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Convergence Criteria of Viscosity Common Fixed Point Iterative Process for Asymptotically Nonexpansive Nonself Mappings in Banach Spaces[1]

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Abstract : Let X be a real arbitrary Banach space and let C be a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. For $i = 1, 2$, let $T_i : C \to X$ be an asymptotically nonexpansive nonself mapping such that $F(T_1) \cap F(T_2) \neq \emptyset$ in C. Let $f : C \to C$ be a contractive mapping and let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0, 1]$. Define $\{x_n\}$ and $\{y_n\}$ to be the iterative sequences

$$\begin{aligned} y_n &= P(\alpha_n f(x_n) + (1-\alpha_n)(\beta_n x_n + (1-\beta_n)T_2(PT_2)^{n-1}x_n)), \\ x_{n+1} &= P(\gamma_n f(y_n) + (1-\gamma_n)(\delta_n y_n + (1-\delta_n)T_1(PT_1)^{n-1}y_n)), \ \ n \geq 1. \end{aligned}$$

Some strong convergence theorems of the sequence $\{x_n\}$ to a common fixed point of T_1 and T_2 are established under appropriate conditions.

Keywords : Asymptotically nonexpansive mapping; Nonexpansive retraction; Banach space; Common fixed point.
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1 Introduction

The concept of asymptotically nonexpansive self-mappings which is a generalization of the class of nonexpansive self-mappings was first introduced in 1972 by Goebel and Kirk [1]. They proved that any asymptotically nonexpansive self-mapping of a nonempty closed convex bounded subset of a uniformly convex Banach space possesses a fixed point. Since then, the weak and strong convergence problems of iterative sequences (with errors) for asymptotically nonexpansive self-mappings have been studied by many authors (see, for examples, [2–7]). In 2003, Chidume et al. [8] introduced the concept of asymptotically nonexpansive nonself-mappings. Such a nonself mapping is defined as follows. Let X be a real normed space, C a nonempty subset of X and $P : X \to C$ the nonexpansive retraction of X onto C. A nonself mapping $T : C \to X$ is called *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq k_n\|x - y\|,$$

for all $x, y \in C$ and $n \geq 1$. They proved the following.

Theorem 1.1. *Let E be a real uniformly convex Banach space, K closed convex nonempty subset of E. Let $T : K \to E$ be completely continuous and asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1, \infty)$ such that $\sum_{n\geq1}({k_n}^2 - 1) < \infty$ and $F(T) \neq \emptyset$. Let $\alpha_n \in (0, 1)$ be such that $\epsilon \leq 1 - \alpha_n \leq 1 - \epsilon$, $\forall n \geq 1$ and some $\epsilon > 0$. From arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by*

$$x_n = P((1 - \alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n), \;\; n \geq 1.$$

Then $\{x_n\}$ converges strongly to some fixed point of T.

They also proved the following theorem which was about the weak convergence of $\{x_n\}$ to some fixed point of T.

Theorem 1.2. *Let E be a real uniformly convex Banach space which has a Frechet differentiable norm, K closed convex nonempty subset of E. Let $T : K \to E$ be asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1, \infty)$ such that $\sum_{n\geq1}({k_n}^2 - 1) < \infty$ and $F(T) \neq \emptyset$. Let $\alpha_n \in (0, 1)$ be such that $\epsilon \leq 1 - \alpha_n \leq 1 - \epsilon$, $\forall n \geq 1$ and some $\epsilon > 0$. From arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by*

$$x_n = P((1 - \alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n), \;\; n \geq 1.$$

Then $\{x_n\}$ converges weakly to some fixed point of T.

Recently, in 2008, Lou et al. [6] studied the viscosity approximation fixed point for asymptotically nonexpansive self-mappings in Banach spaces. They proved the following theorems.

Theorem 1.3. *Let K be a nonempty closed convex subset of a Banach space X which has a uniformly Gâteaux differentiable norm and $T : K \to K$ an asymptotically nonexpansive mapping with $F(T) \neq \emptyset$ and f a contraction on C. Let $\{\alpha_n\}, \{\beta_n\}$ be sequences in $(0,1)$ satisfying*

$$C1 : \lim_{n\to\infty} \alpha_n = 0; \;\; C2 : \lim_{n\to\infty} \frac{k_n - 1}{\alpha_n} = 0.$$

