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A Note on a New Three Variable Analogue of Hermite Polynomials of I Kind

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Abstract : The present paper is a study of a new three variable analogue of Hermite polynomials $H_n(x,y,z)$ which seems more natural than that of Hermite polynomials of three variables defined and studied by Khan and Abukhammash [1].

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1 Hermite polynomials of three variables

Hermite polynomials of three variable $H_n(x,y,z)$ is defined as follows:

$$H_n(x,y,z) = \sum_{r=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^r n!\ H_{n-2r}(x,y)\ (xy)^{2r} z^{n-2r}}{r!\ (n-2r)!}. \tag{1.1}$$

where $H_n(x,y)$ is Hermite polynomial of two variables [2]. The definition (1.1) can also be written as

$$H_n(x,y,z) = \sum_{r=0}^{\left[\frac{n}{2}\right]} \sum_{s=0}^{\left[\frac{n}{2}-r\right]} \frac{(-1)^{r+s}\ n!\ H_{n-2r-2s}(x)\ x^{2r+2s}\ y^{n-2s}\ z^{n-2r}}{r!\ s!\ (n-2r-2s)!} \tag{1.2}$$

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where $H_n(x)$ is the well-known Hermite polynomial of one variable [3].

The definition (1.2) is equivalent to the following explicit representation of $H_n(x,y,z)$:

$$H_n(x,y,z) = (2xyz)^n \sum_{k=0}^{\left[\frac{n}{2}\right]} \sum_{r=0}^{\left[\frac{n}{2}-k\right]} \sum_{s=0}^{\left[\frac{n}{2}-k-r\right]} \frac{(-n)_{2k+2r+2s} \left(-\frac{1}{x^2}\right)^r \left(-\frac{1}{y^2}\right)^s \left(-\frac{1}{z^2}\right)^k}{2^{2k+2r+2s}\; k!\; r!\; s!} \tag{1.3}$$

In terms of triple hypergeometric function, Hermite polynomials of three variables can be written as

$$H_n(x,y,z)$$
$$= (2xyz)^n\, F^{(3)} \left[\begin{array}{c} -\frac{n}{2}, -\frac{n}{2}+\frac{1}{2} :: -;-;- : \; -;-;-; \\ - :: -;-;- : \; -;-;-; \end{array} -\frac{1}{x^2}, -\frac{1}{y^2}, -\frac{1}{z^2} \right] \tag{1.4}$$

where for right hand side of (1.4), it may be recalled that the definition of a general triple hypergeometric $F^{(3)}$ [x, y, z] (ref. Srivastava and Manocha [4, p. 428]) is defined as :

$$\begin{aligned} F^{(3)}\;[\mathrm{x},\ \mathrm{y},\ \mathrm{z}] \;&\equiv\; F^{(3)} \left[\begin{array}{c} (\mathrm{a}) :: (\mathrm{b}) \; ; \; (\mathrm{b}') \; ; \; (\mathrm{b}'') \; : \; (\mathrm{d}) \; ; \; (\mathrm{d}') \; ; \; (\mathrm{d}'') \; ; \\ (\mathrm{a}) :: (\mathrm{g}) \; ; \; (\mathrm{g}') \; ; \; (\mathrm{g}'') \; : \; (\mathrm{h}) \; ; \; (\mathrm{h}') \; ; \; (\mathrm{h}'') \; ; \end{array} \mathrm{x},\ \mathrm{y},\ \mathrm{z} \right] \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \Lambda\,(m,\ \mathrm{n},\ \mathrm{p})\ \frac{\mathrm{x}^{\mathrm{m}}}{m\,!}\ \frac{\mathrm{y}^{\mathrm{n}}}{n\,!}\ \frac{\mathrm{z}^{\mathrm{p}}}{p\,!}\ , \end{aligned} \tag{1.5}$$

where, for convenience

$$\begin{aligned} \Lambda\,(m,\ \mathrm{n},\ \mathrm{p}) = \frac{\prod_{j=1}^{A} (\mathrm{a_j})_{m+n+p} \prod_{j=1}^{B} (\mathrm{b_j})_{m+n} \prod_{j=1}^{B'} (\mathrm{b'_j})_{n+p} \prod_{j=1}^{B''} (\mathrm{b''_j})_{p+m}}{\prod_{j=1}^{E} (\mathrm{e_j})_{m+n+p} \prod_{j=1}^{G} (\mathrm{g_j})_{m+n} \prod_{j=1}^{G'} \left(\mathrm{g'_j}\right)_{n+p} \prod_{j=1}^{G''} \left(\mathrm{g''_j}\right)_{p+m}} \\ \times \frac{\prod_{j=1}^{D} (\mathrm{d_j})_{m} \prod_{j=1}^{D'} (\mathrm{d'_j})_{n} \prod_{j=1}^{D''} (\mathrm{d''_j})_{p}}{\prod_{j=1}^{H} (\mathrm{h_j})_{m} \prod_{j=1}^{H'} \left(\mathrm{h'_j}\right)_{n} \prod_{j=1}^{H''} \left(\mathrm{h''_j}\right)_{p}} \end{aligned} \tag{1.6}$$

