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Minimum Perimeter Developments of the Platonic Solids

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Abstract : A *development* of a convex polyhedron is a connected plane figure obtained by cutting the surface of the polyhedron and unfolding it. In this paper, we determine the length and configuration of a minimum perimeter development for each of the Platonic solids. We show that such developments are obtained by cutting the surface of the polyhedron along a Steiner minimal tree. We introduce the concept of Steiner isomorphism to develop a search algorithm for determining these Steiner minimal trees. Each of these trees is completely symmetric with respect to rotation around a fixed point.

Keywords : Platonic solid; Development of a polyhedron; Minimum spanning tree; Steiner minimal tree.

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1 Introduction

1.1 Developments of Polyhedra

A *development* of a convex polyhedron is a connected plane figure obtained by cutting the surface of the polyhedron and unfolding it. The surface cut can be made both along edges and across faces, and is not restricted to straight cuts (e.g., see Figure 1). If the surface cut consists of straight cuts along edges of the polyhedron, then the development is an *edge-development*.

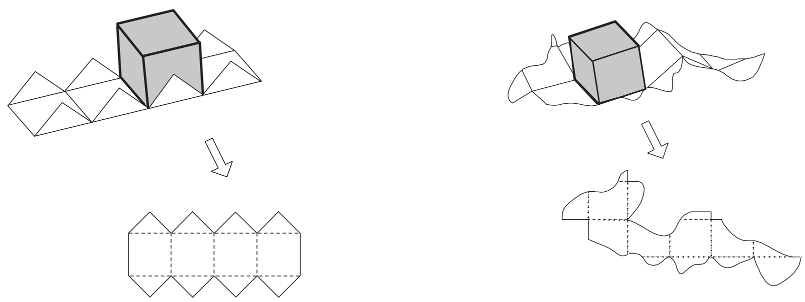


Figure 1: Two Developments of a Cube

Developments of polyhedra have been the subject of recently published papers, which have raised many open problems in discrete geometry. For example, O'Rourke [1] discussed the problem: Does every convex polyhedron have a development which is non-overlapping? The history and progress of this problem makes up part of the new book, *Geometric Folding Algorithms*, by O'Rourke and Demaine [2]. Developments of the tetrahedron have received attention because of their intriguing properties. One of the authors of this paper, Akiyama [3], proved that every development of a regular tetrahedron tiles the plane. Akiyama, Hirata, Kobayashi and Nakamura [4] determined all convex developments of a regular tetrahedron.

1.2 Platonic Solids

A *regular polyhedron* (also known as a *Platonic solid*) is a convex polyhedron with the following properties:

1. All its faces are congruent convex regular polygons,
2. None of its faces intersect except at their edges, and
3. The same number of faces meet at each of its vertices.

There are only five such polyhedra: the tetrahedron, the cube, the octahedron, the icosahedron and the dodecahedron (Figure 2). Euclid gave a complete mathematical description of these polyhedra in the last book of the *Elements*. However, their name is derived from Plato, who mentioned them in his work *Timaeus*.

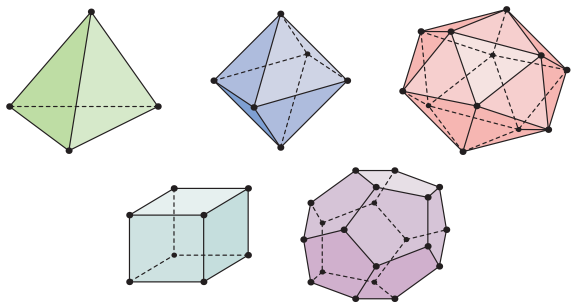
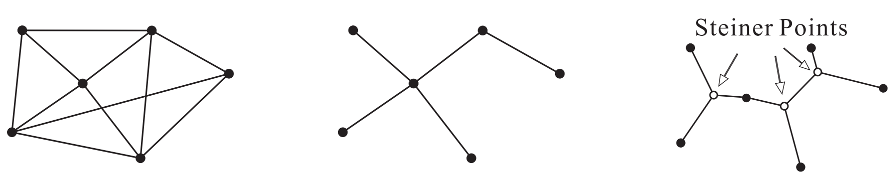


Figure 2: The 5 Platonic Solids

1.3 SMT and MST Problem

Given a fixed set of vertices V on the plane, the *Steiner tree problem* is to determine the geometric graph of shortest length that interconnects the vertices of V. This graph is clearly a *tree* and it is called a *Steiner minimal tree* (SMT). To obtain an SMT, it may be necessary to include vertices other than those in V. These are called *Steiner points*. The Steiner tree problem has a long history dating back to the 17$^{\text{th}}$ century. Hwang and Richards [5] gave an excellent survey of the problem up to 1989, further updated to 1991 by Hwang [6].

The *Minimal spanning tree (MST) problem*, usually defined on a graph $G = (V, E)$, is related to the Steiner tree problem but is distinct from it in that the fixed points are the vertices of a graph, the edges of the tree are edges of the graph and no additional vertices may be added. Figure 3(a), (b) and (c) show, respectively, a graph $G = (V, E)$, an MST on G, and an SMT on the set V of vertices in G. Let L(MST) denote the length of the minimum spanning tree and L(SMT) the length of the Steiner minimal tree. Clearly, L(MST) $\geq$ L(SMT).



(a) $G = (V, E)$ (b) MST on G (c) SMT on V

Figure 3: The MST and SMT on the graph $G = (V, E)$

1.4 Reducing MPD Problem to SMT Problem

The *perimeter* of a development is the length of its exterior boundary. We denote a minimum perimeter development by MPD and its perimeter by L(MPD). In this paper, we determine MPDs for each of the Platonic solids. We begin with the simple observation that the problem of determining an MPD for a Platonic solid (as well as for other convex polyhedra) reduces to a Steiner tree problem on the surface of the polyhedron and involves the vertices of the polyhedron.

A convex polyhedron can be represented by an *edge graph*, *i.e.,* a graph whose vertices and edges are the vertices and edges of the polyhedron. To obtain a development, the surface cut must pass through each vertex of the polyhedron; and to keep the resulting plane figure connected, the cut must not intersect itself. In terms of the edge graph, the surface cut must be made along a tree whose vertices include all the vertices of the graph. For this reason, in this paper we refer to a surface cut of a polyhedron considered in this paper as a *cut tree*. Hence determining an MPD of a polyhedron P is equivalent to determining an SMT, given the vertices of the edge graph (see Figure 4).

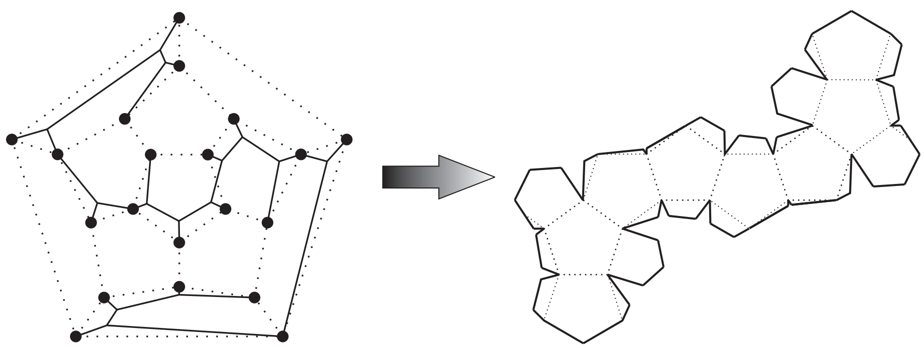


Figure 4: Obtaining an MPD from an SMT of the Dodecahedron

In this paper, we refer to a cut tree on a polyhedron P from which we obtain an MPD of P as an *SMT on P*, and a cut tree from which we obtain an edge-development as an *MST on P*. Note that there could be more than one SMTs and MSTs on P. It is clear that the perimeter of the development is twice the length of the cut tree. Given a convex polyhedron, an upper bound for L(MPD), or 2·L(SMT), is the perimeter length of its edge-development, 2·L(MST). In the case of the graph of a polyhedron with n vertices and whose edges are of unit length, L(MST) $= n - 1$.

The rest of this paper is organized as follows. In Section 2 we give a survey on the general algorithm of obtaining an SMT on some vertex set on the plane, and then show how this, when combined with the symmetry of the Platonic solids, could be arranged to obtain MPDs of the Platonic solids. Section 3 gives the main results for each of the five Platonic solids. Concepts such as Steiner isomorphism and other lemmas are motivated by the simpler cases, *e.g.*, tetrahedron and octahedron, so we introduce them after we examine these earlier examples. We conclude this paper with a conjecture as well as some possible generalization of the results of this paper in Section 4.

