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Common Fixed Point Theorem for Hybrid Generalized Multivalued[1]

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Abstract : In this work, the common fixed point theorems for a pair of hybrid generalized multivalued φ-weak contraction are proven. Consequently, since the concept of hybrid generalized multivalued φ-weak contraction includes almost concepts of the generalizations of Banach contraction principle as special cases, our results can be viewed as a refinement and improvement of the previously known results for metric fixed-point theory.

Keywords : Common fixed points; Hybrid generalized multivalued φ-weak contractions; Bianchini-Grandolfi gauge function; Hausdorff pseudometric.
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1 Introduction and Preliminaries

Let E be a complete metric space with distance $d(\cdot,\cdot)$. Let 2^E denote the family consisting of all nonempty subsets of E. We define the Hausdorff pseudometric, $H : 2^E \times 2^E \to [0,\infty]$ by

$$H(A,B) = \max\{D(a,B), D(A,b)\},$$

where $D(a,B) = \inf\limits_{b\in B} d(a,b)$, $\quad D(A,b) = \inf\limits_{a\in A} d(a,b)$.

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Definition 1.1. Let E be a metric space. A subset $C \subset E$ is said to be *approximative* if the multivalued mapping

$$\mathcal{P}_C(x) = \{c \in C : d(x,c) = D(x,C)\}, \quad \forall x \in E$$

has nonempty values. The multivalued mapping $T : E \to 2^E$ is said to have *approximative values* if $T(x)$ is approximative for each $x \in E$.

Let $\propto \in (0,\infty], \mathcal{R}_{\propto}^{+} = [0,\propto)$. Let $\varphi : \mathcal{R}_{\propto}^{+} \to [0,\infty)$ satisfy

(i) $\varphi(t) < t$ for each $t \in (0,\propto)$;

(ii) φ is nondecreasing on $\mathcal{R}_{\propto}^{+}$;

(iii) φ is upper-semicontinuous.

Define $\Phi[0,\propto) = \{\varphi : \varphi \text{ satisfies (i)-(iii) above}\}$.

From now on, for a metric space E, we let $\Gamma = \sup\{d(x,y) : x,y \in E\}$ and set $\propto = \Gamma$ if $\Gamma = \infty$, and $\propto > \Gamma$ if $\Gamma < \infty$.

Definition 1.2. Let E be a matric space. Suppose that $S,T : E \to 2^E$ and $\varphi \in \Phi[0,\propto)$ satisfy

$$H(Sx,Ty) \leq \varphi(\rho(x,y)),$$

for each $x,y \in E$, where

$$\rho(x,y) = \max\left\{d(x,y), D(Sx,x), D(Ty,y), \frac{1}{2}\left[D(y,Sx) + D(x,Ty)\right]\right\}.$$

Then the pair S,T is called the *hybrid generalized multivalued φ-weak contraction mapping.*

Remark 1.3. *Let E be a Banach algebra with the norm $\|\cdot\|$ and the metric $d(\cdot,\cdot)$ generated by it. In Definition 1.2, let $\rho(x,y) = d(x,y)$; so*

$$H(Tx,Ty) \leq \varphi(d(x,y))$$

for all $x,y \in E$. Then the multivalued mapping T is called a nonlinear D-contraction with a contraction function φ (see [1, 2]). In addition, let $\varphi(t) = kt$ with $k > 0$ and $\rho(x,y) = d(x,y)$; then

$$H(Tx,Ty) \leq \varphi(d(x,y))$$

for all $x,y \in E$. In this case the mapping T is nothing but the multivalued Lipschitz operator defined by [3]. Moreover, if $0 < k < 1$ then the mapping T is called a multivalued contraction on E which was first studied by Markin [4] and Nadler [5].

