Thai Journal of Mathematics
Volume 9 (2011) Number 2 : 405–415

www.math.science.cmu.ac.th/thaijournal
Online ISSN 1686-0209

# Strong Convergence Theorems for Asymptotically Strict Pseudocontractive Mappings in the Intermediate Sense

**Jerry N. Ezeora and Yekini Shehu**[1]

Mathematics Institute,
African University Science and Technology, Abuja, Nigeria
e-mail : jerryflourish@yahoo.ca,
deltanougt2006@yahoo.com

**Abstract :** In this paper, we introduce two hybrid iteration schemes and obtain strong convergence theorems for an infinite family of uniformly continuous asymptotically $\lambda-$strict pseudocontractive mappings in the intermediate sense in a real Hilbert space.

**Keywords :** Strong convergence; Asymptotically strict pseudocontractive mappings; Metric projection; Hilbert spaces.
**2010 Mathematics Subject Classification :** 47H06; 47H09; 47J05; 47J25.

## 1 Introduction

Let $K$ be a nonempty, closed convex subset of a real Hilbert space $H$. A mapping $T : K \to K$ is said to be *nonexpansive* if

$$||Tx - Ty|| \leq ||x - y||, \tag{1.1}$$

for all $x, y \in K$. Many authors have worked extensively on the approximation of fixed points of nonexpansive mappings. For example, the reader can consult the recent monographs of Berinde [1] and Chidume [2].

[1]Corresponding author email: deltanougt2006@yahoo.com (Y. Shehu)

A mapping $T : K \to K$ is said to be *asymptotically nonexpansive* if there exists a sequence $\{\gamma_n\}_{n=1}^{\infty}$ with $\lim \gamma_n = 0$ such that

$$||T^n x - T^n y|| \leq (1 + \gamma_n)||x - y||, \tag{1.2}$$

for all $x, y \in K$, for all $n \geq 1$. The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [3] as a generalization of the class of nonexpansive mappings. They also proved existence of fixed point of such maps in uniformly convex Banach space. Iterative methods for approximation of fixed points of asymptotically nonexpansive mappings have been studied by many authors (see, for example, [3–11], and the references therein). It not difficult to see that the class of nonexpansive mappings is a subclass of asymptotically nonexpansive mappings.

A mapping $T : K \to K$ is said to be *uniformly-L-Lipschitzian* if there exists $L > 0$ such that

$$||T^n x - T^n y|| \leq L||x - y||, \tag{1.3}$$

for all $x, y \in K$, for all $n \geq 1$. We remark here immediately that the class of asymptotically nonexpansive mappings is a subclass of uniformly Lipschitzian mappings.

A mapping $T : K \to K$ is said to be *asymptotically nonexpansive in the intermediate sense* if $T$ is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in K} \Big\{ ||T^n x - T^n y|| - ||x - y|| \Big\} \leq 0. \tag{1.4}$$

Observe that if we define

$$\sigma_n := \max\Big\{0, \sup_{x,y\in K} (||T^n x - T^n y|| - ||x - y||)\Big\}.$$

Then $\sigma_n \to 0, \quad n \to \infty$ and (1.4) reduces to

$$||T^n x - T^n y|| \leq ||x - y|| + \sigma_n, \tag{1.5}$$

for all $x, y \in K$ and for all $n \geq 1$. The class of asymptotically nonexpansive mappings in the intermediate sense was introduced by Bruck, Kuczumow and Reich [12]. Iterative methods for the approximation of such types of non-Lipschitzian mappings have been studied in [12–14] and many others.

A mapping $T : K \to K$ is said to be an *asymptotically $\lambda$-strict pseudocontractive mapping* with sequence $\{\gamma_n\}_{n=1}^{\infty}$ if there exists a constant $\lambda \in [0, 1)$ and a sequence $\{\gamma_n\}_{n=1}^{\infty} \subset [0, \infty)$ with $\lim \gamma_n = 0$ such that

$$||T^n x - T^n y||^2 \leq (1 + \gamma_n)||x - y||^2 + \lambda||x - T^n x - (y - T^n y)||^2,$$

for all $x, y \in K$ and $n \geq 1$. This class of mappings has been studied by several authors (see, for example, [15–18], and it includes the important class of asymptotically nonexpansive maps. It is well known ( see, for example, [15, 16]) that if

$T$ is an asymptotically $\lambda$-strict pseudocontractive mapping, then $T$ is uniformly $L$- Lipschitzian.