where (a) abbreviates, the array of a parameters $a_1,\ a_2,\ \ldots,\ a_A$, with similar interpretations for (b), (b′), (b″), etc. The triple hypergeometric series in (1.5) converges absolutely when

$$\left.\begin{array}{rcl} 1+E+G+G''+H-A-B-B''-C & \geq & 0 \\ 1+E+G+G''+H'-A-B-B'-C'' & \geq & 0 \\ 1+E+G'+G''+H''-A-B'-B''-C'' & \geq & 0 \end{array}\right\} \tag{1.7}$$

where the equalities hold true for suitable constrained values of $|x|, |y|$ and $|z|$.

The paper contains generating functions, recurrence relations, Rodrigues formula, relationship with Hermite polynomials of one variables, some special properties and expansion of Legendre polynomials in terms of Hermite polynomials of two variables.

2 Generating function for $H_n(x,y,z)$

Some generating functions for Hermite polynomials of three variables $H_n(x,y,z)$ are as follows:

$$e^{2xyzt-(x^2y^2+y^2z^2+z^2x^2)t^2} = \sum_{n=0}^{\infty} \frac{H_n(x,y,z)t^n}{n!}, \tag{2.1}$$

$$\sum_{n=0}^{\infty} \frac{(c)_n H_n(x,y,z)t^n}{n!} \cong (1-2xyzt)^{-c} F^{(3)} \left[\begin{matrix} \frac{c}{2}, \frac{c}{2}+\frac{1}{2} :: -; -; - : -; -; -; \\ - :: -; -; - : -; -; -; \end{matrix} \right.$$
$$\left. -\frac{4x^2y^2t^2}{(1-2xyzt)^2}, -\frac{4y^2z^2t^2}{(1-2xyzt)^2}, -\frac{4z^2x^2t^2}{(1-2xyzt)^2} \right], \tag{2.2}$$

$$\sum_{n=0}^{\infty} \frac{H_{n+k}(x,y,z)t^n}{n!} = e^{2xyzt-(x^2y^2+y^2z^2+z^2x^2)t^2}$$
$$\times \sum_{n=0}^{\left[\frac{k}{2}\right]} \frac{(-k)_{2n}}{n!} H_{k-2n}\left[xyz - t(x^2y^2+y^2z^2+z^2x^2)\right]\left[1-(x^2y^2+y^2z^2+z^2x^2)\right]^n, \tag{2.3}$$

$$\sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \frac{H_{n+r+s}(x,y,z)t^n u^r}{r!n!}$$
$$= e^{2xyzt-(x^2y^2+y^2z^2+z^2x^2)t^2} e^{2\left[xyz-t(x^2y^2+y^2z^2+z^2x^2)\right]u-(x^2y^2+y^2z^2+z^2x^2)u^2}$$
$$\times \sum_{n=0}^{\left[\frac{s}{2}\right]} \frac{(-s)_{2n}}{n!} H_{s-2n}\left[xyz - t(x^2y^2+y^2z^2+z^2x^2) - u(x^2y^2+y^2z^2+z^2x^2)\right]$$
$$\times \left[1-(x^2y^2+y^2z^2+z^2x^2)\right]^n. \tag{2.4}$$

3 Special Properties

In this section we obtain some special properties for Hermite polynomials of three variables $H_n(x,y,z)$ as given below:

Consider the identity

$$e^{2xyzt-(x^2y^2+y^2z^2+z^2x^2)t^2} = e^{2x(yzt)-(yzt)^2} e^{-x^2y^2t^2} e^{-x^2z^2t^2}$$

or,

$$\sum_{n=0}^{\infty} \frac{H_n(x,y,z)t^n}{n!} = \sum_{n=0}^{\infty} \sum_{r=0}^{[\frac{n}{2}]} \sum_{s=0}^{[\frac{n}{2}-r]} \frac{(-1)^{r+s}\ H_{n-2r-2s}(x)\ x^{2r+2s} y^{n-2s} z^{n-2r}\ t^n}{r!\ s!\ (n-2r-2s)!}.$$

