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Hyers-Ulam-Rassias Stability of Homomorphisms and Derivations on Normed Lie Triple Systems

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Abstract : We prove the Hyers-Ulam-Rassias stability of homomorphism and derivations on normed Lie triple systems for the following generalized Cauchy-Jensen additive mapping:

$$r_0 f\left(\frac{s\sum_{j=1}^{p} x_j + t\sum_{j=1}^{d} y_j}{r_0}\right) = s\sum_{j=1}^{p} f(x_j) + t\sum_{j=1}^{d} f(y_j)$$

and generalize some results concerning this functional equation.

Keywords : Cauchy-Jensen additive mapping; Normed Lie systems; Homomorphism; Derivation; Generalized Hyers-Ulam stability.
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1 Introduction

The stability problem of functional equations originated from a question of Ulam [1], posed in 1940, concerning the stability of group homomorphism. In 1941, Hyers [2] gave a partial affirmative answer to the question of Ulam in the context of Banach spaces. In 1950, a generalized version of Hyers' theorem for approximate

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additive mappings was given by Aoki [3]. In 1978, Rassias [4] extended the theorem of Hyers by considering the unbounded cauchy difference inequality

$$\|f(x+y)-f(x)-f(y)\| \leq \varepsilon(\|x\|^p+\|y\|^p) \quad (\varepsilon \geq 0,\, p \in [0,1)).$$

Rassias [4] was the first who proved the stability of the linear mappings between Banach spaces. In 1990, Rassias [5] during the 27th international symposium on functional equations asked the question whether such a theorem can also be proved for $p \geq 1$. In 1991, Gajda [6] following the same approach as in Rassias [4] gave an affirmative solution to this question for $p > 1$. It was proved by Gajda [6], as well as by Rassias and Semrl [7] that one cannot prove Rassias' type theorem when $p = 1$. Rassias Theorem for the stability of the linear mappings between Banach spaces provided some influence for the development of the concept of generalized Hyers-Ulam stability, a fact which rekindled interest in the subject of stability of functional equations. This concept is known today as generalized Hyers-Ulam stability or Hyers-Ulam-Rassias stability of functional equations; cf. [8–11]. Several mathematicians worldwide followed the spirit of the approach in the paper of Rassias [4] for the unbounded Cauchy difference obtained various results.

During the last decades several stability problems of functional equations have been investigated by a number of mathematicians; cf. [12–15] and references therein.

One of the interesting functional equations is the following Cauchy-Jensen additive mapping

$$r_0 f\left(\frac{s\sum_{j=1}^{p} x_j + t\sum_{j=1}^{d} y_j}{r_0}\right) = s\sum_{j=1}^{p} f(x_j) + t\sum_{j=1}^{d} f(y_j),$$

where f is a mapping between linear spaces. It is easy to see that a function f satisfies the above Cauchy-Jensen additive type equation if and only if it is additive.

Ternary algebraic operations were considered in 19th century by several mathematicians such as Cayley [16] how introduced the notion of cubic matrix which in turn was generalized by Kapranov et al. [17] in 1990. There are some applications, although still hypothetical, in the fractional quantum Hall effect, the nonstandard statistics, supersymmetric theory, and Yang-Baxter equation. The comments on physical applications of ternary structures can be found in Refs. [18–25].

A normed (Banach) Lie triple system is a normed (Banach) space $(\mathsf{A}; \|.\|)$ with a trilinear mapping $(x,y,z) \longmapsto [x,y,z]$ from $\mathsf{A}\times\mathsf{A}\times\mathsf{A}$ to A satisfying the following axioms

$$\begin{aligned}
&[x,y,z] = -[y,x,z],\\
&[x,y,z]+[y,z,x]+[z,x,y] = 0\\
&[u,v,[x,y,z]] = [[u,v,x],y,z]+[x,[u,v,y],z]+[x,y,[u,v,z]],\\
&\|[x,y,z]\| \leq \|x\|\|y\|\|z\|,
\end{aligned}$$

for all $u, v, x, y, z \in \mathsf{A}$. The concept of lie triple system was first introduced by Lister [26] (see also [27]).