During the last few decades, since the pioneering works of Markin [4] and Nadler [5], an extensive literature has been developed, consisting in many theorems which deal with fixed points for multi-valued mappings (see [6, 7, 8, 9, 10]), or may be related to various classes of φ-contractions, which are obtained for different collection of properties of the function φ (see for example, [11, 12, 13]), especially the monograph of Rus [14, 15], for the good survey and several still open problems. Equally important is the concept of hybrid contractive mapping of the metric fixed-point theory which have been obtained by mathematical researcher, for example [16, 17, 18, 19, 20].

Motivated and spirted by the research going on this field, in this work we prove that there is a common fixed point of hybrid generalized multivalued φ-weak contractions S, T on complete metric spaces E. Since the concept of hybrid generalized multivalued φ-weak contraction includes almost concepts of the generalization of Banach contraction principle as special cases (both singlevalued and multivalued settings), results obtained in this paper continue to hold for those problems. Our results can be viewed as a refinement and improvement of the previously known results for metric fixed-point theory. To reach the goal, we also need the following concepts:

Let J denotes an interval on $[0, \infty)$ containing 0, that is an interval of the form $[0, r], [0, r)$ or $[0, \infty)$, and we use the abbreviation φ^n for the nth iterate of a function φ.

Definition 1.4. A nondecreasing function $\varphi : J \to J$ is said to be a *Bianchini-Grandolfi gauge function* [21] *on* J if $\Sigma_{n=0}^{\infty}\varphi^n(t) < \infty$ for all $t \in J$.

As for the investigations of the Bianchini-Grandolfi gauge function we also refer to [22]. The following lemma is quite important one.

Lemma 1.5 ([23])**.** *Let E be a metric space and B be a nonempty subset of E. Then $D(x, B) \leq d(x, y) + D(y, B)$, for any $x, y \in E$.*

2 Common Fixed Point Theorems

Theorem 2.1. *Let (E, d) be a complete metric space. Let S, T be a pair of hybrid generalized multivalued φ-weak contractions on E. Assume that S, T have the approximative values and $\varphi_{|_J}$ is a Bianchini-Grandolfi gauge function on some interval $J \subset \mathcal{R}^+_\propto$. If there is $x \in E$ such that either $D(x, Sx) \in J$ or $D(x, Tx) \in J$ then the mappings S and T have a common fixed point $u \in E$.*

Proof. Without loss of generality, we will assume that there is $u_0 \in E$ such that $D(u_0, Su_0) \in J$. Take $u_0 \in E$, since Su_0 is approximative it follows that there exists $u_1 \in Su_0$ such that $d(u_0, u_1) = D(u_0, Su_0)$. Next, since Tu_1 is approximative, there exists $u_2 \in Tu_1$ such that $d(u_1, u_2) = D(u_1, Tu_1)$. Moreover,

$$d(u_1, u_2) = D(u_1, Tu_1) \leq \sup_{x \in Su_0} D(x, Tu_1) \leq H(Su_0, Tu_1).$$

It follows that

$$\begin{aligned}
d(u_1,u_2) &\le H(Su_0,Tu_1) \le \varphi(\rho(u_0,u_1)) \\
&= \varphi\left(\max\left\{d(u_0,u_1), D(u_1,Tu_1), D(u_0,Su_0), \frac{1}{2}\left[D(u_0,Tu_1)+D(u_1,Su_0)\right]\right\}\right) \\
&\le \varphi\left(\max\left\{d(u_0,u_1), d(u_1,u_2), d(u_0,u_1), \frac{1}{2}\left[d(u_0,u_1)+d(u_1,u_2)\right]\right\}\right) \\
&\le \varphi\left(\max\{d(u_0,u_1), d(u_1,u_2)\}\right). \qquad (2.1)
\end{aligned}$$