1.5 Related Results

A similar problem is discussed by Smith [7] who generalized the SMT into higher dimensions and determined the SMTs for most of the Archimedean d-polytopes with ≤ 16 vertices. It is well-known that most versions of the Steiner tree problem are NP-complete [5], although there are exact computer solvers (*e.g.,* geosteiner96) which can solve randomly generated problem instances with a few thousand vertices; but in this paper, we determine the minimum perimeter developments of the Platonic solids solely by theoretical arguments, without the help of a digital computer.

2 A Search Procedure for MPDs

2.1 Overview of the Exact Algorithm

To date, all the existing algorithms for the exact solution of the Steiner tree problem in the Euclidean plane are based on the approach given by Melzak [8] in 1961, with some modification. In this section, we show that Melzak's algorithm can also be modified to develop an algorithm for finding an MPD for each of the Platonic solids.

Let T be an SMT with vertices $V \cup S$, where S is a set of Steiner points. It is easily verified that T has the following properties:

1. All vertices of S have degree 3 with respect to the edges in T.
2. There is a $120°$ angle between any pair of the three edges intersecting at each vertex of S.
3. Each pair of the edges of T meet at an angle of $120°$ or greater.

Any tree that satisfies the above conditions is a Steiner tree (although it may need not be a Steiner minimal tree). A *full Steiner tree* (FST) is a tree satisfying the properties above with the additional property that $|S| = |V| - 2$. A *full Steiner minimal tree* (FSMT) is an FST that is also an SMT on a given set of vertices V. These Steiner trees are of basic importance in Melzak's algorithm because of Theorem 2.2. To simplify the statement of the theorem, we first introduce the following definition of a *Steiner partition.*

Definition 2.1. *Let V be a set of vertices. A Steiner partition $\mathcal{P}$ of V is a family of m subsets V_i $(1 \leq i \leq m)$ of V with the following properties:*

1. $\displaystyle\bigcup_{1\leq i\leq m} V_i = V,$

2. *$|V_i \cap V_j| \leq 1$ for all $1 \leq i, j \leq m$ such that $i \neq j$, and*
3. *the intersection graph of $\mathcal{P}$ is a tree.*

It is well known that an SMT on a given set of vertices V can be decomposed into a union of FSMTs with respect to some Steiner partition $\mathcal{P}$ of V. The following decomposition theorem is stated in a paper by Gilbert and Pollak [9]:

Theorem 2.2. *Let T be an SMT on V. Then there exists some Steiner partition $\mathcal{P} = \{V_1, ..., V_m\}$ of V such that T is the union of m subtrees $T_1, ..., T_m$ where each T_i is an FSMT on $V_i \in \mathcal{P}$.*

Theorem 2.2 provides the basis of Melzak's algorithm for solving the Steiner tree problem. The general procedure is as follows:

1. Find all the Steiner partitions of V.
2. For each Steiner partition $\mathcal{P}$ of V, find the FSMT for each of the subsets $V_i \in \mathcal{P}$. If all the subsets $V_i \in \mathcal{P}$ have FSMTs, then the union of these FSMTs gives a Steiner tree on V.
3. Examine all possible Steiner partitions of V and select a shortest Steiner tree on V to be the SMT.

2.2 Pruning Tests for FSTs

In this section, we give a brief description of some tests which can be applied to the FSMTs found in **Step 2** of Melzak's algorithm in order to identify and prune away those that cannot be in any SMT. The two tests of fundamental importance in this direction are the *lune property* and the *bottleneck property*.

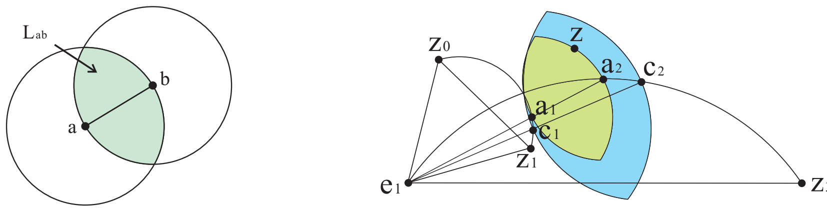
2.2.1 Definitions

We use the term *terminals* to distinguish vertices which are not *Steiner points.* The term *topology* in this paper has a different meaning than in the ordinary sense; it refers to the adjacency structure of a tree interconnecting the terminals and the Steiner points. Thus, a topology specifies the connections but not the locations of the Steiner points. In order to find an FSMT on a set of k terminals Z_k, it is necessary to consider all possible *full topologies*, *i.e.,* all possible ways of interconnecting k terminals and $k-2$ Steiner points such that the *degree condition* (all terminals have degree 1 and all Steiner points have degree 3) is satisfied.

Let $p = (p_x, p_y)$ and $q = (q_x, q_y)$ be two points in the Euclidean plane $\mathbb{R}^2$. The *equilateral point* e_{pq} of p and q is the third corner of the equilateral triangle with the line segment pq as one of its sides, such that the sequence of points $\{p, e_{pq}, q\}$ makes a right turn at e_{pq}. Note that e_{pq} and e_{qp} are distinct equilateral points. The *Euclidean distance* between p and q is $\|p - q\| = \sqrt{|p_x - q_x|^2 + |p_y - q_y|^2}$. The *equilateral circle* of p and q is the circle circumscribing the equilateral triangle $\triangle p e_{pq} q$ and is denoted by C_{pq}. The arc from p to q on C_{pq} is called the *Steiner arc* from p to q, denoted by $\widehat{pq}$.

2.2.2 Lune Property

Let a and b be terminals or Steiner points. A *lune* L_{ab} of the line segment ab is the intersection of two circles both with radius $\|a-b\|$ and centered at a and b, respectively (Figure 5(a)). It is well-known that a necessary condition for the line segment ab to be in any SMT is that L_{ab} contains no terminals [9].



(a) Lune L_{ab} (b) Restricting the Steiner Arc

Figure 5: LUNE PROPERTY

The lune property is often used in restricting the Steiner arc to some feasible subarc where new Steiner points can be located. Suppose that e_1 is an equilateral point of two terminals z_0, z_1, and the projections of a_1 and c_1 on the Steiner arc $\widehat{e_1z_3}$ are respectively a_2 and c_2. Figure 5(b) shows that a feasible subarc of $\widehat{e_1z_3}$ can be reduced by moving c_2 toward e_1.

2.2.3 Bottleneck Property

Construct an MST for the set of terminals Z. The *bottleneck Steiner distance* $b_{z_iz_j}$ is the length of the longest edge on the unique path from a terminal z_i to a terminal z_j. Consider an SMT on Z. The *bottleneck property* can be stated as follows: no edge on the path between a pair of terminals z_i and z_j can be longer than $b_{z_iz_j}$. Note that an immediate consequence of this property is that every edge in an SMT on the edge graph of a platonic solid has length less than 1. This observation will be useful in the proof of Proposition 3.6.

2.3 Symmetry of Platonic Solids

The correctness of Melzak's algorithm is clear from Theorem 2.2; however, the number of possible Steiner partitions of V makes this procedure impractical except for a very small number of points. Imagine, however, that V is the set of vertices on the highly symmetric Platonic solids. The number of possible Steiner partitions of V is reduced considerably because two different Steiner trees S_1, S_2 on a Platonic solid might have the same length, *i.e.,* $L(S_1) = L(S_2)$. Figure 6 shows two different Steiner trees S_1 and S_2 on the cube with the same length.

Lemma 2.3. *Suppose we have a collection of subsets $\{V_1, ..., V_m\}$ of the set V of vertices on the polyhedron such that each V_i has an FSMT. Then a necessary*

Figure 6: Two Steiner Trees on the Cube

condition that they form a Steiner partition $\mathcal{P}$ of V is

$$\sum_{i=1}^{m} |V_i| = |V| + m - 1. \tag{2.1}$$

Note that Lemma 2.3 follows immediately from the definition of a Steiner partition. To overcome the problem of having too many Steiner partitions, our strategy is to find all the subsets V_i of V that have an FSMT and consider only those Steiner partitions that can be constructed from these subsets. Since we are considering the FSMTs on the *surface* of the polyhedron, all the subsets of V that have an FSMT can be thought of as sets of points in the plane (Figure 7). Also note that we are not concerned with the original positions of these subsets on the polyhedron. In Section 3.2, we will give an equivalence relation among these subsets, so that two related subsets will have the same FSMT on the polyhedron.