Let A and B be normed Lie triple systems. A $\mathbb{C}$-linear mapping $H : \mathsf{A} \to \mathsf{B}$ is said to be a homomorphism if $H([x, y, z]) = [H(x), H(y), H(z)]$ for all $x, y, z \in \mathsf{A}$. A $\mathbb{C}$-linear mapping $\mathsf{A} \to \mathsf{B}$ is called a derivation if $D([x, y, z]) = [D(x), y, z] + [x, D(y), z] + [x, y, D(z)]$ for all $x, y, z \in \mathsf{A}$. The third identity asserts that the mappings $D_{u,v} : x \longmapsto [u, v, x]$ are (inner)derivation of A.

Clearly, every Lie algebra is at the same time a Lie triple system via $[x, y, z] := [[x, y], z]$, and our definition of a homomorphism (derivation) coincides with that of prehomomorphism (prederivation) on a Lie algebra [28]. Also, if U is an involutive automorphism of a Lie algebra $(L, [,])$, then the eigenspace $E_{-1}(U)$ is a Lie triple system. Lie triple systems are important since they give the structure of the tangent space of a symmetric space, see [29]. Also some application of Lie triple systems can be found in Nambuòs approach by modifying the Heisenberg equation of motion [30].

In this paper, we have analyzed the Hyers-Ulam-Rassias stability of homomorphism and derivation in Lie triple systems associated with the following generalized Cauchy-Jensen additive mapping

$$r_0 f\left(\frac{s\sum_{j=1}^{p} x_j + t\sum_{j=1}^{d} y_j}{r_0}\right) = s\sum_{j=1}^{p} f(x_j) + t\sum_{j=1}^{d} f(y_j),$$

and then apply our results to study stability of homomorphisms and derivations associated to Cauchy-Jensen additive mapping in normed Lie triple systems, which can be regarded as ternary structures. The reader is referred to [31–33] for some other related results.

Throughout this paper, suppose that A is normed Lie triple system with norm $\|.\|_{\mathsf{A}}$ and that B is a Banach Lie triple system with norm $\|.\|_{\mathsf{B}}$.

2 Stability of Homomorphisms in Normed Lie Triple Systems

In this section, we prove the stability of homomorphisms in normed Lie triple systems associated with the Cauchy-Jensen additive mapping. For given mapping $f : \mathsf{A} :\to \mathsf{B}$ and given subset $\mathbb{E}$ of $\mathbb{C}$, we define

$$\begin{aligned} &J_\lambda f(x_1, \ldots, x_p, y_1, \ldots, y_d) \\ &\qquad := r_0 f\left(\frac{s\sum_{j=1}^{p} \lambda x_j + t\sum_{j=1}^{d} \lambda y_j}{r_0}\right) - s\sum_{j=1}^{p} \lambda f(x_j) - t\sum_{j=1}^{d} \lambda f(y_j), \end{aligned}$$

for all $\lambda \in \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and all $x_1, \ldots, x_p, y_1, \ldots, y_d \in \mathsf{A}$. One can easily show that a mapping $f : \mathsf{A} \to \mathsf{B}$ satisfies

$$J_\lambda f(x_1, \ldots, x_p, y_1, \ldots, y_d) = 0$$

for all $\lambda \in \mathbb{T}$ and all $x_1, \ldots, x_p, y_1, \ldots, y_d \in \mathsf{A}$ if and only if

$$f(\mu x + \lambda y) = \mu f(x) + \lambda f(y)$$

for all $\mu, \lambda \in \mathbb{T}$ and all $x, y \in \mathsf{A}$.

Theorem 2.1. *Let θ be a positive real number, let $r < 3$ and $d \geq 2$. Suppose $f : A \to B$ be a mapping with $f(0) = 0$ such that*

$$\|J_\lambda f(x_1, \ldots, x_p, y_1, \ldots, y_d)\|_B \leq \theta \left(\sum_{j=1}^{p} \|x_j\|_A^r + \sum_{j=1}^{d} \|y_j\|_A^r \right) \tag{2.1}$$

and

$$\|f([x, y, z]) - [f(x), f(y), f(z)]\|_B \leq \theta(\|x\|_A^r + \|y\|_A^r + \|z\|_A^r), \tag{2.2}$$

for all $\lambda \in \mathbb{T}$ and all $x, y, z \in A$. Then there exists a unique homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \leq \frac{\theta}{1 - d^{r-1}} \|x\|_A^r \tag{2.3}$$

for all $x \in A$.