Write $\omega = \max\{d(u_0,u_1), d(u_1,u_2)\}$. Observe that, if $\omega = 0$ then $u_0 = u_1 = u_2$ and it follows that $u_0 = u_1 \in Su_0$ and $u_0 = u_2 \in Tu_1 = Tu_0$, i.e., u_0 is a common fixed point of mappings S and T, and then our proof is completed. On the other hand, if $0 < \omega = d(u_1,u_2)$ then using $\varphi(t) < t$ for $t \in (0,\propto)$, from (2.1) we have

$$d(u_1,u_2) \le \varphi(d(u_1,u_2)) < d(u_1,u_2)$$

which is a contradiction. Therefore, $\omega = d(u_0,u_1)$ and from (2.1) we obtain

$$d(u_1,u_2) \le \varphi(\rho(u_0,u_1)) \le \varphi(d(u_0,u_1)) < d(u_0,u_1). \qquad (2.2)$$

We continue the procedure of constructing u_n inductively, we can choose a sequence $\{u_n\}$ in E such that for all $n \ge 1$, $u_{2n} \in Tu_{2n-1}, u_{2n+1} \in Su_{2n}$ and

$$d(u_{2n},u_{2n+1}) = D(u_{2n},Su_{2n}),\ d(u_{2n+1},u_{2n+2}) = D(u_{2n+1},Tu_{2n+1}).$$

Moreover,

$$D(u_{2n},Su_{2n}) \le \sup_{x\in Tu_{2n-1}} D(x,Su_{2n}) \le H(Tu_{2n-1},Su_{2n}),$$

and

$$D(u_{2n+1},Tu_{2n+1}) \le \sup_{x\in Tu_{2n+1}} D(x,Su_{2n}) \le H(Su_{2n},Tu_{2n+1})$$

for all $n \ge 1$. Therefore, by using an argument similar to the above we get,

$$d(u_{2n},u_{2n+1}) \le \varphi(\rho(u_{2n-1},u_{2n})) < d(u_{2n-1},u_{2n}) \qquad (2.3)$$

and

$$d(u_{2n+1},u_{2n+2}) \le \varphi(\rho(u_{2n},u_{2n+1})) < d(u_{2n},u_{2n+1}) \qquad (2.4)$$

for all $n \ge 1$. Thus, from (2.3) and (2.4), we get

$$d(u_n,u_{n+1}) \le \varphi(\rho(u_{n-1},u_n)) < d(u_{n-1},u_n) \qquad (2.5)$$

for all $n \ge 1$. Using (2.2) and (2.5), we repeat the procedure to obtain

$$d(u_n,u_{n+1}) \le \varphi(\rho(u_{n-1},u_n)) \le \varphi^2(d(u_{n-2},u_{n-1})) \le \cdots \le \varphi^n(d(u_0,u_1))$$

for all $n \geq 1$. Therefore, for positive integers m, k, we get

$$\begin{aligned} d(u_k, u_{k+m}) &\leq d(u_k, u_{k+1}) + d(u_{k+1}, u_{k+2}) + \cdots + d(u_{k+m-1}, u_{k+m}) \\ &\leq \sum_{i=k}^{k+m-1} \varphi^i(d(u_0, u_1)) = \sum_{i=k}^{k+m-1} \varphi^i(D(u_0, Su_0)). \end{aligned}$$

Since $D(u_0, S(u_0)) \in J$ and $\varphi_{|_J}$ is a Bainchini-Grandolfi gauging function on J, the above inequality implies that $\{u_n\}$ is a Cauchy sequence in E. By virtue of the completeness of E, there exists $u \in E$ such that $u_n \to u$ for $n \to \infty$. Now, we prove that $u \in Tu$ and $u \in Su$, i.e., u is a common fixed point of S and T. To do this, we note that

$$\begin{aligned} D(u_{2n}, Su) &\leq H(Tu_{2n-1}, Su) \leq \varphi(\rho(u_{2n-1}, u)) \\ &= \varphi\Big(\max\Big\{d(u_{2n-1}, u), D(u_{2n-1}, Tu_{2n-1}), D(u, Su), \\ &\qquad \frac{1}{2}[D(u_{2n-1}, Su) + D(u, Tu_{2n-1})]\Big\}\Big) \\ &\leq \varphi\Big(\max\Big\{d(u_{2n-1}, u), d(u_{2n-1}, u_{2n}), D(u, Su), \\ &\qquad \frac{1}{2}[d(u_{2n-1}, u) + D(u, Su) + d(u, u_{2n})]\Big\}\Big). \end{aligned}$$