A mapping $T : K \to K$ is said to be an *asymptotically $\lambda$-strict pseudocontractive mapping in the intermediate sense* with sequence $\{\gamma_n\}_{n=1}^{\infty}$ if there exist a constant $\lambda \in [0,1)$ and a sequence $\{\gamma_n\}_{n=1}^{\infty} \subset [0,\infty)$ with $\lim \gamma_n = 0$ such that

$$\limsup_{n\to\infty} \sup_{x,y\in K} \Big\{||T^n x - T^n y||^2 - (1+\gamma_n)||x-y||^2 - \lambda||x - T^n x - (y - T^n y)||^2\Big\} \le 0.$$

If we set

$$c_n := \max\Big\{0, \sup_{x,y\in K}\big(||T^n x - T^n y||^2 - (1+\gamma_n)||x-y||^2 - \lambda||x - T^n x - (y - T^n y)||^2\big)\Big\},$$

then $c_n \ge 0 \;\; \forall \;\; n \ge 1$, $c_n \to 0, \;\; as \;\; n \to \infty$ and the last inequality reduces to

$$||T^n x - T^n y||^2 \le (1+\gamma_n)||x-y||^2 + \lambda||x - T^n x - (y - T^n y)||^2 + c_n$$

for all $x, y \in K$ and $n \ge 1$. We remark here immediately that the class of asymptotically $\lambda$-strict pseudocontractive mappings is a subclass of asymptotically $\lambda$-strict pseudocontractive mappings in the intermediate sense. The class of asymptotically $\lambda$-strict pseudocontractive mappings in the intermediate sense was recently introduced by Sahu et al. [19] in a real Hilbert space. They showed with example that an asymptotically $\lambda$-strict pseudocontractive mapping in the intermediate sense is not necessary uniformly $L$-Lipschitzian and proved the following strong convergence theorem.

**Theorem 1.1** (Sahu et al. [19]). *Let $K$ be a nonempty closed convex subset of a real Hilbert space $H$ and $T : K \to K$ a uniformly continuous asymptotically $\lambda-$ strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}_{n=1}^{\infty}$ such that $F(T)$ is nonempty and bounded. Let $\{\alpha_n\}$ be a sequence in $[0,1]$ such that $0 < \delta \le \alpha_n \le 1-\lambda$ for all $n \ge 1$. Let $\{x_n\}_{n=1}^{\infty}$ be the sequence in $C$ generated by the following algorithm.*

$$\begin{cases} u = x_1 \in C \;\; chosen \;\; arbitrary, \\ y_n = (1-\alpha_n)x_n + \alpha_n T^n x_n, \quad n \ge 1 \\ C_n = \Big\{z \in C : ||y_n - z||^2 \le ||x_n - z||^2 + \theta_n\Big\} \\ Q_n = \Big\{z \in C : \langle x_n - z, u - x_n\rangle \ge 0\Big\} \\ x_{n+1} = P_{C_n \cap Q_n} u, \quad n \ge 1, \end{cases} \tag{1.6}$$

*where $\theta_n = c_n + \gamma_n \Delta_n$, $\Delta_n = \sup\{||x_n - p||^2 : p \in F(T)\} < \infty$. Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to $P_{F(T)}u$.*

Motivated by the results of Sahu et al. [19], we introduce *two hybrid iteration schemes for approximating a common fixed point of a countable infinite family of asymptotically $\lambda-$strict pseudocontractive mappings in the intermediate sense in a real Hilbert space*. Consequently, we show that our iterative schemes converge strongly to a common fixed point of a countable infinite family of asymptotically $\lambda-$strict pseudocontractive mappings in the intermediate sense.

## 2 Preliminaries

Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $|| \cdot ||$ and let $K$ be a nonempty closed convex subset of $H$. The strong convergence of $\{x_n\}_{n=1}^{\infty}$ to $x$ is written $x_n \to x$ as $n \to \infty$.

For any point $u \in H$, there exists a unique point $P_K u \in K$ such that

$$||u - P_K u|| \leq ||u - y||, \quad \forall y \in K. \tag{2.1}$$

$P_K$ is called the *metric projection* of $H$ onto $K$. We know that $P_K$ is a nonexpansive mapping of $H$ onto $K$. It is also known that $P_K$ satisfies

$$\langle x - y, P_K x - P_K y \rangle \geq ||P_K x - P_K y||^2, \tag{2.2}$$

for all $x, y \in H$. Furthermore, $P_K x$ is characterized by the properties $P_K x \in K$ and

$$\langle x - P_K x, P_K x - y \rangle \geq 0, \tag{2.3}$$

for all $y \in K$ and

$$||x - P_K x||^2 \leq ||x - y||^2 - ||y - P_K x||^2, \quad \forall x \in H, \quad y \in K. \tag{2.4}$$

**Lemma 2.1** (Sahu et al. [19])**.** *Let $K$ be a nonempty subset of a real Hilbert space $H$ and $T : K \to K$ be a uniformly continuous asymptotically $\lambda$-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}_{n=1}^{\infty}$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $K$ such that $||x_n - x_{n+1}|| \to 0$ and $||x_n - T^n x_n|| \to 0$, as $n \to \infty$. Then $||x_n - Tx_n|| \to 0, \quad as \quad n \to \infty$.*

**Lemma 2.2.** *Let $H$ be a real Hilbert space, and $K$ a nonempty closed convex subset of $H$. Then*

*(i)* $||x - y||^2 = ||x||^2 - ||y||^2 - 2\langle x - y, y \rangle \; \forall \, x, y \in H$.