Then the sequence $\{z_n\}$ defined by

$$z_{n+1} = \alpha_n f(z_n) + (1-\alpha_n)T^n z_n,$$

converges strongly to the unique solution of the variational inequality:

$$p \in F(T) \text{ such that } \langle (I-f)p, j(p-x^*)\rangle \leq 0 \;\; \forall x^* \in F(T).$$

Theorem 1.4. *Let K be a nonempty closed convex subset of a uniformly convex Banach space X which has a uniformly Gâteaux differentiable norm and $T : K \to K$ an asymptotically nonexpansive mapping with $F(T) \neq \emptyset$ and f a contraction on C. Let $\{\alpha_n\}, \{\beta_n\}$ be sequences in $(0,1)$ satisfying*

$$C1 : \lim_{n\to\infty} \alpha_n = 0; \;\; C2 : \sum_{n=1}^{\infty} \alpha_n = \infty \;\; C3 : \lim_{n\to\infty} \frac{k_n - 1}{\alpha_n} = 0.$$

For arbitrary $x_0 \in K$, let the sequence $\{x_n\}$ be defined iteratively by

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T^n x_n.$$

Assume

(i) $\alpha_n, \beta_n, \gamma_n \in [0,1], \alpha_n + \beta_n + \gamma_n = 1$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iii) *T satisfies the asymptotically regularity;* $\lim_{n\to\infty} \|T^{n+1}x_n - T^n x_n\| = 0$.

Then the sequence $\{x_n\}$ converges strongly to the unique solution of the variational inequality:

$$p \in F(T) \text{ such that } \langle (I-f)p, j(p-x^*)\rangle \leq 0 \;\; \forall x^* \in F(T).$$

2 Preliminaries

In this paper, we study a viscosity approximation for some common fixed point of asymptotically nonexpansive nonself mappings in Banach spaces as follows.

Let X be a real arbitrary Banach space and let C be a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. A mapping $f : C \to C$ is called a *contractive mapping* if there exists a constant $\alpha \in (0,1)$ such that

$$\|f(x) - f(y)\| \leq \alpha \|x - y\|,$$

for all $x, y \in C$. We use $d(x, F)$ for the distance from the point x to the set F and $F(T)$ for the set of all fixed points of the mapping T. For $i = 1, 2$, let $T_i : C \to X$ be an asymptotically nonexpansive nonself mapping such that $F(T_1) \cap F(T_2) \neq \emptyset$. Let $f : C \to C$ be a contractive mapping and let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\delta_n\}$ be real sequences in $[0, 1]$. For arbitrary $x_1 \in C$, let $\{x_n\}$ and $\{y_n\}$ be the iterative sequences defined by

$$\begin{aligned} y_n &= P(\alpha_n f(x_n) + (1-\alpha_n)(\beta_n x_n + (1-\beta_n)T_2(PT_2)^{n-1}x_n)), \\ x_{n+1} &= P(\gamma_n f(y_n) + (1-\gamma_n)(\delta_n y_n + (1-\delta_n)T_1(PT_1)^{n-1}y_n)), \; n \geq 1. \end{aligned} \tag{2.1}$$

Here, for convenience, we use the following definition of asymptotically nonexpansive nonself mapping. A nonself mapping $T : C \to X$ is called *asymptotically nonexpansive* if there exists a sequence $\{r_n\} \subset [0, 1)$ with $r_n \to 0$ as $n \to \infty$ such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq (1 + r_n)\|x - y\|,$$

for all $x, y \in C$ and $n \geq 1$.

We need the following lemmas for the main results in this paper.

Lemma 2.1 ([9, Lemma 2.1]). *Let $\{a_n\}, \{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n \text{ for all } n.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then

(1) $\lim_{n\to\infty} a_n$ *exists.*

(2) $\lim_{n\to\infty} a_n = 0$ *if $\{a_n\}$ has a subsequence converging to zero.*

Lemma 2.2. Let C be a nonempty closed subset of a Banach space X and $T : C \to X$ be an asymptotically nonexpansive nonself mapping with the fixed point set $F(T) \neq \emptyset$. Then $F(T)$ is a closed subset in C.

Proof. Assume that $T : C \to X$ is an asymptotically nonexpansive nonself mapping with respect to $\{r_n\}$. Let $\{p_n\}$ be a sequence in $F(T)$ such that $p_n \to p$ as $n \to \infty$. Since C is closed and $\{p_n\}$ is a sequence in C, we must have $p \in C$. Since $T : C \to X$ is asymptotically nonexpansive, we obtain

$$\|Tp - p_n\| = \|Tp - Tp_n\| \leq (1 + r_1)\|p - p_n\|.$$

Taking limit as $n \to \infty$ and using the continuity of the norm, we obtain $\|Tp - p\| \leq 0$, which implies that $Tp = p$. The proof is complete. ∎

3 Main Results

In this section, we present our main results. The first theorem gives the necessary and sufficient condition for the convergence of the sequence $\{x_n\}$ defined by (2.1).