Denote by

$$\begin{aligned} \alpha(u_n, u) =: \max\Big\{&d(u_{2n-1}, u), d(u_{2n-1}, u_{2n}), D(u, Su), \\ &\frac{1}{2}\left[D(u_{2n-1}, u) + D(u, Su) + d(u, u_{2n})\right]\Big\} \end{aligned}$$

the right hand side of the above inequality. Then, $\alpha(u_n, u) \to D(u, Su)$ as $n \to \infty$. Therefore, in view of Lemma 1.5 and the upper semi-continuity of φ, we get

$$D(u, Su) = \lim_{n\to\infty} D(u_{2n}, Su) \leq \limsup_{n\to\infty} \varphi(\alpha(u_n, u)) \leq \varphi(D(u, Su)).$$

This implies $D(u, Su) = 0$. Since Su is approximative, there exists $y \in Su$ such that $d(u, y) = 0$, i.e., $u = y$. Hence $u \in Su$. As

$$\begin{aligned} D(u, Tu) &\leq H(Su, Tu) \\ &\leq \varphi\left(\max\left\{d(u, u), D(u, Tu), D(u, Su), \frac{1}{2}\left[D(u, Tu) + D(u, Su)\right]\right\}\right) \\ &= \varphi(D(u, Tu)), \end{aligned}$$

which gives $D(u, Tu) = 0$, and this reduces to $u \in Tu$. This completes the proof. $\square$

Remark 2.2. *Under the hypothesis of Theorem 2.1, S and T have a unique common fixed point if the following condition is satisfied:*

$$d(x, y) \leq H(Sx, Ty), \quad \forall x, y \in E. \qquad \textbf{(C)}$$

Proof. Let u and v be common fixed points of S and T. Then, by the condition (**C**), we have

$$
\begin{aligned}
d(u,v) &\le H(Su,Tv) \le \varphi(\rho(u,v)) \\
&= \varphi\left(\max\left\{d(u,v), D(v,Tv), D(u,Tu), \frac{1}{2}[D(u,Tv)+D(v,Su)]\right\}\right) \\
&\le \varphi\left(\max\left\{d(u,v), \frac{1}{2}[d(u,v)+D(v,Tv)+d(v,u)+D(u,Su)]\right\}\right) = \varphi(d(u,v)).
\end{aligned}
$$

Hence $u=v$. The proof is completed. □

By Theorem 2.1, we get the following results immediately.

Corollary 2.3. *Let (E,d) be a complete metric space. Let T be a hybrid generalized multivalued φ-weak contractions on E. Assume that T has the approximative values and $\varphi_{|_J}$ is a Bianchini-Grandolfi gauge function on some interval $J \subset \mathcal{R}^+_\propto$. If there is $x \in E$ such that $D(x,Tx) \in J$ then the mapping T has a fixed point $u \in E$.*

Corollary 2.4. *Let (E,d) be a complete metric space. Let S,T be a pair of hybrid generalized multivalued φ-weak contractions on E. If S,T have the approximative values and $\sum_{i=1}^{\infty}\varphi^i(t) < \infty$ for all $t \in (0,\propto)$, then the pair S,T has a common fixed point $u \in E$.*

3 Further Results

Let $\propto \in (0,\infty], \mathcal{R}^+_\propto = [0,\propto)$. Let $f:[0,\infty)\to[0,\infty)$ satisfy

(i) $f(0)=0$ and $f(t)>0$ for each $t \in (0,\propto)$;

(ii) f is nondecreasing on $\mathcal{R}^+_\propto$;

(iii) f is continuous on $\mathcal{R}^+_\propto$;

(iv) $f(a+b) \le f(a)+f(b)$ for all $a,b \in [0,\infty)$.