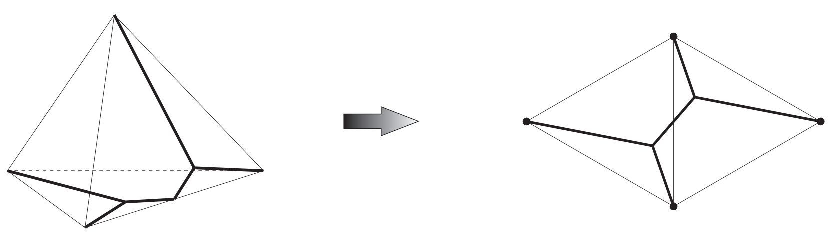


Figure 7: The Vertex Set of the Tetrahedron

Suppose there are m subsets $V_1, ..., V_m$ of V (some of them may appear more than once) that satisfy (2.1). We can avoid the problem of two Steiner partitions of V corresponding the same value of L(SMT) by disregarding the order of arrangement of these subsets when constructing a Steiner partition of V. Hence, the problem of finding all the Steiner partitions of V, which we need to consider, becomes the problem of choosing m subsets of V (where the order does not matter) satisfying (2.1) from the collection of all subsets V_i of V that have an FSMT[2].

[2]We need to make sure that these m subsets can actually *be* arranged on the polyhedron in the final stage.

2.4 The Search Procedure

Let $\mathbb{S}_i$ $(i = 1, ..., 5)$ be the vertex set for each of the Platonic solids: tetrahedron, octahedron, cube, icosahedron, and dodecahedron whose sides are of unit length, respectively. Let $\mathbf{G}$ be the set of graphs that satisfy the properties of a Steiner minimal tree.

For an SMT, $g = (V, S, E) \in \mathbf{G}$ on V, define $V(g), S(g),$ and $E(g)$ to be the original vertex set, the set of Steiner points, and the set of edges in g, respectively. Then the length of g is given by

$$L(g) = \sum_{\{v_i, v_j\} \in E(g)} \|v_i - v_j\|,$$

where $\|v_i - v_j\|$ is the Euclidean distance between the two points v_i and v_j.

Let $F(\mathbb{S}_i)$ be the set of FSMTs for each $\mathbb{S}_i$ (*i.e.*, the full Steiner minimal trees whose vertices are contained in $\mathbb{S}_i$). Define the map $f : F(\mathbb{S}_i) \to \mathbb{Z}$ by the function $f(g) = |V(g)|$. The MPD and L(MPD) for each of the Platonic solids are obtained by performing the following steps:

1. Fix the value of i and list all the elements of $F(\mathbb{S}_i)$.
2. For each $1 \leq m \leq |\mathbb{S}_i| - 1$, partition the integer $|\mathbb{S}_i| + m - 1$ by expressing it as the sum of m integers with summands in the set $f\left(F(\mathbb{S}_i)\right)$. A partition may involve multiples of the same summands in $f\left(F(\mathbb{S}_i)\right)$. List all of the partitions found this way.
3. Each partition determines a corresponding m-tuple $\mathbf{A} = (a_1, a_2, ..., a_m)$ in an obvious way, *i.e.*, the vector whose components are the integers involved in the partition. Without loss of generality, we can reorder the components of $\mathbf{A}$ and assume that
$$a_1 \leq a_2 \leq \cdots \leq a_m.$$
Now, each m-tuple $\mathbf{A} = (a_1, a_2, ..., a_m)$ represents the Steiner partitions[3] of $\mathbb{S}_i$
$$\mathcal{P} = \{V_j \subseteq \mathbb{S}_i \mid 1 \leq j \leq m\} \quad \text{where} \quad |V_j| = a_j \quad \text{for} \quad 1 \leq j \leq m.$$
Note that an m-tuple may represent more than one Steiner partition of $\mathbb{S}_i$. List all of the Steiner partitions $\mathcal{P}$ of $\mathbb{S}_i$ found this way.
4. Now, to each Steiner partition $\mathcal{P} = \{V_j \subseteq \mathbb{S}_i \mid 1 \leq j \leq m\}$ of $\mathbb{S}_i$, there corresponds a shortest Steiner tree in the plane. The length of this tree is
$$\sum_{j=1}^{m} L(g_j) \quad \text{where} \quad g_j \in F(\mathbb{S}_i) \quad \text{and} \quad V(g_j) = V_j \quad \text{for} \quad 1 \leq j \leq m.$$
Compute the lengths of all the Steiner trees found this way.

[3]To be precise, we cannot say that an m-tuple $\mathbf{A} = (a_1, a_2, ..., a_m)$ represents a Steiner partition before we check that we can actually construct (or arrange) these subsets in the Steiner partition on the polyhedron.

5. The Steiner minimal tree on $\mathbb{S}_i$, denoted by SMT($\mathbb{S}_i$), is the Steiner tree with the shortest length found in Step 4 that is constructible on the surface of the Platonic solid. The MPD is obtained by cutting the surface of the Platonic solid along SMT($\mathbb{S}_i$), and its perimeter length is 2·L(SMT($\mathbb{S}_i$)).

We demonstrate this procedure for each of the Platonic solids in Section 3.

3 MPDs of the Platonic Solids

3.1 Tetrahedron ($\mathbb{S}_1$)

Theorem 3.1. *The Steiner minimal tree on the unit tetrahedron SMT($\mathbb{S}_1$) is shown in Figure 8(c) with* $L(SMT(\mathbb{S}_1)) = \sqrt{7}(\risingdotseq 2.64575)$. *Hence* $L(MPD(\mathbb{S}_1)) = 2\sqrt{7}(\risingdotseq 5.29150)$.

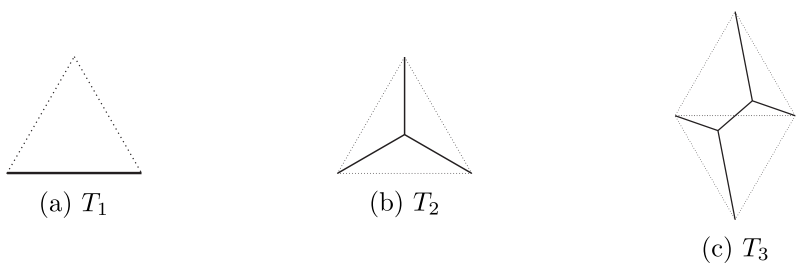


Figure 8: The FSMTs for $\mathbb{S}_1$

Proof. Let G be a planar embedding of the tetrahedron with the vertex set $V = \{v_1, v_2, v_3, v_4\}$. We consider all possible ways of choosing subsets of V. Without loss of generality, we may start by choosing v_1 as our first vertex, hence forming our first (and the only) subset of order 1. Now we wish to choose subsets of V of order 2. Notice that no matter which vertex we choose from $\{v_2, v_3, v_4\}$ as our second vertex, we always end up with a 2-set that *looks the same*[4]. Similarly, suppose we have a 2-set, and wish to choose a 3-set, we also encounter the same situation. Since there is only one way of choosing a 4-set, that is, by choosing all elements of V, we have found all possible ways of choosing subsets of V.

Once we have found the subsets of V, we wish to find the FSMTs on them, and then use these FSMTs as our atomic elements to construct an SMT on the surface of the tetrahedron. For the 2-set, we simply draw a line to connect the two points. For the 3-set, we apply Melzak's method to construct a Steiner point, and by connecting this point to the three vertices we have the FSMT. Similarly, we

[4]The precise meaning of this will be made clear in the comments following the proof.

Table 1: FSMTs for $\mathbb{S}_1$

FSMT	Length
T_1	1.00000
T_2	$\sqrt{3} \fallingdotseq 1.73205$
T_3	$\sqrt{7} \fallingdotseq 2.64575$

construct the FSMT for the 4-set, as shown in Figure 8. Their lengths are shown in Table 1.

By Lemma 2.3, for the potential subsets $\{V_1, \ldots, V_m\}$ forming a Steiner partition of $\mathcal{P}$ of V,

$$\sum_{i=1}^{m} |V_i| = |V| + m - 1 = 4 + m - 1 = 3 + m \quad \text{where} \quad m \in \mathbb{N}.$$

Now if we are to use only one FSMT to form a Steiner partition, *i.e.* $m = 1$, then this FSMT must span all four vertices of the tetrahedron. Similarly, if we are to use two FSMTs, *i.e.* $m = 2$, then the total number of vertices spanned must be five, where the vertices are obtained by considering each FSMT as an individual object. Observe that we must terminate with $m = 3$ as our last possible number of subsets, since this is the case in which we obtain the minimal spanning tree on the vertex set of the tetrahedron, and the resulting development is the edge-development.