Proof. First, we assume that $\|0\|^r = \infty$ for $r < 0$. Let $\lambda = 1, x_1 = \cdots = x_p = 0$, $y_1 = \cdots = y_d = x$ and $t = 1$ in (2.1) we get

$$\|f(dx) - df(x)\|_{\mathsf{B}} \leq d\theta \|x\|_{\mathsf{A}}^r \tag{2.4}$$

for all $x \in \mathsf{A}$. If we replace x by $d^n x$ in (2.4) and divide both sides of (2.4) to d^{n+1}, we get

$$\|\frac{1}{d^{n+1}} f(d^{n+1}x) - \frac{1}{d^n} f(d^n x)\|_{\mathsf{B}} \leq \theta d^{(r-1)n} \|x\|_{\mathsf{A}}^r$$

for all $x \in \mathsf{A}$ and all nonnegative integers n. Therefore, one can use induction to show that

$$\|d^{-n} f(d^n x) - d^{-m} f(d^m x)\|_{\mathsf{B}} \leq \theta \sum_{k=m}^{n-1} d^{(r-1)k} \|x\|_{\mathsf{A}}^r \tag{2.5}$$

for all nonnegative $n > m$ and all $x \in \mathsf{A}$. It follows from the convergence of the series (2.5) that the sequence $\{\frac{f(d^n x)}{d^n}\}$ is a Cauchy sequence. Due to the completeness of B, this sequence is convergent. Now, by define the mapping $H : \mathsf{A} \to \mathsf{B}$ by

$$H(x) := \lim_{n \to \infty} \frac{f(d^n x)}{d^n}$$

for all $x \in \mathsf{A}$. Set $m = 0$ in (2.5) and let n tend to infinity to get (2.3). It follows from (2.1) that

$$\begin{aligned}
&\Big\| r_0 H\Big(\frac{s\sum_{j=1}^{p}\lambda x_j + t\sum_{j=1}^{d}\lambda y_j}{r_0}\Big) - s\sum_{j=1}^{p}\lambda H(x_j\Big) - t\sum_{j=1}^{d}\lambda H(y_j)\Big\|_{\mathsf{B}}\\
&= \lim_{n\to\infty}\frac{1}{d^n}\Big\| r_0 f\Big(d^n\frac{s\sum_{j=1}^{p}\lambda x_j + t\sum_{j=1}^{d}\lambda y_j}{r_0}\Big) - s\sum_{j=1}^{p}\lambda f(d^n x_j) - t\sum_{j=1}^{d}\lambda f(d^n y_j)\Big\|_{\mathsf{B}}\\
&\le \lim_{n\to\infty}\frac{d^{nr}}{d^n}\,\theta\Big(\sum_{j=1}^{p}\|x_j\|_{\mathsf{A}}^r + \sum_{j=1}^{d}\|y_j\|_{\mathsf{A}}^r\Big) = 0,
\end{aligned}$$

for all $\lambda \in \mathbb{T}$ and $x_1, \ldots, x_p, y_1, \ldots, y_d \in \mathsf{A}$. Hence

$$r_0 H\left(\frac{s\sum_{j=1}^{p}\lambda x_j + t\sum_{j=1}^{d}\lambda y_j}{r_0}\right) = s\sum_{j=1}^{p}\lambda H(x_j) + t\sum_{j=1}^{d}\lambda H(y_j)$$

for all $\lambda \in \mathbb{T}$ and $x_1, \ldots, x_p, y_1, \ldots, y_d \in \mathsf{A}$. So $H(\lambda x + \mu y) = \lambda H(x) + \mu H(y)$ for all $\lambda, \mu \in \mathbb{T}$ and all $x, y \in \mathsf{A}$, then H ia an additive mapping. Obviously, $H(0x) = 0 = 0H(x)$. Next, let $\mu \in \mathbb{C}(\mu \neq 0)$, and let M be a natural number greater then $|\mu|$. By an easily geometric argument, one can coincide that there exist two number $\lambda_1, \lambda_2 \in \mathbb{T}$ such that $2\frac{\mu}{M} = \lambda_1 + \lambda_2$. By additivity and also definition of H we get $H(\frac{1}{2}x) = \frac{1}{2}H(x)$ for all $x \in \mathsf{A}$. Therefore