Define $\mathcal{F}[0,\propto) = \{f : f \ \text{ satisfies (i)-(iv) above}\}$.

Example 3.1. *The following examples were partially given in [24]:*

(i) Let ϕ is nonnegative, nondecreasing, Lebesgue integrable on $[0,\propto)$ and satisfies

$$\int_0^t \phi(s)ds > 0, \quad t \in (0,\propto).$$

Define $f(t) = \int_0^t \phi(s)ds$ then $f \in \mathcal{F}[0,\propto)$.

(ii) Let ψ be a nonnegative, Lebesgue integrable on $[0,\propto)$ and satisfies

$$\int_0^t \psi(s)ds > 0, \quad t \in (0,\propto)$$

and θ be a nonnegative, Lebesgue integrable on $[0, \int_0^{\propto} \psi(s)ds)$ and satisfies

$$\int_0^t \theta(s)ds > 0, \quad t \in [0, \int_0^{\propto} \psi(s)ds).$$

If ψ and θ are nondecreasing and we define $f(t) = \int_0^{\int_0^t \psi(s)ds} \theta(\tau)d\tau$, then $f \in \mathcal{F}[0,\propto)$.

Using above concepts, Theorem 2.1 could be further extended to more general results. In fact, the proof of next Theorem is similar to that of Theorem 2.1, however, for the sake of completeness we will present it.

Theorem 3.2. *Let (E,d) be a complete metric space and $S,T : E \to 2^E$ be a pair of multivalued mappings. Suppose that $\varphi \in \Phi[0,\propto)$ and $f \in \mathcal{F}[0,\propto)$ satisfy*

$$f(H(Sx,Ty)) \leq \varphi(f(\rho(x,y))) \tag{3.1}$$

for each $x,y \in E$. Assume that S,T have the approximative values and $\varphi_{|J}$ is a Bianchini-Grandolfi gauge function on some interval $J \subset \mathcal{R}^+_{\propto}$. If there is $x \in E$ such that either $f(D(x,Sx)) \in J$ or $f(D(x,Tx)) \in J$ then the mappings S and T have a common fixed point $u \in E$.

Proof. Without loss of generality, we will assume that there is $u_0 \in E$ such that $f(D(u_0, Su_0)) \in J$. Take $u_0 \in E$, since Su_0 is approximative it follows that there exists $u_1 \in Su_0$ such that $d(u_0,u_1) = D(u_0,Su_0)$. Next, since Tu_1 is approximative, there exists $u_2 \in Tu_1$ such that $d(u_1,u_2) = D(u_1,Tu_1)$. Moreover,

$$d(u_1,u_2) = D(u_1,Tu_1) \leq \sup_{x \in Su_0} D(x,Tu_1) \leq H(Su_0,Tu_1).$$

It follows that

$$\begin{aligned}
&f(d(u_1,u_2)) \leq f(H(Su_0,Tu_1)) \leq \varphi(f(\rho(u_0,u_1)))\\
&= \varphi\left(f\left(\max\left\{d(u_0,u_1), D(u_1,Tu_1), D(u_0,Su_0), \frac{1}{2}\left[D(u_0,Tu_1) + D(u_1,Su_0)\right]\right\}\right)\right)\\
&\leq \varphi\left(f\left(\max\left\{d(u_0,u_1), d(u_1,u_2), d(u_0,u_1), \frac{1}{2}\left[d(u_0,u_1) + d(u_1,u_2)\right]\right\}\right)\right)\\
&\leq \varphi(f(\max\{d(u_0,u_1), d(u_1,u_2)\})).
\end{aligned} \tag{3.2}$$

