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Some Remarks on F-weak Multiplication Modules

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Abstract : Let R be a commutative ring with identity and M be a unitary R-module. In this work we introduce a weak multiplication module of certain kind and consider various properties of such a module. Also a number of results concerning associated and supported prime submodules of a module are proved.

Keywords : Weak multiplication module; Primeful module; Associated prime submodule.
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1 Introduction

In this paper all rings are commutative with identity and all modules are unitary. If K and N are submodules of an R-module M we recall that $(N :_R K) = (N : K) = \{r \in R \mid rK \subseteq N\}$, which is an ideal of R. A proper submodule N of an R-module M is said to be prime if for every $r \in R$, $x \in M$; $rx \in N$ implies that $x \in N$ or $r \in (N : M)$. In such a case $p = (N : M)$ is a prime ideal of R and N is said to be p-prime. An R-module M is called a multiplication module if for any submodule N of M there exists an ideal I of R such that $N = IM$. Clearly M is a multiplication module if and only if $N = (N : M)M$ for any submodule N of M. If S is a non-empty subset of an R-module M, the annihilator of S, denoted by $Ann_R(S)$ or simply $Ann(S)$, is defined as $\{r \in R \mid rS = 0\}$. The set of all prime (maximal) submodules of an R-module M is denoted by $Spec(M)$ ($Max(M)$). If

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R is a ring (not necessarily an integral domain) and M is an R-module, the subset $T(M)$ of M is defined by $T(M) = \{m \in M \mid \exists r \neq 0,\ r \in R \text{ such that } rm = 0\}$. Clearly, if R is an integral domain then $T(M)$ is a submodule of M called the torsion submodule. If $T(M) = 0$ we say that M is torsion-free.

In Section 2, we introduce F-weak multiplication modules and then prove a number different results concerning these modules. Section 3 deals with associated and supported prime ideals (submodules) of a module. Here, among other things, we find the supported prime ideals (submodules) of module $S^{-1}M$ in terms of supported prime ideals (submodules) of M itself. Also some relations between the sets of associated prime submodules, supported prime submodules and $Spec(M)$ has been found.

2 Some basic results

We begin this section with the following definitions which have the main role in the whole work.

Definition 2.1. An R-module M is called *weak multiplication* if $Spec(M) = \emptyset$ or for every prime submodule N of M we have $N = IM$, where I is an ideal of R.

It is clear that every multiplication module is weak multiplication. Also if N is a p-prime submodule of a weak multiplication module M it can be shown that $N = pM$.

Definition 2.2. An R-module M is said to be *F-weak multiplication* if it satisfies the following conditions:

(1) M is weak multiplication;

(2) For every $p \in Spec(R)$, pM is a prime submodule of M and $(pM : M) = p$.

For example we can show that the R-module M is F-weak multiplication in the following cases:

(i) M is a finitely generated multiplication R-module such that $Ann(M) \subseteq p$ for every $p \in Spec(R)$. In a very particular case, when M is a free weak multiplication module, it is F-weak multiplication module.

(ii) In (i) we assume $Ann_R(M) = 0$, that is, M is faithful.

Proposition 2.3. *Let M be an F-weak multiplication R-module, where R is a Noetherian ring. Then the number of minimal prime submodule of M is finite.*

Proof. Let $0 = Q_1 \cap ... \cap Q_n$ be a normal primary decomposition of the zero ideal, where Q_i is p_i-primary ($1 \leq i \leq n$). Then all the minimal prime ideals of R can be found in the set $\{p_1, p_2, ..., p_n\}$. Let $\{p_1, p_2, ..., p_k\}$, where $k \leq n$, be the set of minimal prime ideals of R. We know that there is a one-to-one inclusion preserving correspondence between prime ideals of R and prime submodules of M

in such a way that if $p \in Spec(R)$ corresponds to $N \in Spec(M)$ then $N = pM$ and $p = (N : M)$. This implies that $\{p_1M, ..., p_kM\}$ is the set of all minimal prime submodules of M. □