Theorem 3.1. *Let X be a real arbitrary Banach space and let C be a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. For $i = 1, 2$, let $T_i : C \to X$ be an asymptotically nonexpansive nonself mapping with respect to $\{r_i^{(n)}\}$ such that $F(T_1) \cap F(T_2) \neq \emptyset$ and $\sum_{n=1}^{\infty} r_n < \infty$, where $r_n = max\{r_1^{(n)}, r_2^{(n)}\}$. Let $f : C \to C$ be a contractive mapping and let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\delta_n\}$ be real sequences in $[0, 1]$ such that $\sum_{n=1}^{\infty} \alpha_n < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$. Then, the iterative sequence $\{x_n\}$ defined by (2.1) converges to a common fixed point of T_1 and T_2 if and only if $\liminf_{n\to\infty} d(x_n, F(T_1) \cap F(T_2)) = 0$.*

Proof. The necessity is obvious, so it is omitted. We now prove the sufficiency. Assume that $T_i : C \to X$ is an asymptotically nonexpansive nonself mapping with respect to $\{r_n^{(i)}\}$. Let $p \in F(T_1) \cap F(T_2)$. Note that $T_i(PT_i)^{n-1}p = p$. By assumption, we have

$$\begin{aligned}
\|y_n - p\| &= \|P(\alpha_n f(x_n) + (1-\alpha_n)(\beta_n x_n + (1-\beta_n)T_2(PT_2)^{n-1}x_n)) - Pp\| \\
&\le \|\alpha_n f(x_n) + (1-\alpha_n)(\beta_n x_n + (1-\beta_n)T_2(PT_2)^{n-1}x_n) - p\| \\
&\le \alpha_n\|f(x_n) - p\| + (1-\alpha_n)\beta_n\|x_n - p\| \\
&\quad + (1-\alpha_n)(1-\beta_n)\|T_2(PT_2)^{n-1}x_n - p\| \\
&\le \alpha_n\alpha\|x_n - p\| + \alpha_n\|f(p) - p\| + (1-\alpha_n)\beta_n\|x_n - p\| \\
&\quad + (1-\alpha_n)(1-\beta_n)(1 + r_n^{(2)})\|x_n - p\| \\
&\le \alpha_n\alpha\|x_n - p\| + \alpha_n\|f(p) - p\| + (1-\alpha_n)\beta_n\|x_n - p\| \\
&\quad + (1-\alpha_n)(1-\beta_n)\|x_n - p\| + r_n^{(2)}(1-\alpha_n)(1-\beta_n)\|x_n - p\| \\
&\le (1 - (1-\alpha)\alpha_n + r_n)\|x_n - p\| + \alpha_n\|f(p) - p\| \\
&\le (1 + r_n)\|x_n - p\| + \alpha_n\|f(p) - p\|. \qquad (3.1)
\end{aligned}$$

Similarly we have that

$$\|x_{n+1} - p\| \le (1 + r_n)\|y_n - p\| + \gamma_n\|f(p) - p\|.$$

From this and (3.1), we have

$$\begin{aligned}
\|x_{n+1} - p\| &\le (1 + r_n)\{(1 + r_n)\|x_n - p\| + \alpha_n\|f(p) - p\|\} + \gamma_n\|f(p) - p\| \\
&\le (1 + r_n)(1 + r_n)\|x_n - p\| + [(1 + r_n)\alpha_n + \gamma_n]\|f(p) - p\| \\
&\le (1 + r_n(2 + r_n))\|x_n - p\| + [(1 + r_n)\alpha_n + \gamma_n]\|f(p) - p\| \\
&= (1 + c_n)\|x_n - p\| + b_n, \qquad (3.2)
\end{aligned}$$

where $c_n = r_n(2+r_n)$ and $b_n = [(1+r_n)\alpha_n+\gamma_n]\|f(p)-p\|$. Since $\sum_{n=1}^{\infty} r_n < \infty$, we have that $\{2+r_n\}$ and $\{1+r_n\}$ are bounded. Thus $\sum_{n=1}^{\infty} c_n < \infty$ and $\sum_{n=1}^{\infty} b_n <$

∞ because $\sum_{n=1}^{\infty} \alpha_n < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$. Hence Lemma 2.1 implies that $\lim_{n\to\infty} \|x_n - p\|$ exists. Thus $\{x_n\}$ is bounded and so are $\{T_1(PT_2)^{n-2}x_n\}$ and $\{f(x_n)\}$ because T_1 is asymptotically nonexpansive and f is contractive. Now since $\{x_n\}$ is bounded and from (3.1), we conclude that $\{y_n\}$ is bounded and so are $\{T_1(PT_1)^{n-1}y_n\}$ and $\{f(y_n)\}$.