Equating the coefficients of t^n, we get

$$H_n(x,y,z) = \sum_{r=0}^{[\frac{n}{2}]} \sum_{s=0}^{[\frac{n}{2}-r]} \frac{(-1)^{r+s} n!\ H_{n-2r-2s}(x)\ x^{2r+2s} y^{n-2s} z^{n-2r}}{r!\ s!\ (n-2r-2s)!}$$

or,

$$H_n(x,y,z) = \sum_{r=0}^{[\frac{n}{2}]} \sum_{s=0}^{[\frac{n}{2}-r]} \frac{(-n)_{2r+2s}(-1)^{r+s}}{r!\ s!} H_{n-2r-2s}(x) x^{2r+2s} y^{n-2s} z^{n-2r}. \quad (3.1)$$

Similarly, by considering different identities we obtain the following special properties for $H_n(x,y,z)$:

$$H_n(x,y,z) = \sum_{r=0}^{[\frac{n}{2}]} \sum_{s=0}^{[\frac{n}{2}-r]} \frac{(-1)^{r+s} n!\ H_{n-2r-2s}(y)\ x^{n-2r} y^{2r+2s} z^{n-2s}}{r!\ s!\ (n-2r-2s)!}.$$

or,

$$H_n(x,y,z) = \sum_{r=0}^{[\frac{n}{2}]} \sum_{s=0}^{[\frac{n}{2}-r]} \frac{(-n)_{2r+2s}(-1)^{r+s}}{r!\ s!} H_{n-2r-2s}(y) x^{n-2r} y^{2r+2s} z^{n-2s}. \quad (3.2)$$

$$H_n(x,y,z) = \sum_{r=0}^{[\frac{n}{2}]} \sum_{s=0}^{[\frac{n}{2}-r]} \frac{(-1)^{r+s} n!\ H_{n-2r-2s}(z)\ x^{n-2s} y^{n-2r} z^{2r+2s}}{r!\ s!\ (n-2r-2s)!}$$

or,

$$H_n(x,y,z) = \sum_{r=0}^{[\frac{n}{2}]} \sum_{s=0}^{[\frac{n}{2}-r]} \frac{(-n)_{2r+2s}\ (-1)^{r+s}}{r!\ s!}\ H_{n-2r-2s}(z)\ x^{n-2s} y^{n-2r} z^{2r+2s}. \quad (3.3)$$

$$H_n(x,y,z) = \sum_{r=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^r n!\ H_{n-2r}(x,y)\ (xy)^{2r} z^{n-2r}}{r!\ (n-2r)!}$$

or,

$$H_n(x,y,z) = \sum_{r=0}^{\left[\frac{n}{2}\right]} \frac{(-n)_{2r}\ (-1)^r}{r!}\ H_{n-2r}(x,y)\ (xy)^{2r} z^{n-2r}. \tag{3.4}$$

$$H_n(x,y,z) = \sum_{r=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^r n!\ H_{n-2r}(y,z)\ (yz)^{2r} x^{n-2r}}{r!\ (n-2r)!}$$

or,

$$H_n(x,y,z) = \sum_{r=0}^{\left[\frac{n}{2}\right]} \frac{(-n)_{2r}\ (-1)^r}{r!}\ H_{n-2r}(y,z)\ (yz)^{2r} x^{n-2r}. \tag{3.5}$$

$$H_n(x,y,z) = \sum_{r=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^r n!\ H_{n-2r}(z,x)\ (zx)^{2r} y^{n-2r}}{r!\ (n-2r)!}$$

or,

$$H_n(x,y,z) = \sum_{r=0}^{\left[\frac{n}{2}\right]} \frac{(-n)_{2r}\ (-1)^r}{r!}\ H_{n-2r}(z,x)\ (zx)^{2r} y^{n-2r}. \tag{3.6}$$

$$H_n(x,y,z) = \sum_{r=0}^{n} \sum_{s=0}^{\left[\frac{n-r}{2}\right]} \frac{(-1)^s\ n!\ H_{n-r-2s}(\frac{x}{2})\ H_r(\frac{y}{2})\ x^{r+2s} y^{n-r} z^{n-2s}}{r!s!(n-r-2s)!} \tag{3.7}$$

$$H_n(x,y,z) = \sum_{r=0}^{n} \sum_{s=0}^{\left[\frac{n-r}{2}\right]} \frac{(-1)^s\ n!\ H_{n-r-2s}(\frac{y}{2})\ H_r(\frac{z}{2})\ x^{n-2s} y^{r+2s} z^{n-r}}{r!s!(n-r-2s)!} \tag{3.8}$$