Write $\omega = \max\{d(u_0,u_1), d(u_1,u_2)\}$. Observe that, if $\omega = 0$ then $u_0 = u_1 = u_2$ and it follows that $u_0 = u_1 \in Su_0$ and $u_0 = u_2 \in Tu_1 = Tu_0$, i.e., u_0 is a common fixed point of mappings S and T, and then our proof is completed. On the other hand, if $0 < \omega = d(u_1, u_2)$ then using $\varphi(t) < t$ for $t \in (0, \propto)$, from (3.2) we have

$$f(d(u_1, u_2)) \leq \varphi(f(d(u_1, u_2))) < f(d(u_1, u_2))$$

which is a contradiction. Therefore, $\omega = d(u_0, u_1)$ and from (3.2) we obtain

$$f(d(u_1, u_2)) \leq \varphi(f(\rho(u_0, u_1))) \leq \varphi(f(d(u_0, u_1))). \tag{3.3}$$

We continue the procedure of constructing u_n inductively, we can choose a sequence $\{u_n\}$ in E such that for all $n \geq 1$, $u_{2n} \in Tu_{2n-1}, u_{2n+1} \in Su_{2n}$ and

$$d(u_{2n}, u_{2n+1}) = D(u_{2n}, Su_{2n}),\ d(u_{2n+1}, u_{2n+2}) = D(u_{2n+1}, Tu_{2n+1}).$$

Moreover,

$$D(u_{2n}, Su_{2n}) \leq \sup_{x \in Tu_{2n-1}} D(x, Su_{2n}) \leq H(Tu_{2n-1}, Su_{2n}),$$

and

$$D(u_{2n+1}, Tu_{2n+1}) \leq \sup_{x \in Tu_{2n+1}} D(x, Su_{2n}) \leq H(Su_{2n}, Tu_{2n+1})$$

for all $n \geq 1$. Therefore, by using an argument similar to the above we get,

$$f(d(u_{2n}, u_{2n+1})) \leq \varphi(f(\rho(u_{2n-1}, u_{2n}))) < f(d(u_{2n-1}, u_{2n})), \tag{3.4}$$

and

$$f(d(u_{2n+1}, u_{2n+2})) \leq \varphi(f(\rho(u_{2n}, u_{2n+1}))) < f(d(u_{2n}, u_{2n+1})) \tag{3.5}$$

for all $n \geq 1$. Thus, from (3.4) and (3.5), we get

$$f(d(u_n, u_{n+1})) \leq \varphi(f(\rho(u_{n-1}, u_n))) < f(d(u_{n-1}, u_n)) \tag{3.6}$$

for all $n \geq 1$. Using (3.3) and (3.6), we repeat the procedure to obtain

$$f(d(u_n, u_{n+1})) \leq \varphi(f(\rho(u_{n-1}, u_n))) \leq \varphi^2(f(d(u_{n-2}, u_{n-1}))) \leq \cdots \leq \varphi^n(f(d(u_0, u_1)))$$

for all $n \geq 1$. Therefore, for positive integers m, k, we get

$$\begin{aligned} f(d(u_k, u_{k+m})) &\leq f(d(u_k, u_{k+1}) + d(u_{k+1}, u_{k+2}) + \cdots + d(u_{k+m-1}, u_{k+m})) \\ &\leq f(d(u_k, u_{k+1})) + f(d(u_{k+1}, u_{k+2})) + \cdots + f(d(u_{k+m-1}, u_{k+m})) \\ &\leq \sum_{i=k}^{k+m-1} \varphi^i(f(d(u_0, u_1))) = \sum_{i=k}^{k+m-1} \varphi^i(f(D(u_0, Su_0))). \end{aligned}$$