Table 2: The Steiner Trees on $\mathbb{S}_1$

m	$\sum_{i=1}^{m} \lvert V_i \rvert$	Steiner Trees	Length
1	4	T_3	$\sqrt{7} = 2.64575$
2	5	$T_1 \cup T_2$	$1 + \sqrt{3} = 2.73205$
3	6	$T_1 \cup T_1 \cup T_1$	3.00000

Since we have, $|V_1| = 2$, $|V_2| = 3$, and $|V_3| = 4$, we partition the integers $3 + m$ (where $1 \leq m \leq 3$) by expressing each of them as the sum of 1, 2, and 3 integers, respectively, with summands in the set $\{2, 3, 4\}$. Now to each partition, there corresponds a Steiner partition of the vertex set of the tetrahedron, and consequently a Steiner tree on the surface of the tetrahedron. Table 2 shows the Steiner trees obtained from these Steiner partitions of $\mathbb{S}_1$ and their corresponding lengths. Comparing the lengths of the Steiner trees in Table 2, we see that $\mathrm{SMT}(\mathbb{S}_1) = T_3$, and $\mathrm{L}(\mathrm{SMT}(\mathbb{S}_1)) = \sqrt{7}(\fallingdotseq 2.64575)$. Hence $\mathrm{L}(\mathrm{MPD}(\mathbb{S}_1)) = 2\sqrt{7}(\fallingdotseq 5.29150)$. $\square$

3.2 Steiner Isomorphism

Before we move on to the other Platonic solids, a few observations must be made from this simple case of the tetrahedron. We were not being precise when we said that two subsets of the vertex set of the tetrahedron *looked the same*. The idea is to define an equivalence relation among the subsets of the vertex set, so that under this relation, two subsets in the same equivalence class will have the same FSMT on the surface of the Platonic solid. The use of isomorphisms between subgraphs might be an efficient approach in making the definition. However, as one can observe, two subsets spanned by some isomorphic subgraphs may fail to have the same FSMT on the surface of the Platonic solid. What we need is a special isomorphism which not only preserves the structure of the graph, but also its orientation when considered as a subgraph of the Platonic solid.

What we mean by *orientation* as a subgraph is made precise in what follows.

Definition 3.2. *Let G be a finite, simple, geometric graph, and H a geometric subgraph of G. Three vertices v_1, v_2, v_3 of H are adjacent with the orientation α if*

1. *The two pairs of vertices v_1, v_2 and v_2, v_3 are connected by the edges e_1 and e_2, respectively, and*
2. *e_2 is the αth edge in G coming from v_2 moving counter-clockwise after e_1.*

Now we are ready to impose this additional restriction on the structure of the subgraphs of G to make precise the isomorphism we have been looking for.

Definition 3.3. *Let G be a planar embedding of a convex polyhedron, and $V(G)$ be the resulting vertex set. Two subgraphs, A and B, of G are Steiner isomorphic if one of the following holds:*

1. *There is an isomorphism $f : V(A) \to V(B)$ such that any three vertices v_1, v_2, v_3 of A are adjacent with the orientation α iff $f(v_1), f(v_2), f(v_3)$ are adjacent with the orientation α.*
2. *There is an isomorphism $f : V(A) \to V(B)$ such that any three vertices v_1, v_2, v_3 of A are adjacent with the orientation α iff $f(v_1), f(v_2), f(v_3)$ are adjacent with the orientation $-\alpha$.*

Note that the two possible orientations α and $-\alpha$ in this definition merely account for the possibility that two subgraphs of G with the same FSMT can be *symmetric* precisely in this sense. From this point on, we use the term "isomorphism," to mean "Steiner isomorphism." With this new vocabulary, we can easily define an equivalence relation among the subsets of the vertex set, and then show that two subsets in the same equivalence class will have the same FSMT on the surface of the Platonic solid. However, what is more surprising is that we only need to consider the equivalence classes of those subsets whose associated subgraphs are simple paths (see Proposition 3.2).

Definition 3.4. *Let G be a planar embedding of a convex polyhedron, and $V(G)$ the resulting vertex set. Two subsets, say A and B, of $V(G)$ are related if the following properties are satisfied:*

1. *There is some subgraph of G containing A with the least number of edges (i.e. an MST on A) that is Steiner isomorphic to some MST on B, and*
2. *This Steiner isomorphism is an extension of some bijection between A and B*

We included in the definition some bijection between A and B because of the following simple observation: since we want to show that two related subsets will have the same FSMT, first of all, their size should be equal. We remark that this relation among subsets of $V(G)$ is an equivalence relation, and consequently we obtain a partition of the class of all subsets of $V(G)$. Let G be a planar embedding of a Platonic solid, and (A, B) a pair of related subsets of $V(G)$. If we identify A and B with the original vertex sets of the Platonic solid P, then they have the same FSMT on the surface of P.

3.3 Generating FSTs on Edge-developments

Given a subgraph H of a planar embedding G of a Platonic solid, the *closure* $\mathrm{cl}(H)$ is uniquely constructed from H as follows: for all nonadjacent pairs of vertices u and v, add the shortest paths between them. If $H = \mathrm{cl}(H)$, we say that H is *closed* in G. In Proposition 3.6 we show that a necessary condition for a vertex set to have an FSMT is that it is the vertex set of some closed subgraph.

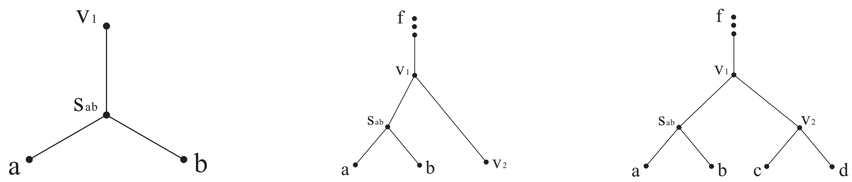
The following lemma provides considerable insights into the structure of FSTs, and is stated in [10]. Because of its importance in proving Proposition 3.6, we give a proof adopting the argument given by Hwang. The *distance* between two vertices in a full topology $\mathcal{T}_n$ is the number of edges on the path connecting the two vertices.

Lemma 3.5. *A full topology $\mathcal{T}_n$ with n terminals, $n \geq 3$, has at least one pair of terminals a and b adjacent to a common Steiner point s_{ab} with the property that if v_1 is the third point adjacent to s_{ab}, then one of the three cases is true (see Figure 9):*

1. *v_1 is a terminal, (i.e. $n = 3$)*
2. *v_1 is adjacent to a terminal v_2, or*
3. *v_1 is adjacent to a Steiner point (other than s_{ab}) with its two other neighbours being terminals.*

Proof. Let f be an arbitrary terminal point and let a be a terminal point farthest away from f. The case when $n = 3$ or 4 is trivial, so we assume that $n \geq 5$. Then the distance from a to f is at least three. Let s_{ab} be the Steiner point on the path from a to f adjacent to a. Since each Steiner point is of degree three, s_{ab} must be adjacent to a third point b, which must be a terminal point (otherwise, b

would be further away from f than a). Let v_1 be the third point adjacent to s_{ab} on the path from a to f. Then v_1 must be adjacent to a third point v_2, which is either a terminal (Figure 9(b)), or a Steiner point adjacent to two other terminals (Figure 9(c)). □



(a) $\mathcal{T}_1$ (b) $\mathcal{T}_2$ (c) $\mathcal{T}_3$

Figure 9: Three Topologies in Lemma 3.5

Proposition 3.6. *Let G be the edge graph of a Platonic solid. If a subset $A \subseteq V(G)$ has an FSMT contained in any SMT on the surface of the Platonic solid, then there is an edge-development E and a closed subgraph H of E such that $V(H) = A$. (i.e., the vertices of H are precisely the vertices in A).*

Proof. We divide the Platonic solids into three categories according to their face shapes. Namely, regular triangles, squares, and regular pentagons. Observe that, in the case of tetrahedron, octahedron, and icosahedron, an FSMT on the surface of a Platonic solid is also an FSMT on some vertex set of the lattice points on the plane. Thus, we can reduce the problem to the FST generation problem on the plane.

Imagine the triangular lattice points on the plane such as those on a triangular grid. Let Z_n be any subset of $V(G)$ with n terminals. When $n = 2$, the corresponding FST is the line segment between the two terminals. Then it is necessary that the two terminals be adjacent in G; otherwise a lune of the two terminals will contain some terminal in G as an interior point (see Figure 10).

By Lemma 3.5, in a full topology $\mathcal{T}_n$ with $n \geq 3$ terminals, we can find a pair of terminals a, b adjacent to a common Steiner point s_{ab}, and the third point v_1 adjacent to s_{ab} with the properties in Lemma 3.5. If a and b are not adjacent vertices in G then $\|a - b\| > 1$, and the Steiner point s_{ab} must be on one of the two Steiner arcs $\widehat{ab}$ and $\widehat{ba}$. But then, either the lune of the line-segment as_{ab} or bs_{ab} will contain some terminal in G (see Figure 10).