We next turn to another calculation for $\|y_n - p\|$ and $\|x_{n+1} - p\|$ as follows.

$$\begin{aligned}
\|y_n - p\| &= \|P(\alpha_n f(x_n) + (1-\alpha_n)(\beta_n x_n + (1-\beta_n)T_2(PT_2)^{n-1}x_n)) - Pp\| \\
&\le \|\alpha_n f(x_n) + (1-\alpha_n)(\beta_n x_n + (1-\beta_n)T_2(PT_2)^{n-1}x_n) - p\| \\
&\le \alpha_n\|f(x_n) - T_2(PT_2)^{n-1}x_n\| + (1-\alpha_n)\beta_n\|x_n - p\| \\
&\quad + (1-\beta_n+\alpha_n\beta_n)\|T_2(PT_2)^{n-1}x_n - p\| \\
&\le \alpha_n\|f(x_n) - T_2(PT_2)^{n-1}x_n\| + (1-\alpha_n)\beta_n\|x_n - p\| \\
&\quad + (1-\beta_n+\alpha_n\beta_n)(1+r_n^{(2)})\|x_n - p\| \\
&\le \alpha_n\|f(x_n) - T_2(PT_2)^{n-1}x_n\| + (1-\alpha_n)\beta_n\|x_n - p\| \\
&\quad + (1-\beta_n+\alpha_n\beta_n)\|x_n - p\| + r_n^{(2)}(1-\beta_n+\alpha_n\beta_n)\|x_n - p\| \\
&= (1 + r_n(1+\alpha_n\beta_n))\|x_n - p\| + \alpha_n\|f(x_n) - T_2(PT_2)^{n-1}x_n\| \\
&\le (1+2r_n)\|x_n - p\| + \alpha_n\|f(x_n) - T_2(PT_2)^{n-1}x_n\|. \qquad (3.3)
\end{aligned}$$

Similarly, we have that

$$\|x_{n+1} - p\| \le (1+2r_n)\|y_n - p\| + \gamma_n\|f(y_n) - T_1(PT_1)^{n-1}y_n\|. \qquad (3.4)$$

Putting (3.3) in (3.4), we obtain that

$$\begin{aligned}
\|x_{n+1} - p\| &\le (1+2r_n)^2\|x_n - p\| + (1+2r_n)\alpha_n\|f(x_n) - T_2(PT_2)^{n-1}x_n\| \\
&\quad + \gamma_n\|f(y_n) - T_1(PT_1)^{n-1}y_n\| \\
&= (1+d_n)\|x_n - p\| + e_n, \qquad (3.5)
\end{aligned}$$

where $d_n = 4r_n(1+r_n)$ and $e_n = (1+2r_n)\alpha_n\|f(x_n) - T_2(PT_2)^{n-1}x_n\| + \gamma_n\|f(y_n) - T_1(PT_1)^{n-1}y_n\|$. By the assumption that $\sum_{n=1}^{\infty} r_n < \infty$, $\sum_{n=1}^{\infty} \alpha_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n < \infty$, and $\{T_2(PT_2)^{n-1}x_n\}$, $\{T_1(PT_1)^{n-1}y_n\}$, $\{f(x_n)\}$ and $\{f(y_n)\}$ are bounded, we have that $\sum_{n=1}^{\infty} d_n < \infty$ and $\sum_{n=1}^{\infty} e_n < \infty$. Hence Lemma 2.1 tells us that $\lim_{n\to\infty} \|x_n - p\|$ exists. Thus $\{\|x_n - p\|\}$ is bounded. Let $L = \sup_n \|x_n - p\|$. We can rewrite (3.5) as

$$\|x_{n+1} - p\| \le \|x_n - p\| + Ld_n + e_n \quad \text{for} \quad n \ge 1. \qquad (3.6)$$