Since $f(D(u_0, S(u_0))) \in J$ and $\varphi_{|_J}$ is a Bainchini-Grandolfi gauging function on J, in light of the continuity of the function f, the above inequality implies that $\{u_n\}$ is a Cauchy sequence in E. By virtue of the completeness of E, there exists $u \in E$ such that $u_n \to u$ for $n \to \infty$. Now, we prove that $u \in Tu$ and $u \in Su$, i.e.,

u is a common fixed point of S and T. Now, since f is a nondecreasing function, we have

$$\begin{aligned} f(D(u_{2n}, Su)) &\leq f(H(Tu_{2n-1}, Su)) \leq \varphi(f(\rho(u_{2n-1}, u))) \\ &= \varphi\Big(f\Big(\max\Big\{d(u_{2n-1}, u), D(u_{2n-1}, Tu_{2n-1}), D(u, Su), \\ &\qquad \frac{1}{2}\left[D(u_{2n-1}, Su) + D(u, Tu_{2n-1})\right]\Big\}\Big)\Big) \\ &\leq \varphi\Big(f\Big(\max\Big\{d(u_{2n-1}, u), d(u_{2n-1}, u_{2n}), D(u, Su), \\ &\qquad \frac{1}{2}\left[d(u_{2n-1}, u) + D(u, Su) + d(u, u_{2n})\right]\Big\}\Big)\Big). \end{aligned}$$

Denote by

$$\alpha(u_n, u) =: \max\Big\{d(u_{2n-1}, u), d(u_{2n-1}, u_{2n}), D(u, Su), \frac{1}{2}\left[D(u_{2n-1}, u) + D(u, Su) + d(u, u_{2n})\right]\Big\}$$

the right hand side of the above inequality. Then $\alpha(u_n, u) \to D(u, Su)$ as $n \to \infty$, and consequently, $f(\alpha(u_n, u)) \to f(D(u, Su))$ as $n \to \infty$. Therefore, in view of Lemma 1.5 and the upper semi-continuity of φ, we get

$$\begin{aligned} f(D(u, Su)) &= f\Big(\lim_{n\to\infty} D(u_{2n}, Su)\Big) = \lim_{n\to\infty} f\left(D(u_{2n}, Su)\right) \\ &\leq \limsup_{n\to\infty} \varphi(f(\alpha(u_n, u))) \leq \varphi(f(D(u, Su))). \end{aligned}$$

Thus $f(D(u, Su)) = 0$, which implies that $D(u, Su) = 0$. Since Su is approximative, there exists $y \in Su$ such that $d(u, y) = 0$, i.e., $u = y$. Hence $u \in Su$. As

$$\begin{aligned} f(D(u, Tu)) &\leq f(H(Su, Tu)) \\ &\leq \varphi\left(f\left(\max\left\{d(u, u), D(u, Tu), D(u, Su), \frac{1}{2}\left[D(u, Tu) + D(u, Su)\right]\right\}\right)\right) \\ &= \varphi(f(D(u, Tu))), \end{aligned}$$

which gives $f(D(u, Tu)) = 0$, and this reduces to $u \in Tu$. This completes the proof. □

Remark 3.3. *Theorem 3.2 is a genuine generalization of Lemma 3.1 of a paper by Hong et al. [25], which is the important result for such paper. However, it has been observed that a proof of such lemma contains an error. The proof of such lemma at line 14, p. 5, presented as:*

$$f(d(u_{m+1}, u_{n+1})) \leq f(H(Tu_m, Tu_n)) \leq (f(\rho(u_m, u_n, \delta))), \tag{3.7}$$

where $\delta \in (0,1]$. It is related to the procedure of constructing the sequence $\{u_n\}$, that we only have

$$d(u_{n-1}, u_n) \leq H(Tu_{n-2}, Tu_{n-1})), \quad for \;\; n = 2, 3, ...$$

Hence, the first inequality is not assuredly hold and this is a point which may break down the conclusion of such lemma. Because, if the first inequality is not true, then the conclusion that $\{u_n\}$ is a Cauchy sequence would be failed, but this result is an important step in the proof of the lemma.

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References

[1] B.C. Dhage, A fixed point theorem for multivalued mappings on ordered Banach spaces with applications I, Nonlinear Anal. Forum 10 (2005) 105–126.