$$\begin{aligned}
H(\mu x) &= H\left(\frac{M}{2}.2.\frac{\mu}{M}x\right) = MH\left(\frac{1}{2}.2.\frac{\mu}{M}x\right)\\
&= \frac{M}{2}H(\lambda_1 x + \lambda_2 x) = \frac{M}{2}(H(\lambda_1 x) + H(\lambda_2 x))\\
&= \frac{M}{2}(\lambda_1 + \lambda_2)H(x) = \frac{M}{2}.2.\frac{\mu}{M}H(x) = \mu H(x),
\end{aligned}$$

for all $x \in \mathsf{A}$, so that H is a $\mathbb{C}$-linear mapping. It follows from (2.2) that

$$\begin{aligned}
&\|H([x,y,z]) - [H(x), H(y), H(z)]\|_{\mathsf{B}}\\
&\qquad = \lim_{n\to\infty}\frac{1}{d^{3n}}\Big\| f\Big([d^n x, d^n y, d^n z]\Big) - [f(d^n x), f(d^n y), f(d^n z)]\Big\|_{\mathsf{B}}\\
&\qquad \le \theta\lim_{n\to\infty}\frac{d^{nr}}{d^{3n}}(\|x\|_{\mathsf{A}}^r + \|y\|_{\mathsf{A}}^r + \|z\|_{\mathsf{A}}^r) = 0
\end{aligned}$$

for all $x, y, z \in \mathsf{A}$. So, $H([x,y,z]) = [H(x), H(y), H(z)]$ for all $x, y, z \in \mathsf{A}$.

Now we prove that H is the unique such additive mapping. Assume that there exists another one, denote by $H' : \mathsf{A} \to \mathsf{B}$. Then there exist a constant ε_1 and r' $(r' < 1)$ with

$$\|f(x) - H'(x)\| \le \varepsilon_1\|x\|_{\mathsf{A}}^{r'}.$$

By the triangle inequality, (2.3) and above inequality we have

$$\begin{aligned}\|H(x)-H'(x)\|_{\mathsf{B}} &= d^{-n}\|H(d^nx)-H'(d^nx)\|_{\mathsf{B}}\\ &\le d^{-n}\Big(\frac{\theta}{1-d^{r-1}}\|d^nx\|_{\mathsf{A}}^r+\varepsilon_1\|d^nx\|_{\mathsf{A}}^{r'}\Big)\\ &= d^{n(r-1)}\frac{\theta}{1-d^{r-1}}\|x\|_{\mathsf{A}}^r+d^{n(r'-1)}\varepsilon_1\|x\|_{\mathsf{A}}^{r'}\end{aligned}$$

for $n\in\mathbb{N}$. By letting $n\to\infty$ we get $H(x)=H'(x)$ for any $x\in\mathsf{A}$. □

Example 2.2. *Let $L: A\to A$ be a norm one homomorphism between normed Lie triple systems, let $f: A\to A$ be defined by*

$$f(x)=\begin{cases} L(x) & \|x\|<1\\ 0 & \|x\|\ge 1\end{cases},$$

let $r=0$ and $\theta=3$. Then

$$\|J_\lambda f(x_1,\ldots,x_p,y_1,\ldots,y_d)\|_B\le 3=\theta,$$

and

$$\|f([x,y,z])-[f(x),f(y),f(z)]\|_B\le 2\le\theta,$$

for all $\lambda\in\mathbb{T}$ and all $x,y,z\in A$. Note also that f in not linear.

By the theorem 2.1 there is a homomorphism H given by $H(x)=\lim_{n\to\infty}\frac{f(d^nx)}{d^n}$. Further, $H(0)=\lim_{n\to\infty}\frac{f(0)}{d^n}=0$ and for $x\neq 0$ we have

$$H(x)=\lim_{n\to\infty}\frac{f(d^nx)}{d^n}=\lim_{n\to\infty}\frac{0}{d^n}=0,$$

since for sufficiently large n, $\|d^nx\|\ge 1$. Thus H is identically zero and

$$\|f(x)-H(x)\|_B\le 1\le\frac{d\theta}{d-1}=\frac{3d}{d-1},$$

for all $x\in A$ and $d\ge 2$.