*(ii)* $||tx + (1-t)y||^2 = t||x||^2 + (1-t)||y||^2 - t(1-t)||x - y||^2$, $\forall x, y \in H$ and $\forall t \in [0, 1]$.

*(iii) For all $x, y, z \in H$ and a real number $a \in \mathbb{R}$, the set $\{v \in K : ||y - v||^2 \leq ||x - v||^2 + \langle z, v \rangle + a\}$ is closed and convex.*

*(iv) If $T : K \to K$ uniformly continuous asymptotically $\lambda-$strict pseudocontractive mapping in the intermediate sense. Then, $F(T) := \{x \in C : Tx = x\}$ is closed and convex.*

*(i) and (ii) are very well known results, (iii) is proved in [10] while (iv) is proved in [19].*

# 3 Main Results

**Theorem 3.1.** *Let $K$ be a nonempty closed convex subset of a real Hilbert space $H$. For each $i = 1, 2, ...,$ let $T_i : K \to K$ be a uniformly continuous asymptotically $\lambda_i-$ strict pseudocontraction in the intermediate sense with sequence $\{\gamma_{n,i}\}_{n=1}^{\infty} \subset [0, \infty)$ such that $\lim_{n\to\infty} \gamma_{n,i} = 0,\ (i = 1, 2, ...)$. Let $F := \bigcap_{i=1}^{\infty} F(T_i)$ be nonempty and bounded. Let $\{y_{n,i}\}_{n=1}^{\infty},\ (i = 1, 2, ...)$ and $\{x_n\}_{n=1}^{\infty}$ be generated iteratively by*

$$\begin{cases} x_1 \in K, \\ y_{n,i} = (1-\alpha_{n,i})x_n + \alpha_{n,i}T_i^n x_n, \quad n \geq 1 \\ C_{n,i} = \Big\{z \in K : ||y_{n,i} - z||^2 \leq ||x_n - z||^2 + \theta_{n,i}\Big\} \\ C_n = \bigcap\limits_{i=1}^{\infty} C_{n,i} \\ Q_1 = K \\ Q_n = \{z \in Q_{n-1} : \langle z - x_n, x_1 - x_n\rangle \leq 0\}, \quad n \geq 2 \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad n \geq 1, \end{cases} \tag{3.1}$$

*where $\theta_{n,i} = c_{n,i} + \gamma_{n,i}\Delta_n$, where $\Delta_n := \sup\{||x_n - p||^2 : p \in F\} < \infty$. Assume that $\{\alpha_{n,i}\}_{n=1}^{\infty} \subset [0, 1),\ (i = 1, 2, ...)$ and $0 < \delta_i \leq \alpha_{n,i} \leq 1 - \lambda_i < 1$. Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to $P_F x_0$.*

*Proof.* We split the proof into seven steps for convenience.

**Step 1.** We show that $P_F x_0$ is well defined. By Lemma 2.2 (iv), we know that $F(T_i)$ is closed convex for each $i = 1, 2, ....$ Hence, $F = \bigcap_{i=1}^{\infty} F(T_i)$ is closed and convex so that $P_F x_0$ is well defined.

**Step 2.** We show that both $C_n$ and $Q_n$ are both closed and convex. Observe that Lemma 2.2 (iii) implies that $C_n$ is closed and convex. Also, $Q_n$ is closed and convex from its definition in (3.1).

**Step 3.** We show that $F \subset C_n \cap Q_n$. We first show that $F \subset C_n$. Let $x^* \in F$ be arbitrary. For $n \geq 1$, then, from (3.1) we have