From this and by induction, we obtain, for $m, n \ge 1$ and $p \in F(T_1) \cap F(T_2)$, that

$$\|x_{n+m} - p\| \le \|x_n - p\| + L\sum_{i=n}^{n+m-1} d_i + \sum_{i=n}^{n+m-1} e_i. \qquad (3.7)$$

Also from (3.6), we obtain

$$d(x_{n+1}, F(T_1) \cap F(T_2)) \le d(x_n, F(T_1) \cap F(T_2)) + Ld_n + e_n.$$

But, the assumption $\liminf_{n\to\infty} d(x_n, F(T_1)\cap F(T_2)) = 0$ implies that there exists a subsequence of $\{d(x_n, F(T_1)\cap F(T_2))\}$ converging to zero. From this and because $\sum_{n=1}^{\infty}(Ld_n + e_n) < \infty$, Lemma 2.1 tells us that

$$\lim_{n\to\infty} d(x_n, F(T_1)\cap F(T_2)) = 0. \tag{3.8}$$

We now show that $\{x_n\}$ is a Cauchy sequence in X. Let $\epsilon > 0$. From (3.8), $\sum_{n=1}^{\infty} d_n < \infty$ and $\sum_{n=1}^{\infty} e_n < \infty$, there exists n_0 such that, for $n \geq n_0$, we have

$$d(x_n, F(T_1)\cap F(T_2)) < \epsilon/6, \sum_{i=n}^{\infty} d_i < \epsilon/(3L) \text{ and } \sum_{i=n}^{\infty} e_i < \epsilon/3. \tag{3.9}$$

By the first inequality in (3.9) and the definition of infimum, there exists $p_0 \in F(T_1)\cap F(T_2)$ such that

$$\|x_{n_0} - p_0\| < \epsilon/6. \tag{3.10}$$

Combining (3.6), (3.9) and (3.10), we obtain

$$\begin{aligned} \|x_{n_0+m} - x_{n_0}\| &\leq \|x_{n_0+m} - p_0\| + \|x_{n_0} - p_0\| \\ &\leq 2\|x_{n_0} - p_0\| + L\sum_{i=n_0}^{n_0+m-1} d_i + \sum_{i=n_0}^{n_0+m-1} e_i \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon, \end{aligned}$$

which implies that $\{x_n\}$ is a Cauchy sequence in X. But X is a Banach space, so there must be some $q \in X$ such that $x_n \to q$. Since C is closed and $\{x_n\}$ is a sequence in C, we have that $q \in C$. Now $d(x_n, F(T_1)\cap F(T_2)) \to 0$ and $x_n \to q$ as $n \to \infty$, the continuity of $d(\cdot, F(T_1)\cap F(T_2))$ implies that $d(q, F(T_1)\cap F(T_2)) = 0$. Thus $q \in F(T_1)\cap F(T_2)$ because $F(T_1)\cap F(T_2)$ is closed, by Lemma 2.2. Therefore $\{x_n\}$ converges to a common fixed point of T_1 and T_2, as desired. □

If $T_1 = T_2 = T$, then the iterative sequences (2.1) become

$$\begin{aligned} y_n &= P(\alpha_n f(x_n) + (1-\alpha_n)(\beta_n x_n + (1-\beta_n)T(PT)^{n-1}x_n)), \\ x_{n+1} &= P(\gamma_n f(y_n) + (1-\gamma_n)(\delta_n y_n + (1-\delta_n)T(PT)^{n-1}y_n)), \quad n \geq 1. \end{aligned} \tag{3.11}$$

We then have the following result for a fixed point of a single asymptotically nonexpansive nonself mapping.

Corollary 3.2. *Let X be a real Banach space and let C be a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T : C \to X$ be an asymptotically nonexpansive nonself mapping with respect to $\{r_n\}$ such that $F(T) \neq \emptyset$ in C and $\sum_{n=1}^{\infty} r_n < \infty$. Let $f : C \to C$ be a contractive mapping and let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\delta_n\}$ be real sequences in $[0,1]$ such that $\sum_{n=1}^{\infty} \alpha_n < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$. Then, the sequence $\{x_n\}$, defined by (3.11), converges to a fixed point of T if and only if $\liminf_{n\to\infty} d(x_n, F(T)) = 0$.*

We also have the following results involving asymptotic regularity as in Lou et al. [6] and an auxiliary strictly increasing nonnegative function as in Ayaragarnchanakul [10].