Assume that $a, b \in Z_n$ are adjacent in G and s_{ab} is their common Steiner point. Let v_1 denote the point adjacent to s_{ab}. If v_1 is a terminal in Z_n, then v_1 must be the equilateral point e_{ab} on the same side of ab as s_{ab} (see Figure 11); otherwise we would have $\|v_1 - s_{ab}\| > 1$. Suppose now that v_1 is a Steiner point adjacent to a terminal $c \in Z_n$. Then this new terminal must also be the equilateral point e_{ab}

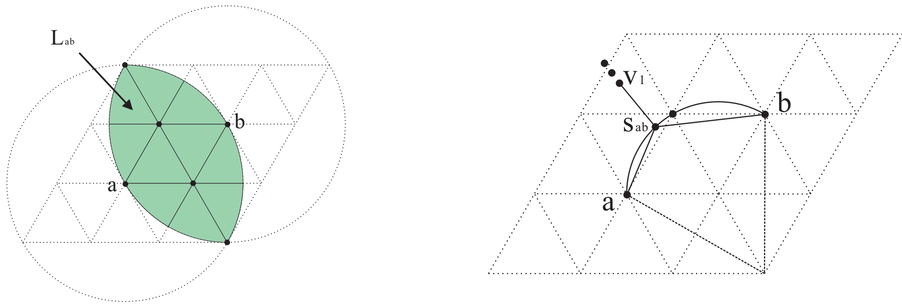


Figure 10: Impossible Situations in Proposition 3.6

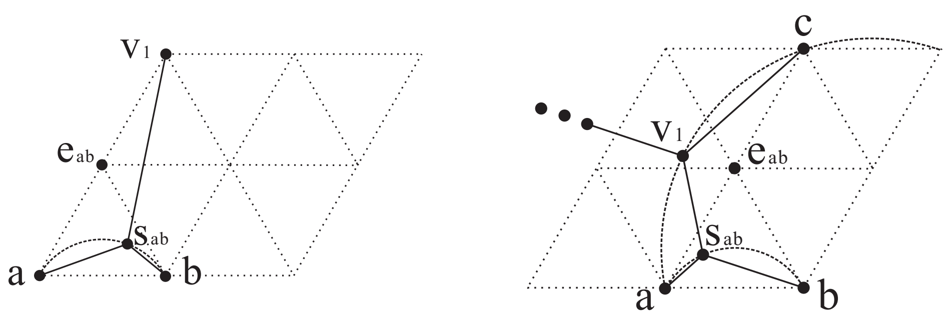


Figure 11: Impossible Situations in Proposition 3.6

(see Figure 11), otherwise either the lune of $v_1 s_{ab}$ or $v_1 c$ will contain e_{ab} in the interior. Alternatively, we can also show this by the equations

$$\|v_1 - s_{ab}\| > 1 \qquad \text{or} \qquad \|v_1 - c\| > 1.$$

Let v_2 be the third point adjacent to v_1 besides c and s_{ab}. If v_2 is a terminal in Z_n, then v_2 must be the equilateral point e_{ac} on the same side of ac as v_1, by the same argument as above.

Suppose v_1 is adjacent to a Steiner point v_2 with its two other neighbors being terminals $c, d \in Z_n$. Without loss of generality, assume that a, b, v_2 are on the same side of cd. By construction we can show that only the following full Steiner tree is possible (see Figure 12), or the lune property will be violated. Let v_3 be the third point adjacent to v_1. Again, if v_3 is a terminal in Z_n, then v_3 must be the equilateral point e_{ab}.

Summarizing, we have shown that in a full topology $\mathcal{T}_n$ with $3 \leq n \leq 5$, we have the property required by the Proposition. It is not so hard to repeat this process by combining another tree with the same topology to yield the result for $n \leq 9$. Note that we only need to repeat this at most twice since another turn will give us the result for $n \leq 17$, and there are only 12 vertices in the icosahedron.

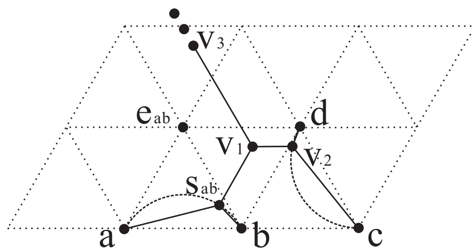


Figure 12: THE ONLY FST FOR A SPECIFIED FULL TOPOLOGY IN PROPOSITION 3.6

A modification of the above argument certainly works if we replace the triangular lattice points by square lattice points such as those on a chessboard. Thus, this proves the Proposition for the case of the cube, tetrahedron, octahedron, and icosahedron. For the case of the dodecahedron, the same argument still holds if we attach the pentagonal faces one by one whenever we wish to construct full Steiner trees on them. Note that this adds another pruning test to the process of generating full topologies; if we are in a situation where we cannot attach the pentagonal faces, then the full topology being considered is not contained in the SMT. □

Note that the FSTs generated in the proof of Proposition 3.6 have a rather nice property, namely, there is a subgraph of E that is a simple path on the vertices of A. Recall that a *simple path* is a sequence of adjacent vertices such that no vertex is repeated. Thus, a simple path is a *path graph*, *i.e.,* a tree such that two of its vertices have degree 1 and all others (if any) have degree 2. As a consequence, Proposition 3.6 may be restated as follows:

Proposition 3.7. *Let G be the edge graph of a Platonic solid. If a subset $A \subseteq V(G)$ has an FSMT contained in any SMT on the surface of the Platonic solid, then there is an edge-development E and a simple path H in E such that $V(H) = A$. (i.e., the vertices of H are precisely the vertices in A).*

Given Proposition 3.7, it suffices to classify all the isomorphic simple paths on G and construct the associated FSMTs. We demonstrate this procedure when we consider the other four Platonic solids, where we first consider the more general equivalence classes: those whose associated subgraphs are trees. Then we restrict our attention to simple paths with the bracket notation which will be introduced in Section 3.5.

3.4 Octahedron ($\mathbb{S}_2$)

Theorem 3.8. *The Steiner minimal tree on the unit octahedron SMT($\mathbb{S}_2$) is shown in Figure 13(d) with $L(SMT(\mathbb{S}_2)) = \sqrt{19} (\fallingdotseq 4.35890)$. Hence $L(MPD(\mathbb{S}_2)) = 2\sqrt{19} (\fallingdotseq 8.71780)$.*

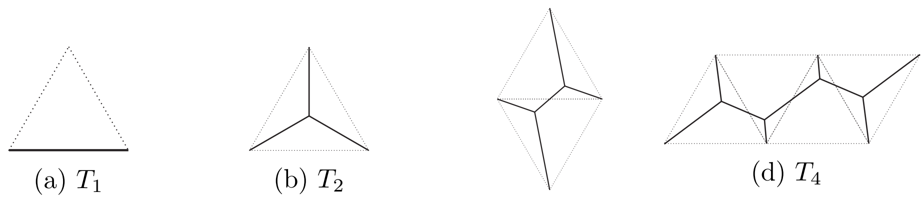

(a) T_1 (b) T_2 (c) T_3 (d) T_4

Figure 13: The FSMTs for $\mathbb{S}_2$

Proof. Let G be a planar embedding of the octahedron with the vertex set V. By Proposition 3.6, we only need to consider the equivalence classes of subsets of $V(G)$ which are spanned by trees. Without loss of generality, we may start by choosing any vertex, say v_1, as our first vertex. It is clear that we only have one Steiner isomorphic tree with two vertices, a line connecting them. Suppose we have a tree with two vertices, and wish to find all Steiner isomorphic trees with 3 vertices. Simply observe that any vertex of G has degree 4, and hence we only have two equivalence classes in this case, namely, one in which the three vertices are adjacent with the orientation 1 or $3 \equiv -1 \pmod 4$, and the other one in which the orientation is 2.

Similarly, we consider the Steiner isomorphic trees with four vertices. A simple observation shows that in general there are two types of trees with four vertices, one in which the degrees are 1, 1, 2, 2, and the other one in which the degrees are 1, 1, 1, 3. We consider the two cases separately. In the first case, we extend our 3-trees (trees with three vertices) by adding another vertex to either leaf of the tree. There are two possibilities for the orientation of this vertex if the orientation of the first three vertices is 1 or $3 \equiv -1 \pmod 4$, and three possibilities if it is 2. We write these out in ordered pairs where the components indicate the orientations as follows:

1. $(1, 2) \cong (3, 2)$; $(1, 3) \cong (3, 1)$
2. $(2, 1) \cong (2, 3)$; $(2, 2)$

Note that this bracket notation is symmetric in the following sense:

$$(m, n) \cong (-n, -m) \cong (n, m).$$

The first Steiner isomorphism follows from constructing a bijection by reversing the order of the vertices, and the second isomorphism follows directly from Definition 3.3. Hence we obtain $(1, 2) \cong (2, 1)$, so that there are at most three[5] equivalence classes in the first case. In the second case, it is clear that there is

[5] It actually turns out to be two, because the vertices spanned by $(1, 2)$ and $(1, 3)$ are equivalent.

only one Steiner isomorphic tree since any vertex of G has degree 4, so that any tree with degrees 1, 1, 1, 3 must be Steiner isomorphic.