$$\begin{aligned} & ||y_{n,i} - x^*||^2 = ||(1-\alpha_{n,i})(x_n - x^*) + \alpha_{n,i}(T_i^n x_n - x^*)||^2 \\ = \ & (1-\alpha_{n,i})||x_n - x^*||^2 + \alpha_{n,i}||T_i^n x_n - x^*||^2 - \alpha_{n,i}(1-\alpha_{n,i})||x_n - T_i^n x_n||^2 \\ \leq \ & (1-\alpha_{n,i})||x_n - x^*||^2 + \alpha_{n,i}[(1+\gamma_{n,i})||x_n - x^*||^2 + \lambda_i||x_n - T_i^n x_n||^2 + c_{n,i}] \\ - \ & \alpha_{n,i}(1-\alpha_{n,i})||x_n - T_i^n x_n||^2 \\ \leq \ & ||x_n - x^*||^2 + \gamma_{n,i}||x_n - x^*||^2 - \alpha_{n,i}((1-\alpha_{n,i}) - \lambda_i)||x_n - T_i^n x_n||^2 + c_{n,i} \\ \leq \ & ||x_n - x^*||^2 + \theta_{n,i}. \end{aligned}$$

which shows that $x^* \in C_{n,i},\ \forall n \geq 1,\ \forall i = 1, 2, ....$ Thus, $F \subset C_{n,i},\ \forall n \geq 1,\ \forall i = 1, 2, ....$ Hence, it follows that $F \subset C_n,\ \forall n \geq 1$. Next, we show that $F \subset Q_n$ for all $n \geq 1$. Observe that for $n = 1$, $Q_1 = K$ so that $F \subset Q_1$. Assume now that $F \subset Q_m$, we show that $F \subset Q_{m+1}$. Since $x_{m+1}$ is the projection of $x_1$ onto $C_m \cap Q_m$, it follows from (2.3) that

$$\langle x_{m+1} - z, x_1 - x_{m+1}\rangle \geq 0,\ \forall z \in C_m \cap Q_m.$$

Since $F \subset C_m \cap Q_m$, we have

$$\langle x_{m+1} - p, x_1 - x_{m+1} \rangle \geq 0, \ \forall p \in F.$$

Thus $F \subset Q_{m+1}$, and by induction we have $F \subset Q_n, \ \forall n \geq 1$. Therefore, $F \subset C_n \cap Q_n$.

**Step 4.** We show that $\lim_{n\to\infty} ||x_n - x_1||$ exists. Since the definition of $Q_n$ implies that $x_n = P_{Q_n} x_1$, we have

$$||x_1 - P_{Q_n} x_1|| \leq ||x_1 - y||, \ \forall y \in Q_n.$$

Thus

$$||x_n - x_1|| \leq ||x_1 - y||, \ \forall y \in Q_n.$$

Since $F \subset Q_n$, we have

$$||x_n - x_1|| \leq ||x_1 - p||, \ \forall p \in F,$$

and in particular

$$||x_n - x_1|| \leq ||x_1 - q||, \ q = P_F x_1.$$

Since $x_{n+1} \in Q_n$, we have $\langle x_{n+1} - x_n, x_n - x_1 \rangle \geq 0$ and using Lemma 2.2 (i), we obtain

$$\begin{aligned} ||x_{n+1} - x_n||^2 &= ||(x_{n+1} - x_1) - (x_n - x_1)||^2 \\ &= ||x_{n+1} - x_1||^2 - ||x_n - x_1||^2 - 2\langle x_{n+1} - x_n, x_n - x_1 \rangle \\ &\leq ||x_{n+1} - x_1||^2 - ||x_n - x_1||^2. \end{aligned}$$

Thus

$$||x_n - x_1|| \leq ||x_{n+1} - x_1||.$$

Since $\{||x_n - x_1||\}_{n=1}^{\infty}$ is also bounded, we have that $\lim_{n\to\infty} ||x_n - x_1||$ exists. Furthermore, $\{x_n\}_{n=1}^{\infty}$ is bounded, and so also are $\{T_i^n x_n\}_{n=1}^{\infty}$ and $\{y_{n,i}\}_{n=1}^{\infty}, \ i = 1, 2, ....$

**Step 5.** We show that $x_n \to z \in K$. For $m > n \geq 1$, from the definition of $Q_n$, we have that $x_m = P_{Q_m} x_1 \in Q_m \subset Q_n$. By (2.4), we obtain

$$||x_m - x_n||^2 \leq ||x_n - x_1||^2 - ||x_m - x_1||^2. \tag{3.2}$$

Letting $m, n \to \infty$ and taking the limit in (3.2), we have $||x_m - x_n|| \to 0, \ m, n \to \infty$, which shows that $\{x_n\}_{n=1}^{\infty}$ is Cauchy. In particular, $\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0$. Since $\{x_n\}_{n=1}^{\infty}$ is Cauchy, there exists $z \in K$ such that $x_n \to z, \ n \to \infty$.