Corollary 3.3. *Let X, C, T_i $(i = 1, 2)$ and the iterative sequence $\{x_n\}$ be as in Theorem 3.1. Suppose that the conditions in Theorem 3.1 hold and*

(1) the mapping T_i $(i = 1, 2)$ is asymptotically regular in x_n, i.e.,

$$\liminf_{n\to\infty} \|x_n - T_i x_n\| = 0, \quad i = 1, 2;$$

(2) $\liminf_{n\to\infty} \|x_n - T_i x_n\| = 0$ implies that

$$\liminf_{n\to\infty} d(x_n, F(T_1) \cap F(T_2)) = 0.$$

Then the sequences $\{x_n\}$ converges to a common fixed point of T_1 and T_2.

Theorem 3.4. *Let X, C, T_i $(i = 1, 2)$ and the iterative sequence $\{x_n\}$ be as in Theorem 3.1. Suppose that the conditions in Theorem 3.1 hold, the mapping T_i is asymptotically regular in x_n, and there exists an increasing function $g : R^+ \to R^+$ with $g(r) > 0$ for all $r > 0$ such that for $i = 1, 2$,*

$$\|x_n - T_i x_n\| \geq g(d(x_n, F(T_1) \cap F(T_2)), \quad \forall n \geq 1.$$

Then the sequence $\{x_n\}$ converges to a common fixed point of T_1 and T_2.

Proof. To apply Theorem 3.1, we prove that $\liminf_{n\to\infty} d(x_n, F(T_1) \cap F(T_2)) = 0$. From the assumption that $\|x_n - T_i x_n\| \geq g(d(x_n, F(T_1) \cap F(T_2)))$ for $i = 1, 2$ and for all $n \geq 1$, we have

$$\frac{1}{2}\sum_{i=1}^{2} \|x_n - T_i x_n\| \geq g(d(x_n, F(T_1) \cap F(T_2))),$$

for all $n \geq 1$. Since T_i is asymptotically regular in x_n, this implies that

$$\liminf_{n\to\infty} g(d(x_n, F(T_1) \cap F(T_2))) = 0. \tag{3.12}$$

Suppose that $\liminf_{n\to\infty} d(x_n, F(T_1) \cap F(T_2)) = L > 0$. By definition of infimum, there exists an N such that

$$\left| \inf_{n\geq m} d(x_n, F(T_1) \cap F(T_2)) - L \right| < \frac{L}{2}, \text{ for all } m \geq N.$$

Equivalently,

$$d(x_n, F(T_1) \cap F(T_2)) > \frac{L}{2}, \text{ for all } n \geq m \geq N.$$

Since g is increasing, we have that

$$g(d(x_n, F(T_1) \cap F(T_2))) \geq g\left(\frac{L}{2}\right), \quad \text{for all } n \geq m \geq N.$$

This implies that

$$\liminf_{n\to\infty} g(d(x_n, F(T_1) \cap F(T_2))) \geq g\left(\frac{L}{2}\right) > 0,$$

which contradicts (3.12). Hence $\liminf_{n\to\infty} d(x_n, F(T_1) \cap F(T_2)) = 0$, as desired. □

If T_i is a self-mapping, then the iterative sequences (2.1) become

$$\begin{aligned} y_n &= \alpha_n f(x_n) + (1-\alpha_n)(\beta_n x_n + (1-\beta_n)T_2^n x_n), \\ x_{n+1} &= \gamma_n f(y_n) + (1-\gamma_n)(\delta_n y_n + (1-\delta_n)T_1^n y_n), \quad n \geq 1. \end{aligned} \tag{3.13}$$

We have the following theorem for common fixed point of two asymptotically nonexpansive self-mappings.

Corollary 3.5. *Let X be a real Banach space and let C be a nonempty closed convex subset of X. For $i = 1, 2$, let $T_i : C \to C$ be an asymptotically nonexpansive self-mapping with respect to $\{r_i^{(n)}\}$ such that $F(T_1) \cap F(T_2) \neq \emptyset$ and $\sum_{n=1}^{\infty} r_n < \infty$, where $r_n = max\{r_1^{(n)}, r_2^{(n)}\}$. Let $f : C \to C$ be a contractive mapping and let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\delta_n\}$ be real sequences in $[0,1]$ such that $\sum_{n=1}^{\infty} \alpha_n < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$. Then, the iterative sequence $\{x_n\}$ defined by (3.13) converges to a common fixed point of T_1 and T_2 if and only if $\liminf_{n\to\infty} d(x_n, F(T_1) \cap F(T_2)) = 0$.*

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