Theorem 2.3. *Let θ be a positive real number, let $r>3$ and $d\ge 2$. Suppose $f: A\to B$ be a mapping with $f(0)=0$ such that*

$$\|J_\lambda f(x_1,\ldots,x_p,y_1,\ldots,y_d)\|_B\le\theta\left(\sum_{j=1}^{p}\|x_j\|_A^r+\sum_{j=1}^{d}\|y_j\|_A^r\right)\tag{2.6}$$

and

$$\|f([x,y,z])-[f(x),f(y),f(z)]\|_B\le\theta(\|x\|_A^r+\|y\|_A^r+\|z\|_A^r),\tag{2.7}$$

for all $\lambda=1,\mathbf{i}$ and all $x,y,z\in A$. Assume that for each fixed $x\in A$ the function $t\to f(tx)$ is continuous on $\mathbb{R}$. Then there exists a unique homomorphism $H: A\to B$ such that

$$\|f(x)-H(x)\|_B\le\frac{\theta}{d^{r-1}-1}\|x\|_A^r\tag{2.8}$$

for all $x\in A$.

Proof. Applying the same argument as in the proof of 2.1 one can deduce the existence of a unique additive mapping $H : \mathsf{A} \to \mathsf{B}$ given by

$$H(x) := \lim_{n\to\infty} d^n f(d^{-n}x)$$

satisfying the required inequalities. By the same reasoning as in the proof of the theorem of [4], the additive mapping H is $\mathbb{R}$-linear.

Letting $x_1 = x, x_2 = \cdots, x_p = y_1 = \cdots = y_d = 0$ and $t = s = 1$ in (2.6), we get

$$\|f(\lambda x) - \lambda f(x)\|_{\mathsf{B}} \le \theta \|x\|_{\mathsf{A}}^r$$

then it follows that $\|f(\mathbf{i}x) - \mathbf{i}f(x)\| \le \theta\|x\|_{\mathsf{A}}^r$, for all $x \in \mathsf{A}$. Hence $d^n\|f(d^{-n}\mathbf{i}x) - \mathbf{i}f(d^{-n}x) \le \frac{\theta}{d^{n(r-1)}}\|x\|_{\mathsf{A}}^r$, for all $n \in \mathbb{N}$ and all $x \in \mathsf{A}$. The right hand side tends to zero as $n \to \infty$, so that

$$H(\mathbf{i}x) = \lim_{n\to\infty} d^n f(\frac{\mathbf{i}x}{d^n}) = \lim_{n\to\infty} \mathbf{i}d^n f(\frac{x}{d^n}) = \mathbf{i}H(x),$$

for all $x \in \mathsf{A}$. For each $\mu \in \mathbb{C}, \mu = \lambda_1 + \mathbf{i}\lambda_2 (\lambda_1, \lambda_2 \in \mathbb{R})$. Hence

$$\begin{aligned} H(\mu x) &= H(\lambda_1 x + \mathbf{i}\lambda_2 x) = \lambda_1 H(x) + \lambda_2 H(\mathbf{i}x) \\ &= \lambda_1 H(x) + \mathbf{i}\lambda_2 H(x) = (\lambda_1 + \mathbf{i}\lambda_2)H(x) \\ &= \mu H(x), \end{aligned}$$

thus H is $\mathbb{C}$-linear. Note that inequality (2.6) implies that $f(0) = 0$. It follows from (2.7) that

$$\begin{aligned} &\|H([x,y,z]) - [H(x),H(y),H(z)]\|_{\mathsf{B}} \\ &\qquad = \lim_{n\to\infty} d^{3n}\Big\|f\Big(\frac{[x,y,z]}{d^{3n}}\Big) - \Big[f\Big(\frac{x}{d^n}\Big), f\Big(\frac{y}{d^n}\Big), f\Big(\frac{z}{d^n}\Big)\Big]\Big\|_{\mathsf{B}} \\ &\qquad \le \theta \lim_{n\to\infty} \frac{d^{3n}}{d^{nr}}(\|x\|_{\mathsf{A}}^r + \|y\|_{\mathsf{A}}^r + \|z\|_{\mathsf{A}}^r) = 0 \end{aligned}$$

for all $x, y, z \in \mathsf{A}$. So, $H([x,y,z]) = [H(x), H(y), H(z)]$ for all $x, y, z \in \mathsf{A}$.

The reminder of proof is similar to Theorem 2.1. □

3 Stability of Derivations on Normed Lie Triple Systems

In this section, we prove the stability of derivations in normed Lie triple systems associated with the Cauchy-Jensen additive mapping.