Next we consider the Steiner isomorphic trees with five vertices. Observe that a subset of $V(G)$ with five vertices can be obtained by deleting any one vertex from $V(G)$, so that any two subsets of $V(G)$ with five vertices are in the same equivalence class as defined by Definition 3.4.

Lastly, observe that although there may be many Steiner isomorphic trees with six vertices, there is only one trivial equivalence class of subsets of $V(G)$ with six vertices, namely $V(G)$ itself. Hence we do not need to consider the Steiner isomorphic trees any further. The lengths of the FSMTs for the spanning sets of these trees are shown in Table 3.

Table 3: FSMTs for $\mathbb{S}_2$

FSMT	Length
T_1	1.00000
T_2	$\sqrt{3} \fallingdotseq 1.73205$
T_3	$\sqrt{7} \fallingdotseq 2.64575$
T_4	$\sqrt{19} \fallingdotseq 4.35890$

Since we have found all the equivalence classes of subsets of $V(G)$, we can construct the FSMTs on them, and then use them to construct an SMT on the surface of the octahedron, so that all vertices of G are spanned and the length of the resulting tree is as short as possible. We use the partition method as described in the case of the tetrahedron. Table 4 shows the Steiner trees corresponding to these Steiner partitions of $\mathbb{S}_2$ and their corresponding lengths. Comparing the lengths of the Steiner trees in Table 4, we see that T_4 is the Steiner tree with the shortest length that is constructible on the surface of the octahedron. Hence $\mathrm{SMT}(\mathbb{S}_2) = T_4$, $\mathrm{L}(\mathrm{SMT}(\mathbb{S}_2)) = \sqrt{19} (\fallingdotseq 4.35890)$, and $\mathrm{L}(\mathrm{MPD}(\mathbb{S}_2)) = 2\sqrt{19} (\fallingdotseq 8.71780)$. □

Table 4: The Steiner Trees on $\mathbb{S}_2$

m	$\sum_{i=1}^{m} \|V_i\|$	Steiner Trees	Length
1	6	T_4	$\sqrt{19} \fallingdotseq 4.35890$
2	7	$T_2 \cup T_3$	$\sqrt{3} + \sqrt{7} \fallingdotseq 4.37780$
3	8	$T_1 \cup T_2 \cup T_2$	$1 + 2\sqrt{3} \fallingdotseq 4.46410$
		$T_1 \cup T_1 \cup T_3$	$2 + \sqrt{7} \fallingdotseq 4.64575$
4	9	$T_1 \cup T_1 \cup T_1 \cup T_2$	$3 + \sqrt{3} \fallingdotseq 4.73205$
5	10	$T_1 \cup T_1 \cup T_1 \cup T_1 \cup T_1$	5.00000

3.5 Bracket Notation

In the proof of Theorem 3.8, we introduced the bracket notation to represent the orientations of the vertices in a particular type of a tree (*i.e.,* a simple path), and also pointed out a few properties about Steiner isomorphism of trees using this notation. We give a definition of the bracket notation for the general case of a simple path as follows.

Definition 3.9. *Let G be a simple path with vertices $v_1, v_2, \ldots, v_n$, where* $\deg v_1 = \deg v_n = 1$ *and* $\deg v_i = 2$ *for all* $2 \leq i \leq n-1$. *G has the bracket notation* $(\alpha_2, \alpha_3, \ldots, \alpha_{n-1})$ *if v_{i-1}, v_i, v_{i+1} are adjacent with the orientation α_i for all* $2 \leq i \leq n-1$.

Note that for a simple path, the bracket notation determines completely the structure of the path, so we can simply identify each simple path with its bracket notation. We summarize some properties of this notation which has already appeared in the proof of Theorem 3.8 in a separate lemma below.

Lemma 3.10. *Let G be a simple path with bracket notation* $(\alpha_1, \alpha_2, \ldots, \alpha_n)$. *Then G also has the following bracket notation:*

1. $-(\alpha_1, \alpha_2, \ldots, \alpha_n) = (-\alpha_1, -\alpha_2, \ldots, -\alpha_n)$
2. $-(\alpha_n, \alpha_{n-1}, \ldots, \alpha_1) = (-\alpha_n, -\alpha_{n-1}, \ldots, -\alpha_1)$
3. $(\alpha_n, \alpha_{n-1}, \ldots, \alpha_1)$

Proof. (*1*) is the definition of Definition 3.3. (*2*) is obtained by reversing the order of the vertices, and lastly, (*3*) follows from (*1*) and (*2*). □

3.6 Cube ($\mathbb{S}_3$)

Theorem 3.11. *The Steiner minimal tree on the unit cube SMT($\mathbb{S}_3$) is shown in Figure 14 with* $L(SMT(\mathbb{S}_3)) = 2\sqrt{3} + 3 (\fallingdotseq 6.46410)$. *Hence* $L(MPD(\mathbb{S}_3)) = 2(2\sqrt{3} + 3) (\fallingdotseq 12.9282)$.

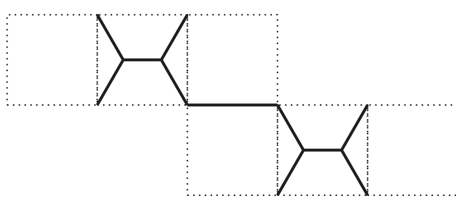


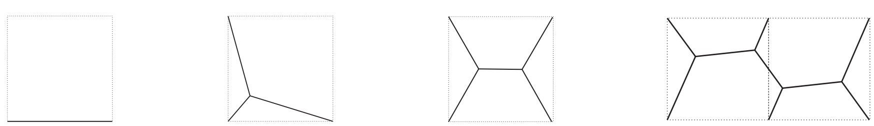
Figure 14: A Steiner Minimal Tree on the Cube

Table 5: Isomorphic Simple Paths for $\mathbb{S}_3$

n	Isomorphic Simple Paths
1	Any vertex
2	Any two adjacent vertices
3	$(1) \cong (2)$
4	$(1,1) \cong (2,2)$
	$(1,2) \cong (2,1)$
5	$(1,1,2) \cong (2,2,1) \cong (2,1,1) \cong (1,2,2)$
	$(1,2,1) \cong (2,1,2)$
6	$(1,1,2,1) \cong (2,2,1,2) \cong (1,2,1,1) \cong (2,1,2,2)$
	$(1,1,2,2) \cong (2,2,1,1)$
	$(1,2,1,2) \cong (2,1,2,1)$
	$(1,2,2,1) \cong (2,1,1,2)$
7	Deleting any vertex
8	All vertices in G

Proof. Let G be a planar embedding of the cube and consider the equivalence classes of subsets of $V(G)$ which are spanned by simple paths in G. Table 5 shows all the isomorphic simple paths of G of degree n, where $1 \leq n \leq 8$, using the bracket notation introduced in Section 3.5.

It turns out that the vertices spanned by $(1,1,2,1)$ and $(1,1,2,2)$ are equivalent. For simplicity, Figure 15 and Table 6 shows only the FSMTs corresponding to the following simple paths: the case when $n = 2$, $(1) \cong (2)$, $(1,1) \cong (2,2)$ and $(1,1,2,2)$, and their lengths, respectively. We use the partition method as usual, and Table 7 shows the Steiner trees corresponding to these Steiner partitions of $\mathbb{S}_3$ and their corresponding lengths. Comparing the lengths of the Steiner trees in Table 7, we see that $T_1 \cup T_3 \cup T_3$ is the Steiner tree with the shortest length that is constructible on the surface of the cube. Hence $\mathrm{SMT}(\mathbb{S}_3) = T_1 \cup T_3 \cup T_3$, $\mathrm{L}(\mathrm{SMT}(\mathbb{S}_3)) = 2\sqrt{3} + 3 (\fallingdotseq 6.46410)$, and $\mathrm{L}(\mathrm{MPD}(\mathbb{S}_2)) = 2(2\sqrt{3} + 3) (\fallingdotseq 12.9282)$. □



(a) T_1 (b) T_2 (c) T_3 (d) T_4

Figure 15: The FSMTs for $\mathbb{S}_3$

Table 6: FSMTs for $\mathbb{S}_3$

FSMT	Length
T_1	1.00000
T_2	$\sqrt{6}/2+\sqrt{2}/2 \fallingdotseq 1.93185$
T_3	$\sqrt{3}+1 \fallingdotseq 2.73205$
T_4	$\sqrt{2(6\sqrt{3}+11)} \fallingdotseq 6.54099$