**Step 6.** We show that $\lim_{n\to\infty} ||x_n - T_i x_n|| = 0$. Since $x_{n+1} \in C_n$, then

$$||y_{n,i} - x_{n+1}||^2 \leq ||x_n - x_{n+1}||^2 + \theta_{n,i}. \tag{3.3}$$

But $y_{n,i} = (1 - \alpha_{n,i}) x_n + \alpha_{n,i} T_i^n x_n$ implies that

$$\begin{aligned} ||y_{n,i} - x_{n+1}||^2 &= (1 - \alpha_{n,i}) ||x_n - x_{n+1}||^2 + \alpha_{n,i} ||T_i^n x_n - x_{n+1}||^2 \\ &\quad - \alpha_{n,i}(1 - \alpha_{n,i}) ||x_n - T_i^n x_n||^2. \end{aligned} \tag{3.4}$$

Putting (3.4) into (3.3), we have

$$\alpha_{n,i}||T_i^n x_n - x_{n+1}||^2 \leq \alpha_{n,i}(1-\alpha_{n,i})||x_n - T_i^n x_n||^2 + ||x_n - x_{n+1}||^2 + \theta_{n,i}.$$

Thus,

$$\begin{aligned} ||T_i^n x_n - x_{n+1}||^2 &\leq (1-\alpha_{n,i})||x_n - T_i^n x_n||^2 + \frac{1}{\alpha_{n,i}}||x_n - x_{n+1}||^2 + \frac{1}{\alpha_{n,i}}\theta_{n,i} \\ &\leq (1-\alpha_{n,i})||x_n - T_i^n x_n||^2 + \frac{1}{\delta_i}||x_n - x_{n+1}||^2 + \frac{1}{\delta_i}\theta_{n,i}. \quad (3.5) \end{aligned}$$

But

$$\begin{aligned} ||T_i^n x_n - x_{n+1}||^2 &= ||x_{n+1} - x_n||^2 + 2\langle x_{n+1} - x_n, x_n - T_i^n x_n\rangle \\ &\quad + ||x_n - T_i^n x_n||^2. \quad (3.6) \end{aligned}$$

Putting (3.6) into (3.5), after some manipulations, we have

$$\begin{aligned} \delta_i||x_n - T_i^n x_n||^2 &\leq \alpha_{n,i}||x_n - T_i^n x_n||^2 \\ &\leq \frac{1}{\delta_i}||x_n - x_{n+1}||^2 - 2\langle x_{n+1} - x_n, x_n - T_i^n x_n\rangle + \frac{1}{\delta_i}\theta_{n,i} \\ &\leq \frac{1}{\delta_i}||x_n - x_{n+1}||^2 + 2||x_{n+1} - x_n||||x_n - T_i^n x_n|| + \frac{1}{\delta_i}\theta_{n,i}. \end{aligned}$$

Hence, $\lim_{n\to\infty}||x_n - T_i^n x_n|| = 0,\ i = 1, 2, ...$ (noting that $\lim_{n\to\infty}\theta_{n,i} = 0$). Thus by Lemma 2.1, we have $||x_n - T_i x_n|| \to 0,\ n \to \infty$. Furthermore, by $\lim_{n\to\infty}||x_n - z|| = 0,\ \lim_{n\to\infty}||x_n - T_i x_n|| = 0$ and uniform continuity of $T_i$ for each $i = 1, 2, ...$, we have that $z \in \bigcap_{i=1}^{\infty} F(T_i)$.

**Step 7.** We show that $z = P_F x_1$. Noting that $F \subset Q_n$, we have that

$$\langle x_1 - x_n, y - x_n\rangle \leq 0,$$

for all $y \in F$. Since $\lim_{n\to\infty}||x_n - z|| = 0$, we obtain from the above inequality that

$$\langle x_1 - z, y - z\rangle \leq 0,$$

for all $y \in F$. By (2.3), we conclude that $z = P_F x_1$. This completes the proof. □

**Corollary 3.1.** *Let $K$ be a nonempty closed convex subset of a real Hilbert space $H$. For each $i = 1, 2, ...$, let $T_i : K \to K$ be an asymptotically nonexpansive mapping with sequence $\{\gamma_{n,i}\}_{n=1}^{\infty} \subset [0, \infty), (i = 1, 2, ...)$ such that $\lim_{n\to\infty}\gamma_{n,i} = 0, (i = 1, 2, ...)$. Let $F := \bigcap_{i=1}^{\infty} F(T_i)$ be nonempty and bounded. Let $\{y_{n,i}\}_{n=1}^{\infty}$, $(i = 1, 2, ...)$ and $\{x_n\}_{n=1}^{\infty}$ be generated iteratively by*

$$\begin{cases} x_1 \in K, \\ y_{n,i} = (1-\alpha_{n,i})x_n + \alpha_{n,i}T_i^n x_n, \quad n \geq 1 \\ C_{n,i} = \left\{z \in K : ||y_{n,i} - z||^2 \leq ||x_n - z||^2 + \theta_{n,i}\right\} \\ C_n = \overset{\infty}{\underset{i=1}{\cap}} C_{n,i} \\ Q_1 = K \\ Q_n = \{z \in Q_{n-1} : \langle z - x_n, x_1 - x_n\rangle \leq 0\}, \quad n \geq 2 \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad n \geq 1, \end{cases} \quad (3.7)$$