Theorem 3.1. *Let θ and $r, r', s', t' \in \mathbb{R}^+$ with $r' + s' + t' > 3$, and $f : A \to B$ be a mapping such that*

$$\|J_\lambda f(x_1, \ldots, x_p, y_1, \ldots, y_d)\|_B \le \theta \prod_{j=1}^{p} \|x_j\|_A^r \cdot \prod_{j=1}^{d} \|y_j\|_A^r \tag{3.1}$$

and

$$\|f([x,y,z]) - [f(x),y,z] - [x,f(y),z] - [x,y,f(z)]\|_B \leq \theta.\|x\|_A^{r'}.\|y\|_A^{s'}.\|z\|_A^{t'} \quad (3.2)$$

for $\lambda = 1, \mathbf{i}$ *and all* $x, y, z, x_1, \dots, x_p, y_1, \dots, y_d \in A$. *Assume that for each fixed* $x \in A$ *the function* $t \to f(tx)$ *is continuous on* $\mathbb{R}$. *Then there exists a unique derivation* $D : A \to B$ *such that*

$$\|f(x) - D(x)\|_B \leq \frac{2^{(p+d)r}\theta}{\Big(2(p+2d)^{(p+d)r} - 2^{(p+d)r}(p+2d)\Big)}\|x\|_A^{(p+d)r} \quad (3.3)$$

for all $x \in A$.

Proof. Note that inequality (3.1) implies $f(0) = 0$, put $\lambda = 1$ and $x_1, \dots, x_p = y_1 = \dots = y_d = x$ and $s = 1, t = 2$ in (3.1), we obtain

$$\|f((p+2d)x) - (p+2d)f(x)\|_{\mathsf{B}} \leq \theta\|x\|_{\mathsf{A}}^{(p+d)r} \quad (3.4)$$

for all $x \in \mathsf{A}$. Therefore

$$\left\|f(x) - (p+2d)f\left(\frac{x}{p+2d}\right)\right\|_{\mathsf{B}} \leq \frac{\theta}{(p+2d)^{(p+d)r}}\|x\|_{\mathsf{A}}^{(p+d)r}$$

for all $x \in \mathsf{A}$. Then

$$\begin{aligned}&\left\|(p+2d)^n f\left(\frac{x}{(p+2d)^n}\right) - (p+2d)^m f\left(\frac{x}{(p+2d)^m}\right)\right\|_{\mathsf{B}} \\ &\leq \sum_{k=m}^{n-1}\left\|(p+2d)^k f\left(\frac{x}{(p+2d)^k}\right) - (p+2d)^{k+1} f\left(\frac{x}{(p+2d)^{k+1}}\right)\right\|_{\mathsf{B}} \\ &\leq \frac{\theta}{(p+2d)^{(p+d)r}}\sum_{k=m}^{n-1}\frac{(p+2d)^k}{(p+2d)^{(p+d)rk}}\|x\|_{\mathsf{A}}^{(p+d)r} \quad (3.5)\end{aligned}$$

for all nonnegative integer $n > m$ and $x \in \mathsf{A}$. It follows from the convergence of the series (3.5) that the sequence $\{(p+2d)^n f(\frac{x}{(p+2d)^n})\}$ is a Cauchy sequence. From the completeness of B, this sequence converges. So we can define the mapping $D : \mathsf{A} \to \mathsf{B}$ by

$$D(x) := \lim_{n\to\infty}(p+2d)^n f\left(\frac{x}{(p+2d)^n}\right)$$

for all $x \in \mathsf{A}$.

Now by considering $m = 0$ and taking the limit as $n \to \infty$ in (3.6), we obtain

(3.3). It follows from (3.2) that

$$\begin{aligned}
&\|D([x,y,z]) - [D(x),y,z] - [x,D(y),z] - [x,y,D(z)]\|_{\mathsf{B}} \\
&= \lim_{n\to\infty}(p+2d)^{3n}\Big\| f\Big(\frac{[x,y,z]}{(p+2d)^{3n}}\Big) \\
&\quad - \Big[f\Big(\frac{x}{(p+2d)^n}\Big), \frac{y}{(p+2d)^n}, \frac{z}{(p+2d)^n}\Big] \\
&\quad - \Big[\frac{x}{(p+2d)^n}, f\Big(\frac{y}{(p+2d)^n}\Big), \frac{z}{(p+2d)^n}\Big] \\
&\quad - \Big[\frac{x}{(p+2d)^n}, \frac{y}{(p+2d)^n}, f\Big(\frac{z}{(p+2d)^n}\Big)\Big]\Big\|_{\mathsf{B}} \\
&= \lim_{n\to\infty}\frac{\theta(p+2d)^{3n}}{(p+2d)^{n(r'+s'+t')}}(\|x\|_{\mathsf{A}}^{r'}.\|y\|_{\mathsf{A}}^{s'}.\|z\|_{\mathsf{A}}^{t'}) = 0
\end{aligned}$$