Proposition 2.4. *Let R be a non-trivial ring and $M \neq 0$ be an F-weak multiplication R-module. Then M has a maximal submodule.*

Proof. We know that R has a maximal ideal $\underline{m}$-say. But $\underline{m} \in Spec(R)$ implies that $\underline{m}M \in Spec(M)$ and $(\underline{m}M : M) = \underline{m}$. Let a submodule H of M be such that $\underline{m}M \subseteq H \subsetneqq M$. By [1, Proposition 3], H is an $\underline{m}$-prime submodule of M. Since M is F-weak multiplication, $H = \underline{m}M$ and so $\underline{m}M$ is a maximal submodule of M. □

Remark 2.5. *Let R be a non-trivial ring and M be an F-weak multiplication R-module, then $IM \neq M$ for each proper ideal I of R.*

Proof. Let I be an arbitrary proper ideal of R, then there exists $\underline{m} \in Max(R)$ containing I. Then $IM \subseteq \underline{m}M \subset M$, since $\underline{m}M$ is a prime submodule of M. □

Proposition 2.6. *Let R be an integral domain and M be an F-weak multiplication R-module. Then M is torsion-free.*

Proof. Let $T(M) \neq 0$ so there exists a non-zero element $x \in T(M)$. Since $Ann(x) \neq 0$ there exists $c \in R$, $c \neq 0$ such that $cx = 0$. We know that $(0) \in Spec(R)$ and so $(0)M = 0 \in Spec(M)$. Now $cx = 0$ implies that $x \in (0)M = 0$ or $c \in ((0)M : M) = Ann_R(M) = (0)$. But $c \neq 0, x \neq 0$, a contradiction. Therefore $T(M) = 0$, that is, M is torsion-free. □

Note that under the hypotheses of above proposition we also conclude that M is a faithful R-module.

Corollary 2.7. *Let R be an integral domain and M be an F-weak multiplication R-module. Then every proper direct summand of M is prime. Hence M is indecomposable.*

Proof. By the preceding proposition M is torsion-free and by [1, Result 1], every direct summand of M is a prime submodule. Now we show that M is indecomposable. If $M = M_1 \oplus M_2$ where $M_1,\ M_2 \neq 0$ then by the current form of the corollary, M_1 is a p-prime for some prime ideal p of R. Thus $M_1 = pM = pM_1 \oplus pM_2$. Hence $pM_2 = 0$. Since M is torsion-free and $M_2 \neq 0$, we have $p = 0$ and hence $M_1 = 0$, a contradiction. □

Proposition 2.8. *Let M be an F-weak multiplication R-module and let $I \trianglelefteq R$, $p \in Spec(R)$. If $IM \subseteq pM$ then $I \subseteq p$.*

Proof. If $IM \subseteq pM$ then $(IM : pM) \subseteq (pM : M)$. But $(pM : M) = p$ and clearly $I \subseteq (IM : M)$. Therefore $I \subseteq p$. □

Definition 2.9. An R-module M is called *primeful*, if $M = 0$ or the natural map of $Spec(M)$ is surjective.

We recall that the natural map of $Spec(M)$ is defined as follows:

$$\psi : Spec(M) \to Spec\left(\frac{R}{Ann_R(M)}\right) \text{ such that } \psi(P) = \frac{(P:M)}{Ann_R(M)}, \ \forall P \in Spec(M).$$

Proposition 2.10. *Let M be an F-weak multiplication R-module. Then M is a primeful R-module.*

Proof. Let $M \neq 0$ and ψ be the natural map of $Spec(M)$. We show that ψ is a surjection. Let $\frac{p'}{Ann_R(M)} \in Spec(\frac{R}{Ann_R(M)})$, where $p' \in Spec(R)$ is such that $Ann_R(M) \subseteq p'$. Thus $p'M \in Spec(M)$ and $(p'M : M) = p'$. Hence $\frac{p'}{Ann_R(M)} = \frac{(p'M:M)}{Ann_R(M)} = \psi(p'M)$ and therefore M is primeful. □