*where* $\theta_{n,i} = \gamma_{n,i}\Delta_n$, *where* $\Delta_n = \sup\{||x_n - p||^2 : p \in F\} < \infty$. *Assume that* $\{\alpha_{n,i}\}_{n=1}^{\infty} \subset [0,1), (i = 1,2,...)$ *and* $0 < \delta_i \leq \alpha_{n,i} < 1$. *Then* $\{x_n\}_{n=1}^{\infty}$ *converges strongly to* $P_F(x_1)$.

For our next result, we give a simpler iterative scheme for finding a common fixed point of countable infinite family of uniformly continuous asymptotically $\lambda-$ strict pseudocontractive mappings in the intermediate sense. In particular, we have the following theorem

**Theorem 3.2.** *Let* $K$ *be a nonempty closed convex subset of a real Hilbert space* $H$. *For each* $i = 1,2,...$, *let* $T_i : K \to K$ *be a uniformly continuous asymptotically* $\lambda_i-$ *strict pseudocontraction in the intermediate sense with sequence* $\{\gamma_{n,i}\}_{n=1}^{\infty} \subset [0,\infty)$ *such that* $\lim_{n\to\infty} \gamma_{n,i} = 0, (i = 1,2,...)$. *Let* $F := \bigcap_{i=1}^{\infty} F(T_i)$ *be nonempty and bounded. Let* $\{y_{n,i}\}_{n=1}^{\infty}, \ (i = 1,2,...)$ *and* $\{x_n\}_{n=1}^{\infty}$ *be generated iteratively by*

$$\begin{cases} x_1 \in K \\ C_{1,i} = K, \quad C_1 = \bigcap\limits_{i=1}^{\infty} C_{1,i} \\ y_{n,i} = (1-\alpha_{n,i})x_n + \alpha_{n,i}T_i^n x_n, \quad n \geq 1 \\ C_{n+1,i} = \{z \in C_{n,i} : ||y_{n,i} - z||^2 \leq ||x_n - z||^2 + \theta_{n,i}\}, \quad n \geq 1 \\ C_{n+1} = \bigcap\limits_{i=1}^{\infty} C_{n+1,i} \\ x_{n+1} = P_{C_{n+1}}x_1, \quad n \geq 1, \end{cases} \tag{3.8}$$

*where* $\theta_{n,i} := c_{n,i} + \gamma_{n,i}\Delta_n$, *where* $\Delta_n := \sup\{||x_n - x^*||^2 : x^* \in F\} < \infty$. *Assume that* $\{\alpha_{n,i}\}_{n=1}^{\infty} \subset (0,1), \ (i = 1,2,...)$ *such that* $0 < \delta_i \leq \alpha_{n,i} \leq 1 - \lambda_i < 1$. *Then,* $\{x_n\}_{n\geq 1}$ *converges strongly to* $P_F(x_1)$.

*Proof.* Let $n = 1$, then $C_{1,i} = K$ is closed convex for each $i = 1,2,....$ Now assume that $C_{n,i}$ is closed convex for some $n > 1$. Then, from definition of $C_{n+1,i}$, we know that $C_{n+1,i}$ is closed convex for the same $n > 1$. Hence, $C_{n,i}$ is closed convex for $n \geq 1$ and for each $i = 1,2,....$ This implies that $C_n$ is closed convex for $n \geq 1$ and for each $i = 1,2,....$ Furthermore, for $n = 1$, $F \subset K = C_{1,i}$. For $n \geq 2$, let $x^* \in F$. Then, from (3.8), we obtain

$$\begin{aligned} & ||y_{n,i} - x^*||^2 = ||(1-\alpha_{n,i})(x_n - x^*) + \alpha_{n,i}(T_i^n x_n - x^*)||^2 \\ = \ & (1-\alpha_{n,i})||x_n - x^*||^2 + \alpha_{n,i}||(T_i^n x_n - x^*)||^2 - \alpha_{n,i}(1-\alpha_{n,i})||x_n - T_i^n x_n||^2 \\ \leq \ & (1-\alpha_{n,i})||x_n - x^*||^2 + \alpha_{n,i}[(1+\gamma_{n,i})||x_n - x^*||^2 + \lambda_i||x_n - T_i^n x_n||^2 + c_{n,i}] \\ - \ & \alpha_{n,i}(1-\alpha_{n,i})||x_n - T_i^n x_n||^2 \\ \leq \ & ||x_n - x^*||^2 + \gamma_{n,i}||x_n - x^*||^2 - \alpha_{n,i}((1-\alpha_{n,i}) - \lambda_i)||x_n - T_i^n x_n||^2 + c_{n,i} \\ \leq \ & ||x_n - x^*||^2 + \theta_{n,i}, \end{aligned}$$