for all $x \in \mathsf{A}$. So, $D([x,y,z]) = [D(x),y,z] + [x,D(y),z] + [x,y,D(z)]$ for all $x,y,z \in \mathsf{A}$. Similar to $H(x)$ in Theorem 2.3, one can show that the mapping $D(x)$ is a $\mathbb{C}$-linear too, and also the reminder is similar to the proof of Theorem 2.3. □

Theorem 3.2. *Let θ and $r, r', s', t' \in \mathbb{R}^+$ with $r' + s' + t' < 1$, and $f : A \to B$ be a mapping such that*

$$\|J_\lambda f(x_1,\dots,x_p,y_1,\dots,y_d)\|_B \le \theta\prod_{j=1}^{p}\|x_j\|_A^r.\prod_{j=1}^{d}\|y_j\|_A^r$$

and

$$\|f([x,y,z]) - [f(x),y,z] - [x,f(y),z] - [x,y,f(z)]\|_B \le \theta.\|x\|_A^{r'}.\|y\|_A^{s'}.\|z\|_A^{t'}$$

for $\lambda \in \mathbb{T}$ and all $x,y,z,x_1,\dots,x_p,y_1,\dots,y_d \in A$. Then there exists a unique derivation $D : A \to B$ such that

$$\|f(x) - D(x)\|_B \le \frac{2^{(p+d)r}\theta}{\Big(2^{(p+d)r}(p+2d) - 2(p+2d)^{(p+d)r}\Big)}\|x\|_A^{(p+d)r}$$

for all $x \in A$.

Proof. First let us assume that $\|0\|_{\mathsf{A}}^p = \infty$ for $p < 0$. We can define the mapping $D : \mathsf{A} \to \mathsf{B}$

$$D(x) := \lim_{n\to\infty}(p+2d)^{-n}f((p+2d)^n x)$$

for all $x \in \mathsf{A}$. The rest of the proof is similar to the proof of Theorem 3.1. □

Theorem 3.3. *Let θ and $r', s', t', r_1, \dots, r_p, s_1, \dots, s_d$ be positive real numbers, with $r' + s' + t' \neq 3$, and $f : A \to B$ be a mapping such that*

$$\|J_\lambda f(x_1,\dots,x_p,y_1,\dots,y_d)\|_B \le \theta\prod_{j=1}^{p}\|x_j\|_A^{r_j}\cdot\prod_{j=1}^{d}\|y_j\|_A^{s_j} \tag{3.6}$$

and

$$\|f([x,y,z]) - [f(x),y,z] - [x,f(y),z] - [x,y,f(z)]\|_B \leq \theta.\|x\|_A^{r'}.\|y\|_A^{s'}.\|z\|_A^{t'}$$

for all $\lambda \in \mathbb{T}$ *and all* $x, y, z, x_1, \ldots, x_p, y_1, \ldots, y_d \in A$. *In case* $\lambda = 1, \mathbf{i}$ *assume that for each fixed* $x \in A$ *the function* $t \to f(tx)$ *is continuous on* $\mathbb{R}$. *Then there exists a unique derivation* $D : A \to B$ *such that*

$$\|f(x) - D(x)\|_B \leq \frac{2^{(p+d)r}\theta}{|(2(p+2d)^{(p+d)r} - 2^{(p+d)r}(p+2d))|}\|x\|_A^{(p+d)r}$$

for all $x \in A$.

Proof. Letting $\lambda = 1$ and $x_1, \ldots, x_p = y_1 = \cdots = y_d = x$ and $s = 1, t = 2$ in (3.6), then $r_1 = \cdots = r_p = s_1 = \cdots = s_d = r$, so (3.6) is same (3.1) in Theorem 3.1 and 3.2. Therefore, we can continue the proof similar to the proofs of Theorems 3.1 and 3.2. □

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