Proposition 2.11. *Let M be a non-zero free R-module. Then M is a primeful R-module.*

Proof. It is clear that $Ann_R(M) = 0$. Now we use [2, Result 1.4] to see that the natural map of $Spec(M)$ is surjective. □

Theorem 2.12. *Let M be an R-module and $\psi : Spec(M) \to Spec(\frac{R}{Ann_R(M)})$ be the natural map of $Spec(M)$. Then M is F-weak multiplication in the following cases:*

(i) M is a free R-module and ψ is injective.

(ii) M is a faithful weak multiplication R-module and ψ is surjective.

Proof. (i) Since M is free, for every $p \in Spec(R)$ we have $pM \in Spec(M)$ and $(pM : M) = p$. It remains to show that M is weak multiplication. It is clear that $Ann_R(M) = 0$ and so by the hypothesis $\psi : Spec(M) \to Spec(R)$ is injective. Let $P \in Spec(M)$. We show that $P = (P : M)M$. Since $\psi(P) = (P : M) \in Spec(R)$ and M is free we have $(P : M)M \in Spec(M)$ and hence $\psi(P) = \psi((P : M)M)$. But ψ is injective and hence $P = (P : M)M$.

(ii) It is enough to show that for every $p \in Spec(R)$, $pM \in Spec(M)$ and $(pM : M) = p$. Since $Ann_R(M) = 0$, by the hypothesis $\psi : Spec(M) \to Spec(R)$ is surjective and hence for every $p \in Spec(R)$ there exists $P \in Spec(M)$ such that $\psi(P) = (P : M) = p$. But $P = (P : M)M = pM$ and so $P = pM \in Spec(M)$. Also $(pM : M) = (P : M) = p$ and the proof is complete. □

Lemma 2.13. *Let M be an F-weak multiplication R-module such that every prime submodule of M is finitely generated. Then M is a Noetherian module.*

Proof. We assume that $M \neq 0$. By Proposition 2.4, M has a maximal submodule L-say. Since $L \subsetneqq M$ there exists $x \in M \setminus L$ and by the maximal property of L we have $M = L + Rx$. By [1, Proposition 4], L is a prime submodule of M and as a result finitely generated. Therefore $M = L + Rx$ is also finitely generated. Now by [3, Theorem 2.7], M is a multiplication R-module. The result follows by [4, Theorem 3.2]. □

Definition 2.14. An R-module M is called a *prime cancellation module* or a *p-cancellation module* if for every $p, q \in Spec(R)$, $pM = qM$ implies that $p = q$.

Proposition 2.15. *Let M be an F-weak multiplication R-module. Then M is a p-cancellation module.*

Proof. This is a particular case of Proposition 2.8. □

Theorem 2.16. *Let M be an F-weak multiplication R-module and let M' be an R-module. Let $\phi : M \to M'$ be an epimorphism such that* $\ker \phi$ *is contained in every prime submodules of M. Then M' is an F-weak multiplication R-module.*

Proof. First, let L' be an arbitrary prime submodules of M'. Then there exists a prime submodule L of M such that $\phi(L) = L'$ and so $\phi^{-1}(L') = L$. Since M is F-weak multiplication, there exists an ideal $p \in Spec(R)$ such that $pM = L$. Hence $L = pM = \phi^{-1}(L')$ implies that $\phi(pM) = L'$, that is, $p\phi(M) = L'$ which means $pM' = L'$. Therefore M' is a weak multiplication R-module.