Table 7: The Steiner Trees on $\mathbb{S}_3$

m	$\sum_{i=1}^{m}\lvert V_i\rvert$	Steiner Trees	Length
2	9	$T_2 \cup T_4$	$\frac{\sqrt{6}}{2}+\frac{\sqrt{2}}{2}+\sqrt{6\sqrt{3}+11} \fallingdotseq 6.55703$
3	10	$T_1 \cup T_1 \cup T_4$	$2+\sqrt{6\sqrt{3}+11} \fallingdotseq 6.62518$
		$T_2 \cup T_2 \cup T_3$	$\sqrt{6}+\sqrt{2}+\sqrt{3}+1 \fallingdotseq 6.59575$
		$T_1 \cup T_3 \cup T_3$	$2\sqrt{3}+3 \fallingdotseq 6.46410$
4	11	$T_1 \cup T_2 \cup T_2 \cup T_2$	$\frac{3\sqrt{6}}{2}+\frac{3\sqrt{2}}{2}+1 \fallingdotseq 6.79555$
		$T_1 \cup T_1 \cup T_2 \cup T_3$	$\frac{\sqrt{6}}{2}+\frac{\sqrt{2}}{2}+\sqrt{3}+3 \fallingdotseq 6.66390$
5	12	$T_1 \cup T_1 \cup T_1 \cup T_1 \cup T_3$	$\sqrt{3}+5 \fallingdotseq 6.73205$
		$T_1 \cup T_1 \cup T_1 \cup T_2 \cup T_2$	$\sqrt{6}+\sqrt{2}+3 \fallingdotseq 6.86370$
6	13	$T_1 \cup T_1 \cup T_1 \cup T_1 \cup T_1 \cup T_1 \cup T_2$	$\frac{\sqrt{6}}{2}+\frac{\sqrt{2}}{2}+5 \fallingdotseq 6.93185$
7	14	$T_1 \cup T_1 \cup T_1 \cup T_1 \cup T_1 \cup T_1 \cup T_1 \cup T_1$	7.00000

3.7 Icosahedron ($\mathbb{S}_4$)

Theorem 3.12. *The Steiner minimal tree on the unit icosahedron SMT($\mathbb{S}_4$) is shown in Figure 16 with* $L(SMT(\mathbb{S}_4)) = \sqrt{37}+2\sqrt{3}(\fallingdotseq 9.54686)$. *Hence* $L(MPD(\mathbb{S}_4)) = 2(\sqrt{37}+2\sqrt{3})(\fallingdotseq 19.09370)$.

Proof. The proof is completely analogous to those for the previous three Platonic solids; we consider the equivalence classes of subsets of the vertex set of the icosahedron which are spanned by simple paths. All the simple paths can be obtained by, first, listing all the n-tuples, $1 \leq n \leq 8$, whose components are the integers $1, 2, 3$ and 4, so in total we obtain

$$4+4^2+4^3+\cdots+4^8 = \frac{4^9-1}{3} - 1 = 87380$$

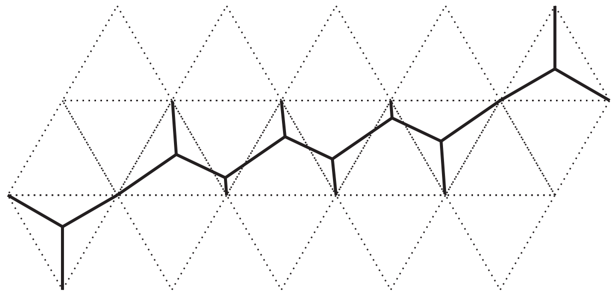


Figure 16: A Steiner Minimal Tree on the Icosahedron

candidates. Then we seek the isomorphisms among these candidates by the relations

$$(1) \cong (4) \qquad \text{and} \qquad (2) \cong (3)$$

together with the results in Lemma 3.10. Note that some sequence of numbers such as $(1,1)$ and $(1,2,1)$ cannot appear in the components of any n-tuples because they form a cycle. Also, as in the previous examples, it is quite possible for two non-isomorphic simple paths to span some vertex set in the same equivalence class. Nevertheless, this is not a problem because the determination of all the isomorphic simple paths guarantees that we will find all the equivalence classes.

Table 8: FSMTs for $\mathbb{S}_4$

FSMT	Length
T_1	1.00000
T_2	$\sqrt{3} \fallingdotseq 1.73205$
T_3	$\sqrt{7} \fallingdotseq 2.64575$
T_4	$\sqrt{19} \fallingdotseq 4.35890$
T_5	$\sqrt{37} \fallingdotseq 6.08276$
T_6	$\sqrt{61} \fallingdotseq 7.81025$

Figure 17 shows the FSMTs corresponding to the following simple paths: the case when $n = 2$, (1), $(2,1)$, $(2,3,1,2)$, $(2,3,3,1,2,3)$ and $(2,3,3,3,1,2,3,3)$. We use the partition method as usual, and Table 9 shows the Steiner trees corresponding to these Steiner partitions and their corresponding lengths. Comparing the lengths of the Steiner trees in Table 9, we see that $T_2 \cup T_2 \cup T_5$ is the Steiner tree with the shortest length that is constructible on the surface of the icosahedron. Hence $\mathrm{SMT}(\mathbb{S}_4) = T_2 \cup T_2 \cup T_5$, $\mathrm{L}(\mathrm{SMT}(\mathbb{S}_4)) = \sqrt{37} + 2\sqrt{3} (\fallingdotseq 9.54686)$, and $\mathrm{L}(\mathrm{MPD}(\mathbb{S}_4)) = 2(\sqrt{37} + 2\sqrt{3}) (\fallingdotseq 19.09370)$. $\square$

Table 9: The Steiner Trees on $\mathbb{S}_4$

m	$\sum_{i=1}^{m} \|V_i\|$	Steiner Trees	Length
2	13	$T_2 \cup T_6$	$\sqrt{3}+\sqrt{61} \fallingdotseq 9.54230$
3	14	$T_1 \cup T_3 \cup T_5$	$\sqrt{37}+\sqrt{7}+1 \fallingdotseq 9.72851$
		$T_1 \cup T_1 \cup T_6$	$\sqrt{61}+2 \fallingdotseq 9.81025$
		$T_1 \cup T_4 \cup T_4$	$2\sqrt{19}+1 \fallingdotseq 9.71780$
		$T_2 \cup T_2 \cup T_5$	$\sqrt{37}+2\sqrt{3} \fallingdotseq 9.54686$
		$T_3 \cup T_3 \cup T_4$	$\sqrt{19}+2\sqrt{7} \fallingdotseq 9.65040$
4	15	$T_1 \cup T_1 \cup T_2 \cup T_5$	$\sqrt{37}+\sqrt{3}+2 \fallingdotseq 9.81481$
		$T_1 \cup T_2 \cup T_3 \cup T_4$	$\sqrt{19}+\sqrt{7}+\sqrt{3}+1 \fallingdotseq 9.73670$
		$T_2 \cup T_2 \cup T_2 \cup T_4$	$\sqrt{19}+3\sqrt{3} \fallingdotseq 9.55505$
		$T_2 \cup T_3 \cup T_3 \cup T_3$	$3\sqrt{7}+\sqrt{3} \fallingdotseq 9.66930$
5	16	$T_1 \cup T_1 \cup T_1 \cup T_1 \cup T_5$	$\sqrt{37}+4 \fallingdotseq 10.08276$
		$T_1 \cup T_1 \cup T_2 \cup T_2 \cup T_4$	$\sqrt{19}+2\sqrt{3}+2 \fallingdotseq 9.82300$
		$T_1 \cup T_1 \cup T_3 \cup T_3 \cup T_3$	$3\sqrt{7}+2 \fallingdotseq 9.93725$
		$T_1 \cup T_1 \cup T_1 \cup T_3 \cup T_4$	$\sqrt{19}+\sqrt{7}+3 \fallingdotseq 10.00465$
		$T_2 \cup T_2 \cup T_2 \cup T_2 \cup T_3$	$\sqrt{7}+4\sqrt{3} \fallingdotseq 9.57395$
6	17	$T_1 \cup T_1 \cup T_1 \cup T_2 \cup T_3 \cup T_3$	$2\sqrt{7}+\sqrt{3}+3 \fallingdotseq 10.02355$
		$T_1 \cup T_2 \cup T_2 \cup T_2 \cup T_2 \cup T_2$	$5\sqrt{3}+1 \fallingdotseq 9.66025$
		$T_1 \cup T_1 \cup T_2 \cup T_2 \cup T_2 \cup T_3$	$\sqrt{7}+3\sqrt{3}+2 \fallingdotseq 9.84190$
		$T_1 \cup T_1 \cup T_1 \cup T_1 \cup T_2 \cup T_4$	$\sqrt{19}+\sqrt{3}+4 \fallingdotseq 10.09095$
7	18	$T_1 \cup T_1 \cup T_1 \cup T_2 \cup T_2 \cup T_2 \cup T_2$	$4\sqrt{3}+3 \fallingdotseq 9.92820$
		$T_1 \cup T_1 \cup T_1 \cup T_1 \cup T_2 \cup T_2 \cup T_3$	$\sqrt{7}+2\sqrt{3}+4 \fallingdotseq 10.10985$
		$T_1 \cup T_1 \cup T_1 \cup T_1 \cup T_1 \cup T_3 \cup T_3$	$2\sqrt{7}+5 \fallingdotseq 10.29150$
		$T_1 \cup T_1 \cup T_1 \cup T_1 \cup T_1 \cup T_1 \cup T_4$	$\sqrt{19}+6 \fallingdotseq 10.35890$
8	19	$T_1 \cup T_1 \cup T_1 \cup T_1 \cup T_1 \cup T_2 \cup T_2 \cup T_2$	$3\sqrt{3}+5 \fallingdotseq 10.19615$
		$T_1 \cup T_1 \cup T_1 \cup T_1 \cup T_1 \cup T_1 \cup T_2 \cup T_3$	$\sqrt{7}+\sqrt{3}+6 \fallingdotseq 10.37780$
9	20	$T_1 \cup \cdots \cup T_1 \cup T_2 \cup T_2$	$2\sqrt{3}+7 \fallingdotseq 10.46410$
		$T_1 \cup \cdots \cup T_1 \cup T_3$	$\sqrt{7}+8 \fallingdotseq 10.64575$
10	21	$T_1 \cup \cdots \cup T_1 \cup T_2$	$\sqrt{3}+9 \fallingdotseq 10.73205$
11	22	$T_1 \cup \cdots \cup T_1$	11.00000

