) and Lemma 1.2 that

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \left\| (1 - \alpha_n)(x_n - x^*) + \alpha_n (T_{i(n)}^{h(n)} y_n - x^*) \right\|^2 \\ &\leq (1 - \alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n \langle T_{i(n)}^{h(n)} y_n - x^*, j(x_{n+1} - x^*) \rangle \\ &\leq (1 - \alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n \{ k_{h(n)} \|x_{n+1} - x^*\|^2 - \phi(\|x_{n+1} - x^*\|) \} \\ &\quad + 2\alpha_n L \|y_n - x_{n+1}\| \|x_{n+1} - x^*\| . \end{aligned} \tag{2.1}$$

Note that

$$\begin{aligned} \|x_{n+1} - y_n\| &= \left\| (1 - \alpha_n)(x_n - y_n) + \alpha_n (T_{i(n)}^{h(n)} y_n - y_n) \right\| \\ &\leq (1 - \alpha_n) \|x_n - y_n\| + \alpha_n \left\| T_{i(n)}^{h(n)} y_n - x^* + x^* - y_n \right\| \\ &\leq (1 - \alpha_n) \|x_n - y_n\| + \alpha_n (1 + L) \|x^* - y_n\| \\ &\leq (1 - \alpha_n) \|x_n - y_n\| + \alpha_n (1 + L)(\|x_n - y_n\| + \|x_n - x^*\|) \\ &= (1 + L\alpha_n) \|x_n - y_n\| + \alpha_n (1 + L) \|x_n - x^*\| \\ &= (1 + L\alpha_n)\beta_n \left\| x_n - T_{i(n)}^{h(n)} x_n \right\| + \alpha_n (1 + L) \|x_n - x^*\| \\ &\leq (1 + L\alpha_n)\beta_n (1 + L) \|x_n - x^*\| + \alpha_n (1 + L) \|x_n - x^*\| \\ &= c_n \|x_n - x^*\| , \end{aligned} \tag{2.2}$$

where $c_n = (1 + L)\{(1 + L\alpha_n)\beta_n + \alpha_n\}$. Substituting (2.2) into (2.1), we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq (1 - \alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n \{ k_{h(n)} \|x_{n+1} - x^*\|^2 - \phi(\|x_{n+1} - x^*\|) \} \\ &\quad + 2\alpha_n L c_n \|x_n - x^*\| \|x_{n+1} - x^*\| \\ &\leq (1 - \alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n \{ k_{h(n)} \|x_{n+1} - x^*\|^2 - \phi(\|x_{n+1} - x^*\|) \} \\ &\quad + \alpha_n L c_n (\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) \end{aligned} \tag{2.3}$$

and hence

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\leq \frac{A_n}{B_n}\|x_n-x^*\|^2 - \frac{2\alpha_n\phi(\|x_{n+1}-x^*\|)}{B_n}\\ &= \left\{1+\frac{2\alpha_n(k_{h(n)}-1)+2Lc_n\alpha_n+\alpha_n^2}{B_n}\right\}\|x_n-x^*\|^2\\ &\quad - \frac{2\alpha_n\phi(\|x_{n+1}-x^*\|)}{B_n} \qquad (2.4)\end{aligned}$$

where $A_n = 1-2\alpha_n+\alpha_n^2+\alpha_nLc_n, B_n = 1-(2\alpha_nk_{h(n)}+\alpha_nLc_n)$.

Since $\alpha_n \to 0$ as $n \to \infty$, there exists a positive integern_0 such that $\frac{1}{2} < B_n \leq 1$ for all $n \geq n_0$. Therefore, it follows from (2.4) that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 \leq \{1+2[2\alpha_n(k_{h(n)}-1)+2Lc_n\alpha_n+\alpha_n^2]\}\|x_n-x^*\|^2\\ -2\alpha_n\phi(\|x_{n+1}-x^*\|), \forall n \geq n_0 \qquad (2.5)\end{aligned}$$

and so

$$\|x_{n+1}-x^*\|^2 \leq \{1+2[2\alpha_n(k_{h(n)}-1)+2Lc_n\alpha_n+\alpha_n^2]\}\|x_n-x^*\|^2, \forall n \geq n_0. \quad (2.6)$$

By the conditions $(a1)-(a3)$, we know that

$$\sum_{n=1}^{\infty} c_n\alpha_n < \infty. \qquad (2.7)$$

It follows from the condition $(a2)$, $(a4)$ and (2.7) that

$$2\sum_{n=1}^{\infty}[2\alpha_n(k_{h(n)}-1)+2Lc_n\alpha_n+\alpha_n^2] < \infty.$$

So, we obtain by Lemma1.4 that $\lim_{n\to\infty}\|x_n-x^*\|$ exists. Therefore, there exists a positive constant M such that $\|x_n-x^*\|^2 \leq M$.

Secondly, we prove that $x_n \to x^*$. Taking $\theta_n = \|x_n-x^*\|, \lambda_n = 2\alpha_n$ and $\sigma_n = 2[2\alpha_n(k_{h(n)}-1)+2Lc_n\alpha_n+\alpha_n^2]M$, by the conditions $(a1)-(a3)$ and Lemma 1.3, we have $\|x_n-x^*\| \to 0$, that is, $x_n \to x^*$ as $n \to \infty$. This completes the proof. □

The following theorem can be obtained from Theorem 2.1 by taking $\beta_n = 0, \forall n$.

Theorem 2.2. *Let K be a nonempty closed convex subset of a real Banach space E. Let $N \geq 1$ be an integer, for each $1 \leq i \leq N, T_i : K \to K$ be a uniformly L_i-Lipschitzian for $L_i > 0$ and $L = \max\{L_i : 1 \leq i \leq N\}$. Assume that the common fixed point set $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and x^* be a point in $\bigcap_{i=1}^N F(T_i)$. Let $\{k_{h(n)}\} \subset [1,\infty)$ be a sequence with $k_{h(n)} \to 1$. For any $x_1 \in K$, let $\{x_n\}$ be the sequence generated by the cyclic algorithm:*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_nT_{i(n)}^{h(n)}x_n$$

Let $\{\alpha_n\}$ be a sequence in $[0,\frac{1}{2}]$ satisfying the following conditions:

(b1) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(b2) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$;

(b3) $\sum_{n=1}^{\infty} \alpha_n (k_{h(n)} - 1) < \infty$.

If there exists a strict increasing function $\phi : [0, \infty) \to [0, \infty)$ *with* $\phi(0) = 0$ *such that*

$$\langle T_{i(n)}^{h(n)} - x^*, j(x - x^*)\rangle \leq k_{h(n)} \|x - x^*\|^2 - \phi(\|x - x^*\|)$$

for all $j(x - x^*) \in J(x - x^*)$ *and* $x \in K, i = 1, 2, \ldots, N$*, then* $\{x_n\}$ *converges strongly to* x^*.

Remark 2.3.

(1) *Theorem 2.1 extends and improves ([15, Theorem 3.10]) for a two-step implicit iteration scheme of two parametric sequences (without error terms) in the setting of an arbitrary real Banach space.*

(2) *Theorem 2.2 provides strong convergence analogue of Theorem 2.1 in [11] for a two-step implicit iteration scheme of one parametric sequence in a real Banach space.*

(3) *For strong convergence of an implicit scheme of one-step to a common fixed point of a finite family of uniformly continuous mappings on a uniformly convex Banach space, we refer the reader to [16].*

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