which shows that $x^* \in C_{n,i}, \ \forall n \geq 2, \ \forall i = 1,2,....$ Thus, $F \subset C_{n,i}, \ \forall n \geq 1, \ \forall i = 1,2,....$ Hence, it follows that $F \subset C_n, \ \forall n \geq 1$. Since $x_{n+1} \in C_{n+1} \subset C_n, \ \forall n \geq 1, \ x_n = P_{C_n}x_1, \ \forall n \geq 1$, we have

$$||x_n - x_1|| \leq ||x_{n+1} - x_1||, \quad \forall n \geq 1. \tag{3.9}$$

Also, as $F \subset C_n$ by (2.1), it follows that

$$||x_n - x_1|| \leq ||z - x_1||, \quad z \in F, \quad \forall n \geq 1. \tag{3.10}$$

From (3.9) and (3.10), we have that $\lim_{n\to\infty} ||x_n - x_1||$ exists and so $\{||x_n - x_1||\}_{n=1}^{\infty}$ is bounded. Consequently, $\{x_n\}_{n=1}^{\infty}$, $\{T_i^n x_n\}_{n=1}^{\infty}$ and $\{y_{n,i}\}_{n=1}^{\infty}$, $i = 1, 2, ...$ are all bounded. For $m > n \geq 1$, we have that $x_m = P_{C_m} x_1 \in C_m \subset C_n$. By (2.4), we obtain

$$||x_m - x_n||^2 \leq ||x_n - x_1||^2 - ||x_m - x_1||^2. \tag{3.11}$$

Letting $m, n \to \infty$ and taking the limit in (3.11), we have $||x_m - x_n|| \to 0$, $m, n \to \infty$, which shows that $\{x_n\}_{n=1}^{\infty}$ is Cauchy. In particular, $\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0$. Since $\{x_n\}_{n=1}^{\infty}$ is Cauchy, there exists $z \in K$ such that $x_n \to z$, $n \to \infty$.

By repeating the same arguments as in step 6 of the proof of Theorem 3.1, we can show that $\lim_{n\to\infty} ||x_n - T_i^n x_n|| = 0$, $i = 1, 2, ....$ Consequently, by Lemma 2.1, we have $\lim_{n\to\infty} ||x_n - T_i x_n|| = 0$, $i = 1, 2, ....$ Furthermore, by $\lim_{n\to\infty} ||x_n - z|| = 0$, $\lim_{n\to\infty} ||x_n - T_i x_n|| = 0$ and uniform continuity of $T_i$ for each $i = 1, 2, ...$ we have that $z \in \bigcap_{i=1}^{\infty} F(T_i)$. Noting that $F \subset Q_n$, we have that

$$\langle x_1 - x_n, y - x_n \rangle \leq 0,$$

for all $y \in F$. Since $\lim_{n\to\infty} ||x_n - z|| = 0$, we obtain from the above inequality that

$$\langle x_1 - z, y - z \rangle \leq 0,$$

for all $y \in F$. By (2.3), we conclude that $z = P_F x_1$. This completes the proof. □

**Corollary 3.3.** *Let $K$ be a nonempty closed convex subset of a real Hilbert space $H$. For each $i = 1, 2, ...$, let $T_i : K \to K$ be an asymptotically nonexpansive mapping with sequence $\{\gamma_{n,i}\}_{n=1}^{\infty} \subset [0, \infty)$ such that $\lim_{n\to\infty} \gamma_{n,i} = 0, (i = 1, 2, ...)$. Let $F := \bigcap_{i=1}^{\infty} F(T_i)$ be nonempty and bounded. Let $\{y_{n,i}\}_{n=1}^{\infty}$, $(i = 1, 2, ...)$ and $\{x_n\}_{n=1}^{\infty}$ be generated iteratively by*

$$\begin{cases} C_{1,i} = K, \quad C_1 = \bigcap_{i=1}^{\infty} C_{1,i} \\ y_{n,i} = (1 - \alpha_{n,i}) x_n + \alpha_{n,i} T_i^n x_n, \quad n \geq 1 \\ C_{n+1,i} = \{z \in C_{n,i} : ||y_{n,i} - z||^2 \leq ||x_n - z||^2 + \theta_{n,i}\} \\ C_{n+1} = \bigcap_{i=1}^{\infty} C_{n+1,i} \\ x_{n+1} = P_{C_{n+1}} x_1, \quad n \geq 1, \end{cases} \tag{3.12}$$