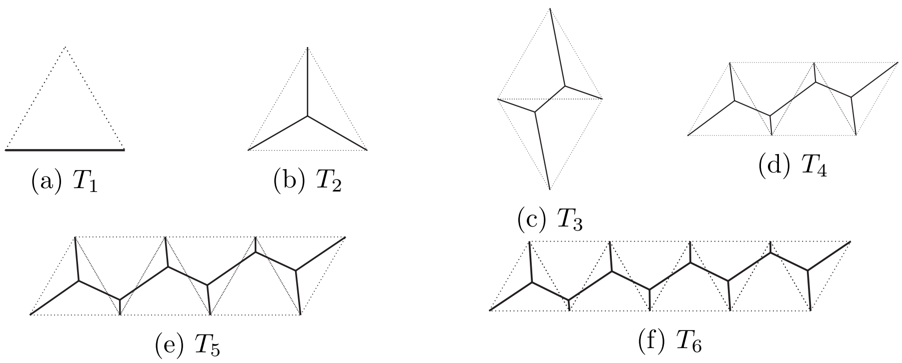


Figure 17: The FSMTs for $\mathbb{S}_4$

3.8 Dodecahedron ($\mathbb{S}_5$)

Theorem 3.13. *The Steiner minimal tree on the unit dodecahedron $SMT(\mathbb{S}_5)$ is shown in Figure 18 with* $L(SMT(\mathbb{S}_5)) = 2 \cdot L(T_2) + 2 \cdot L(T_4) + L(T_6) (\fallingdotseq 18.59864)$, *where the lengths of T_2, T_4, and T_6 are shown in Table 10. Hence* $L(MPD(\mathbb{S}_4)) = 2 \cdot L(SMT(\mathbb{S}_5)) (\fallingdotseq 37.19729)$.

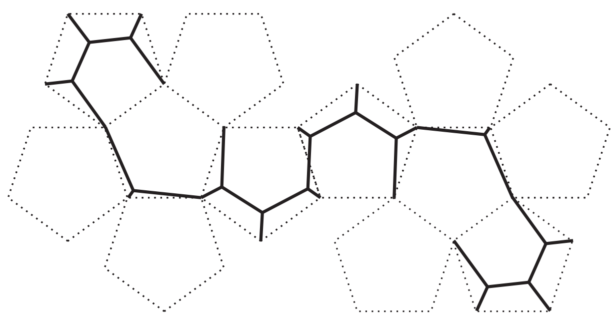


Figure 18: A Steiner Minimal Tree on the Dodecahedron

Proof. The proof of this case is also analogous to those for the previous examples, and we simply follow the same procedure. For simplicity, we only state results. Figure 19 shows the FSMTs corresponding to the following simple paths: the case when $n = 2$, (1), $(1,1)$, $(1,1,1)$, $(1,2,1,1)$, and $(1,1,1,2,2,2)$. Comparing the lengths of the Steiner trees corresponding to the Steiner partitions obtained from these FSMTs, we get the conclusion of the theorem. $\square$

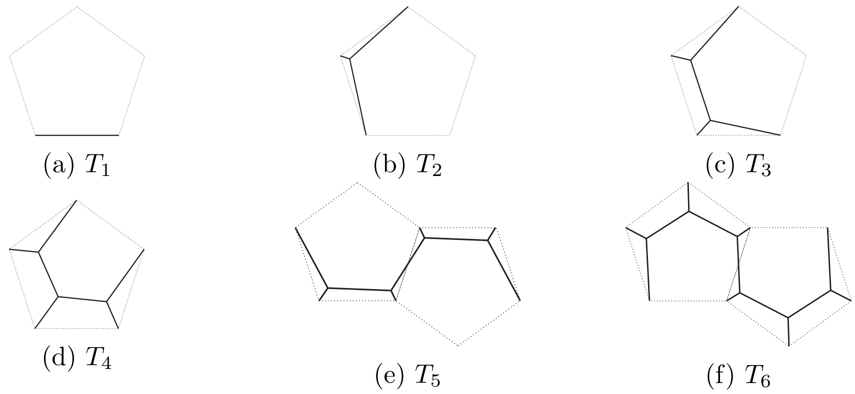


Figure 19: The FSMTs for $\mathbb{S}_5$

Table 10: FSMTs for $\mathbb{S}_5$

FSMT	Length
T_1	1.00000
T_2	$\dfrac{\sqrt{3-\tau}+\sqrt{3}\tau}{2} \risingdotseq 1.98904$
T_3	$1+2\cos\dfrac{\pi}{15} \risingdotseq 2.95630$
T_4	$\dfrac{\sqrt{4\tau+3}+\sqrt{3}(1+2\cos(\pi/15))}{2} - \sin\dfrac{\pi}{15} \risingdotseq 3.89116$
T_5	$\sqrt{(4ab+c)^2+(4ad+e)^2} \risingdotseq 4.93882$ $\left(a=\cos\dfrac{\pi}{30}, b=\sin\dfrac{\pi}{5}, c=\cos\dfrac{2\pi}{5}, d=\cos\dfrac{\pi}{5}, e=\sin\dfrac{2\pi}{5}\right)$
T_6	$2\sqrt{\left(a\cos\dfrac{\pi}{30}+\dfrac{\sqrt{2(\sqrt{5}+5)}}{8}\right)^2+\left(\dfrac{\sqrt{5}-1}{8}\right)^2} \risingdotseq 6.83824$ $\left(a=\sqrt{4\cos^2\dfrac{\pi}{30}+\cos\dfrac{\pi}{30}\sqrt{2\left(\sqrt{5}+5\right)}+1}\right)$

4 Concluding Remarks

In this paper, we have presented an exact algorithm for determining the length and configuration of a minimum perimeter development for each of the Platonic solids. The algorithm is based on Melzak's algorithm for the solution of the Steiner tree problem. The symmetries on the Platonic solids were used to overcome the problem of Melzak's algorithm resulting in too many Steiner partitions. The cut trees from which we obtain the minimum perimeter developments are completely symmetric with respect to rotation around a fixed point in the tree. We are presently engaged in determining the minimum perimeter developments of the Archimedean solids and the Catalan solids. Preliminary results indicate the following conjecture:

Conjecture 4.1. *Let P, $SMT(P)$, and $MST(P)$ be the vertex set of a polyhedron, an SMT on P, and an MST on P, respectively. Then*

$$\frac{L(SMT(P))}{L(MST(P))} < 1.$$

Note that it is easily verified that $\inf_P \frac{L(SMT(P))}{L(MST(P))} = \frac{\sqrt{3}}{2}$, (where the infimum is taken over all vertex sets P of polyhedra in $\mathbb{R}^3$) is exactly the value attained by using $P =$ the vertices of the pyramid, *i.e.*, the upper half of a regular octahedron.

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