Second, let $p \in Spec(R)$ be an arbitrary prime ideal, we must prove that $pM' \in Spec(M')$ and $(pM' : M') = p$. But $pM' = p\phi(M) = \phi(pM) \leq M'$. Since M is F-weak multiplication, then $pM \in Spec(M)$ and so $pM' = \phi(pM) \in Spec(M')$. Now we must prove that $(pM' : M') = p$. Obviously, $p \subseteq (pM' : M')$. We show that $(pM' : M') \subseteq p$. But $(pM' : M') = (p\phi(M) : \phi(M)) = (\phi(pM) : \phi(M))$. Let $r \in (pM' : M') = (\phi(pM) : \phi(M))$, so $r\phi(M) \subseteq \phi(pM)$, that is, $\phi(rM) \subseteq \phi(pM)$. Since $rM \subseteq \phi^{-1}(\phi(rM)) \subseteq \phi^{-1}(\phi(pM)) = \phi^{-1}(p\phi(M)) = \phi^{-1}(pM') = p\phi^{-1}(M') = pM$, then $rM \subseteq pM$ and so $r \in (pM : M) = p$. Therefore $(pM' : M') \subseteq p$.

Hence, $(pM' : M') = p$ and so M' is an F-weak multiplication R-module. □

Corollary 2.17. *Let M be an F-weak multiplication R-module and N be a submodule of M such that N is contained in every prime submodule of M. Then $\frac{M}{N}$ is an F-weak multiplication R-module.*

Proof. The proof is clear by the above theorem. □

Corollary 2.18. *Let $\{M_i\}, 1 \leq i \leq n$, be a collection of R-modules. If $M = \bigoplus_{i=1}^{n} M_i$ is a weak multiplication R-module, then for every $1 \leq i \leq n$, M_i is a weak multiplication R-module.*

Proof. We define the map ϕ_i as follows:

$$\phi_i : M = \bigoplus_{i=1}^{n} M_i \longrightarrow M_i \ \ , \ (\forall i = 1, ..., n) \ \text{ by}$$

$$\phi_i(m_1, \ldots, m_n) = m_i \ , \ \ \forall (m_1, \ldots, m_n) \in \bigoplus_{i=1}^{n} M_i.$$

Since ϕ_i is an epimorphism, the result follows by the first part of the proof of Theorem 2.16. □

3 Associated and supported primes

We recall some definitions and notions which are needed in the sequel.

Definition 3.1. Let M be an R-module.

(i) The prime ideal p of R is called an *associated prime ideal* of M if for some non-zero $x \in M$, $p = (0 : x) = Ann_R(x)$. The set of all associated prime ideals of M is denoted by $Ass_R(M)$.

(ii) The prime ideal p of R is called a *supported prime ideal* of M if $M_p \neq 0$. The set of all such prime ideals is denoted by $Supp_R(M)$, that is, $Supp_R(M) = \{p \in Spec(R) \mid M_p \neq 0\}$.

It can be proved that

$$Supp_R(M) = \{p \in Spec(R) \mid p \supseteq (0 : x) \ for\ some\ x \in M, \ x \neq 0\}.$$

It is clear that $Ass_R(M) \subseteq Supp_R(M)$. Also for a Noetherian ring R, $p \in Supp_R(M)$ if and only if $p \supseteq q$ for some $q \in Ass_R(M)$, see [5, Chapter IV, Proposition 7].

Definition 3.2. Let M be an R-module and p be a prime ideal of R. We define

$$M(p) = \{x \in M \mid sx \in pM \ \ for\ some \ \ s \in R \backslash p\}.$$

Clearly $M(p)$ is a submodule of M. Also we recall that an R-module M is said to be *weakly finitely generated* if for any $p \in Supp_R(M)$ the submodule $M(p)$ is proper. In this situation it can be shown that $M(p)$ is a p-prime submodule of M.