*where $\theta_{n,i} := \gamma_{n,i} \Delta_n$,where $\Delta_n = \sup\{||x_n - x^*||^2 : x^* \in F\} < \infty$. Assume that $\{\alpha_{n,i}\}_{n=1}^{\infty} \subset (0, 1)$ $(i = 1, 2, ...)$ such that $0 < \delta_i \leq \alpha_{n,i} < 1$. Then, $\{x_n\}_{n=1}^{\infty}$ converges strongly to $P_F(x_1)$.*

## References

[1] V. Berinde, Iterative approximation of fixed points, Lecture Notes in Mathematics 1912, Springer Berlin 2007.

[2] C.E. Chidume, Geometric properties of Banach spaces and nonlinear iterations, Springer Verlag Series: Lecture Notes in Mathematics Vol. 1965 (2009), XVII, 326p, ISBN 978-1-84882-189-7.

[3] K. Goebel, W.A. Kirk, Fixed point theorems for asymptotically nonexpansive mappings, Proc. Amer. math. Soc. 35 (1) (1972) 171–174.

[4] C.E. Chidume, B. Ali, Approximation of common fixed points of finite family of nonself asymptotically nonexpansive mappings in Banach spaces, J. Math. Anal. Appl. 326 (2) (2007) 960–973.

[5] C.E. Chidume, E.U. Ofoedu, H. Zegeye, Strong and weak convergence theorems for asymptotically nonexpansive mappings, J. Math. Anal. Appl. 280 (2003) 364–374.

[6] W. Cholamjiak, S. Suantai, Monotone hybrid projection algorithms for an infinitely countable family of Lipschitz generalized asymptotically quasi-nonexpansive mappings, Abstract and Applied Analysis, Volume 2009, Article ID 297565, 16 pages.

[7] T.H. Kim, H.K. Xu, Strong convergence of modified Mann's iteration method for asymptotically nonexpansive mappings and semigroup, Nonlinear Anal. 64 (2006) 1140–1152.

[8] T.C. Lim, H.K. Xu, Fixed point theorems for asymptotically nonexpansive mappings, Nonlinear Anal. 2 (1994) 1345–1355.

[9] H. Oka, A nonlinear ergodic theorem for asymptotically nonexpansive mappings in Banach spaces, Proc. Japan Acad. Ser. A 65 (1998) 284–287.

[10] C. Matinez-Yanes, H.K. Xu, Strong convergence of the $CQ$ method for fixed point processes, Nonlinear Anal. 64 (2006) 2400–2411.

[11] L. Qihou, Convergence theorems of the sequence of iterates for asymptotically demicontractive and hemicontractive mappings, Nonlinear Anal. 26 (11) (1996) 1835–1842.

[12] R.E. Bruck, T. Kuczumow, S. Reich, Convergence of iterates of asymptotically nonexpansive mappings in Banach spaces with uniform Opial property, Colloq. Math. 65 (1993) 169–179.

[13] R.P. Agarwal, D. O'Regan, D.R. Sahu, Iterative construction of fixed points of nearly asymptotically nonexpansive mappings. J. Nonlinear Convex. Anal. 8 (1) (2007) 61–79.

[14] C.E. Chidume, N. Shahzad, H. Zegeye, Convergence theorems for mappings which are asymptotically nonexpansive in the intermediate sense, Numer. Funct. Anal. Optim. 477 (2004) 239–257.

[15] T.H. Kim, H.K. Xu, Convergence of modified Mann's iteration method for asymptotically strict pseudocontraction, Nonlinear Anal. 68 (2008) 2828–2836.

[16] M.O. Osilike, A. Udomene, D.I. Igbokwe, B.G. Akuchu, Demiclosedness principle and convergence theorems for strictly asymptotically pseudocontractive maps, J. Math. Anal. Appl. 326 (2007) 1334–1345.

[17] M.O. Osilike, S.C. Aniagbosor, B.G. Akuchu, Fixed points of asymptotically demicontractive mappings in arbitrary Banach spaces, Panamerican Math. J. 12 (2) (2002) 77–88.

[18] M.O. Osilike, Y. Shehu, Explicit averaging cyclic algorithm for common fixed points of asymptotically strictly pseudocontractive maps, Appl. Math. Comp. 213 (2009) 548–553.

[19] D.R. Sahu, H.K. Xu, J.C. Yao, Asymptotically pseudocontractive mappings in the intermediate sense, Nonlinear Anal. 71 (2009) 3502–3511.

(Received 17 February 2011)
(Accepted 18 March 2011)

**Thai J. Math.** Online @ http://www.math.science.cmu.ac.th/thaijournal