[2] B.C. Dhage, Hybrid fixed point theory for strictly monotone increasing multivalued mappings with applications, Comput. Math. Appl. 53 (2007) 803–824.

[3] B.C. Dhage, Multi-valued operators and fixed point theorems in Banach algebras I, Taiwanese J. Math. 10 (2006) 1025–1045.

[4] J.T. Markin, A fixed point theorem for set-valued mappings, Bull. Amer. Math. Soc. 74 (1968) 639–640.

[5] S.B. Nadler Jr., Multivalued contraction mappings, Pacific, J. Math. 30 (1969) 475–488.

[6] N.A. Assad, W.A. Kirk, Fixed point theorems for set-valued mappings of contractive type, Pacific J. Math. 43 (1972) 553–562.

[7] L.B. Ciric, J.S. Ume, Common fixed point theorems for multi-valued non-self mappings, Publ. Math. Debrecen 60 (3-4) (2002) 359–371.

[8] P.Z. Daffer, H. Kaneko, Fixed points of generalized contractive multi-valued mappings, J. Math. Anal. Appl. 192 (1995) 655–666.

[9] S. Itoh, Multi-valued generalized contractions and fixed point theorems, Comment. Math. Univ. Carolin. 18 (1977) 247–258.

[10] N. Mizoguchi, W. Takahashi, Fixed point theorems for multi-valued mappings on complete metric spaces, J. Math. Anal. Appl. 141 (1989) 177–188.

[11] Ya.I. Alber, S. Guerre-Delabriere, Principles of weakly contractive maps in Hilbert spaces, in: I. Gohberg, Yu. Lyubich (Eds.), New Results in Operator Theory, in: Advances and Appl., vol. 98, Birkhuser, Basel, 1997, 7–22.

[12] B.E. Rhoades, Some theorems on weakly contractive maps, Nonlinear Anal. 47 (2001) 2683–2693.

[13] D.W. Boyd, T.S.W. Wong, On nonlinear contractions, Proc. Amer. Math. Soc. 20 (1969) 458–464.

[14] I.A. Rus, Principles and Applications of the Fixed Point Theory, Editura Dacia, Cluj-Napoca, (1979) (in Romanian).

[15] I.A. Rus, Generalized Contractions and Applications, Cluj University Press, Cluj-Napoca, 2001.

[16] M.A. Al-Thagafi, N. Shahzad, Noncommuting selfmaps and invariant approximations, Nonlinear Anal. 64 (2006) 2778–2786.

[17] N. Hussain, G. Jungck, Common fixed point and invariant approximation results for noncommuting generalized (f, g)-nonexpansive maps, J. Math. Anal. Appl. 321 (2006) 851–861.

[18] Y. Song, Common fixed points and invariant approximations for generalized $(f; g)$-nonexpansive mappings, Commun. Math. Anal. 2 (2007) 17–26.

[19] Y. Song, S. Xu, A note on common fixed-points for Banach operator pairs, Int. J. Contemp. Math. Sci. 2 (2007) 1163–1166.

[20] Q. Zhang, Y. Song, Fixed point theory for generalized φ-weak contractions, Appl. Math. Lett. 22 (2009) 75–78.

[21] R.M. Bianchini, M. Grandolfi, Transformazioni di tipo contracttivo generalizzato in uno spazio metrico, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fiz. Mat. Natur. 45 (1968) 212–216.

[22] V. Ptak, The rate of convergence of Newtons process, Numer. Math. 25 (1976) 279–285.

[23] H.M. Abu-Donia, Common fixed points theorems for fuzzy mappings in metric space under φ-contraction condition. Chaos, Solitons & Fractals 34 (2007) 538–543.

[24] X. Zhang, Common fixed point theorems for some new generalized contractive type mappings, J. Math. Anal. Appl. 333 (2007) 780–786.

[25] S. Hong, D. Guan, L. Wang, Hybrid fixed points of multivalued operators in metric spaces with applications, Nonlinear Analysis 70 (11) (2008) 4106–4117.

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