Definition 3.3. Let M be a weakly finitely generated R-module. The sets of *associated* and *supported prime submodules* of M are defined, respectively, as follows:

$$Ass_P(M) = \{M(p) \mid p \in Ass_R(M)\} \text{ and } Supp_P(M) = \{M(p) \mid p \in Supp_R(M)\}.$$

Lemma 3.4. *Let R be a Noetherian ring and M be an R-module. Then the sets of minimal elements of $Ass_R(M)$ and that of $Supp_R(M)$ are equal.*

Proof. It is clear that $Ass_R(M) \subseteq Supp_R(M)$. If $p_0 \in Ass_R(M)$ is minimal in $Supp_R(M)$, then p_0 is minimal in $Ass_R(M)$. Because if $p \in Ass_R(M)$ and $p \subset p_0$, since $p_0 \in Supp_R(M)$, this contradicts the minimality of p_0 in $Supp_R(M)$. Let $p \in Ass_R(M)$ be minimal in $Ass_R(M)$. If there exists $q_0 \in Supp_R(M)$ such that $q_0 \subset p$, then there exists $p_0 \in Ass_R(M)$ such that $p_0 \subseteq q_0$. But then $p_0 \subset p$, a contradiction to minimality of p in $Ass_R(M)$. Therefore p is minimal in $Supp_R(M)$. Finally, we can show that no element of $Supp_R(M) \backslash Ass_R(M)$ can be minimal in $Supp_R(M)$. □

Theorem 3.5. *Let M be an F-weak multiplication R-module. Then:*

(i) $Spec(R) = Supp_R(M)$, $Spec(M) = Supp_P(M)$ and the map $p \longmapsto pM$ is an order preserving bijection from $Supp_R(M)$ to $Supp_P(M)$, under which $Ass_R(M)$ is mapped to $Ass_P(M)$.

(ii) If R is an integral domain, then $Ass_P(M) = 0$.

(iii) If R is Noetherian, then minimal elements of $Supp_P(M)$ and $Ass_P(M)$ coincide.

Proof. (i): Let $\phi : Spec(R) \to Spec(M)$ be the map defined by $\phi(p) = pM$. By the definition of an F-weak multiplication module and Proposition 2.15, it is clear that ϕ is an order preserving bijection. Also for every prime ideal p of R, we have $M(p) = pM$ which is a prime submodule, thus $\phi(Supp_R(M)) = Supp_P(M)$ and $\phi(Ass_R(M)) = Ass_P(M)$. Thus to prove (i) we just need to show that $Supp_R(M) = Spec(R)$. Let p be a prime ideal of R. Since pM is p-prime, $(pM)_p$ is a prime (and hence proper) submodule of M_p. Therefore $M_p \neq 0$ and $p \in Supp_R(M)$.

(ii): By (i), it is sufficient to show that $Ass_R(M) = 0$. But by Proposition 2.6, M is torsion-free and hence $Ann_R(m) = 0$ for every $0 \neq m \in M$.

(iii): It follows from part (i) and Lemma 3.4. □

Corollary 3.6. *Let M be a finitely generated multiplication R-module. Then $Supp_P(M) = Spec(M)$.*

Proof. If M is finitely generated multiplication as an R-module, then it is so as an $\frac{R}{Ann_R(M)}$-module. Also $Spec(M)$ and $Supp_P(M)$ remain the same if we consider M as $\frac{R}{Ann_R(M)}$-module. Thus we just need to prove the claim for faithful modules. But a faithful finitely generated multiplication module is F-weak multiplication and hence the claim holds by (i) of the above theorem. □

In the rest of our work we prove some results in which the R-module M is not necessarily F-weak multiplication.

Lemma 3.7. *Let S be a multiplicatively closed subset of a ring R and let M be an R-module. Then the set of supported prime ideals of the $S^{-1}R$-module $S^{-1}M$ is equal to:*

$$Supp_{S^{-1}R}(S^{-1}M) = \{S^{-1}p \mid p \in Supp_R(M) \ \text{ and } \ p \cap S = \emptyset\}.$$

Proof. We recall that

$$Supp_R(M) = \{p \in Spec(R) \mid p \supseteq (0 : x) = Ann_R(x) \text{ for some } 0 \neq x \in M\}.$$

Let $p \in Supp_R(M)$ and $p \cap S = \emptyset$. Then $p \supseteq Ann_R(x)$ for some $x \in M$, $x \neq 0$. Hence $S^{-1}p \supseteq S^{-1}(Ann_R(x))$ and $S^{-1}(Ann_R(x)) = S^{-1}(Ann_R(Rx)) = Ann(S^{-1}(Rx))$. It is easy to show that $Ann(S^{-1}(Rx)) = Ann(\frac{x}{s})$, where $s \in S$ (here $\frac{x}{s} \neq 0$ since $p \cap S = \emptyset$). Therefore $S^{-1}p \supseteq Ann(\frac{x}{s})$ and so $S^{-1}p \in Supp_{S^{-1}R}(S^{-1}M)$.

Conversely, let $p' \in Supp_{S^{-1}R}(S^{-1}M)$ then $p' \supseteq Ann(\overline{x})$ for some $\overline{x} \in S^{-1}M$, $\overline{x} \neq 0$. We know there exists a prime ideal p of R such that $S^{-1}p = p'$ (p is the contraction of p' in R). We have $p' \supseteq Ann(S^{-1}(Rx)) = S^{-1}(Ann(Rx)) = S^{-1}(Ann(x))$ and so $S^{-1}p \supseteq S^{-1}(Ann(x))$. This implies $p^S \supseteq (Ann(x))^S$, the S-components of p and $Ann(x)$, respectively. But $p^S = p$ and hence $p \supseteq (Ann(x))^S$. But $Ann_R(x) \subseteq (Ann_R(x))^S$ and consequently $p \in Supp_R(M)$. □

Theorem 3.8. *Let S be a multiplicatively closed subset of a ring R and let M be a weakly finitely generated R-module. Then the set of supported prime submodules of the $S^{-1}R$-module $S^{-1}M$ is equal to:*

$$Supp_P(S^{-1}M) = \{S^{-1}Q \mid Q \in Supp_P(M),\ (Q : M) \cap S = \emptyset\}.$$

Proof. We recall that

$$Supp_P(M) = \{M(p) \mid p \in Supp_R(M)\},$$

$$Supp_P(S^{-1}M) = \{M(p') \mid p' \in Supp_{S^{-1}R}(S^{-1}M)\}.$$

Since M is a weakly finitely generated R-module hence for every $p \in Supp_R(M)$ the submodule $M(p)$ is p-prime. Let $M(p') \in Supp_P(S^{-1}M)$. But $p' \in Supp_{S^{-1}R}(S^{-1}M)$ and so $p' = S^{-1}p$, where $p \in Supp_R(M)$ and $p \cap S = \emptyset$. We have

$$\begin{aligned} M(S^{-1}p) &= M(p') \\ &= \{x' \in S^{-1}M \mid s'x' \in S^{-1}pS^{-1}M \text{ for some } s' \in S^{-1}R\backslash S^{-1}p\} \\ &= \{x' \in S^{-1}M \mid s'x' \in S^{-1}(pM) \text{ for some } s' \in S^{-1}(R\backslash p)\} \\ &= \Big\{\frac{x}{s} \in S^{-1}M \mid x \in M,\ s \in S\ ,\ \frac{\sigma}{\gamma}.\frac{x}{s} \in S^{-1}(pM) \\ &\qquad \text{for some } \sigma \in R\backslash p,\ \gamma \in S\Big\}. \end{aligned}$$

Now $\sigma x \in pM$, $\sigma \in R\backslash p$ and $x \in M$ imply that $x \in M(p)$. Hence

$$M(S^{-1}p) = \{\frac{x}{s} \in S^{-1}M(p) \mid x \in M(p),\ s \in S\} = S^{-1}\overline{p},$$

where $\overline{p} = M(p) \in Supp_P(M)$. Finally we have $(\overline{p} : M) \cap S = (M(p) : M) \cap S = p \cap S = \emptyset$. □

Proposition 3.9. *Let M be a weakly finitely generated R-module and let $\{p_1, ..., p_n\}$ be a subset of the set of minimal elements of $Supp_R(M)$. If $p_1...p_nM = 0$ then $p_1, ..., p_n$ are the only minimal elements of $Supp_R(M)$.*

Proof. Since $p_1...p_nM = 0$ we have $p_1...p_n \subseteq Ann_R(M)$. Now let p be any minimal element of $Supp_R(M)$. Then $p \supseteq Ann_R(x)$ for some $x \in M$, $x \neq 0$. Hence $p_1...p_n \subseteq Ann_R(M) \subseteq Ann_R(x) \subseteq p$. This implies $p_i \subseteq p$ for some $1 \leq i \leq n$. But by minimality of p we have $p = p_i$. □

Theorem 3.10. *Let R be a Noetherian ring and let $\{M_i\}_{i\in I}$ be a family of R-modules in which M_i is weakly finitely generated for every $i \in I$. Then:*

$$Ass_P\left(\bigoplus_{i\in I} M_i\right) = \left\{\bigoplus_{i\in I} M_i(p) \mid M_j(p) \in Ass_P(M_j) \textit{ for some } j \in I\right\}.$$

Proof. Since each M_i is weakly finitely generated, for any $p \in Supp_R(M_i)$, the submodule $M_i(p)$ is p-prime. Now we have,

$$\begin{aligned} Ass_P\left(\bigoplus_{i\in I} M_i\right) &= \left\{\left(\bigoplus_{i\in I} M_i\right)(p) \mid p \in Ass_R\left(\bigoplus_{i\in I} M_i\right)\right\} \\ &= \left\{\bigoplus_{i\in I} M_i(p) \mid p \in Ass_R\left(\bigoplus_{i\in I} M_i\right)\right\} \\ &= \left\{\bigoplus_{i\in I} M_i(p) \mid p \in \bigcup_{i\in I} Ass_R(M_i)\right\} \\ &= \left\{\bigoplus_{i\in I} M_i(p) \mid p \in Ass_R(M_j) \text{ for some } j \in I\right\}, \end{aligned}$$

and by using the definition of $Ass_P(M)$, for an R-module M, we have

$$Ass_P\left(\bigoplus_{i\in I} M_i\right) = \left\{\bigoplus_{i\in I} M_i(p) \mid M_j(p) \in Ass_P(M_j) \text{ for some } j \in I\right\}.$$

The proof is now complete. □

Here we recall that an R-module M is called a *quasi multiplication module* if $M(p) = pM$, for all $p \in Supp_R(M)$. Also it is clear that every F-weak multiplication R-module is a quasi multiplication module.

Theorem 3.11. *Let R be a Noetherian ring and M be a quasi multiplication R-module. Let $p \in Spec(R)$ be such that $M(p) \in Spec(M)$. Then $M(p) \in Supp_P(M)$ if and only if $M(p) \supseteq Q$ for some $Q \in Ass_P(M)$.*

Proof. Let $M(p) \in Supp_P(M)$ then $M(p) = pM$ and $p \in Supp_R(M)$. Since R is Noetherian, $p \supseteq q$ for some $q \in Ass_R(M)$. But $Q = M(q) = qM \in Ass_P(M)$. Hence $pM \supseteq qM$ implies that $M(p) \supseteq Q$. On the other hand, let $M(p) \supseteq Q$ for some $Q \in Ass_P(M)$. Then $Q = M(q) = qM$ for some $q \in Ass_R(M)$. But $M(p) \supseteq Q$ implies that $(M(p) : M) \supseteq (Q : M)$, that is, $p \supseteq q$. Also $q = Ann_R(x)$ for some $x \in M$, $x \neq 0$. Therefore we have $p \in Supp_R(M)$ and hence $M(p) \in Supp_P(M)$. ☐

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