$$H_n(x,y,z) = \sum_{r=0}^{n} \sum_{s=0}^{\left[\frac{n-r}{2}\right]} \frac{(-1)^s\ n!\ H_{n-r-2s}(\frac{z}{2})\ H_r(\frac{x}{2})\ x^{n-r} y^{n-2s} z^{r+2s}}{r!s!(n-r-2s)!} \tag{3.9}$$

$$H_n(x,y,z) = \sum_{r=0}^{n} \sum_{s=0}^{\left[\frac{n-r}{2}\right]} \frac{(-1)^s\ n!\ x^r z^n [x^2 z^2 - (1-y^2)(x^2+z^2)]^s}{r!s!(n-r-2s)!}$$

$$\times H_{n-r-2s}\left(\frac{xy}{2}\right) H_r\left(\frac{yz}{2}\right) \qquad (3.10)$$

$$H_n(x,y,z) = \sum_{r=0}^{n}\sum_{s=0}^{\left[\frac{n-r}{2}\right]} \frac{(-1)^s\ n!\ x^n y^r[x^2y^2-(1-z^2)(x^2+y^2)]^s}{r!s!(n-r-2s)!} \times H_{n-r-2s}\left(\frac{yz}{2}\right) H_r\left(\frac{zx}{2}\right) \qquad (3.11)$$

$$H_n(x,y,z) = \sum_{r=0}^{n}\sum_{s=0}^{\left[\frac{n-r}{2}\right]} \frac{(-1)^s\ n!\ y^n z^r[y^2z^2-(1-x^2)(y^2+z^2)]^s}{r!s!(n-r-2s)!} \times H_{n-r-2s}\left(\frac{zx}{2}\right) H_r\left(\frac{xy}{2}\right) \qquad (3.12)$$

$$H_n(x,y,z) = \sum_{r=0}^{n}\sum_{s=0}^{\left[\frac{n-r}{2}\right]} \frac{(-1)^r\ n!\ (2yz)^r(1-x)^{r+s}(1+x)^s(y^2+z^2)^s}{r!s!(n-r-2s)!} \times H_{n-r-2s}(1,y,z) \qquad (3.13)$$

$$H_n(x,y,z) = \sum_{r=0}^{n}\sum_{s=0}^{\left[\frac{n-r}{2}\right]} \frac{(-1)^r\ n!\ (2zx)^r(1-y)^{r+s}(1+y)^s(z^2+x^2)^s}{r!s!(n-r-2s)!} \times H_{n-r-2s}(x,1,z) \qquad (3.14)$$

$$H_n(x,y,z) = \sum_{r=0}^{n}\sum_{s=0}^{\left[\frac{n-r}{2}\right]} \frac{(-1)^r\ n!\ (2xy)^r(1-z)^{r+s}(1+z)^s(x^2+y^2)^s}{r!s!(n-r-2s)!} \times H_{n-r-2s}(x,y,1) \qquad (3.15)$$

$$H_n(x_1+x_2,y,z) = \sum_{r=0}^{n}\sum_{s=0}^{\left[\frac{n-r}{2}\right]} \frac{(-1)^s n!\ [2x_1x_2(y^2+z^2)-y^2z^2]^s}{r!s!(n-r-2s)!} \times H_{n-r-2s}(x_1,y,z)\, H_r(x_2,y,z) \qquad (3.16)$$

$$H_n(x, y_1+y_2, z) = \sum_{r=0}^{n} \sum_{s=0}^{\left[\frac{n-r}{2}\right]} \frac{(-1)^s n! [2y_1y_2(z^2+x^2) - z^2x^2]^s}{r! s! (n-r-2s)!} \times H_{n-r-2s}(x, y_1, z) H_r(x, y_2, z) \quad (3.17)$$

$$H_n(x, y, z_1+z_2) = \sum_{r=0}^{n} \sum_{s=0}^{\left[\frac{n-r}{2}\right]} \frac{(-1)^s n! [2z_1z_2(x^2+y^2) - x^2y^2]^s}{r! s! (n-r-2s)!} \times H_{n-r-2s}(x, y, z_1) H_r(x, y, z_2) \quad (3.18)$$

$$H_n(\lambda x, y, z) = \sum_{r=0}^{n} \sum_{s=0}^{\left[\frac{n-r}{2}\right]} \frac{(-1)^r n! (1-\lambda)^{r+s} (1+\lambda)^s (2xyz)^r (x^2y^2+z^2x^2)^s}{r!\ s!\ (n-r-2s)!} \times H_{n-r-2s}(x, y, z) \quad (3.19)$$

$$H_n(x, \mu y, z) = \sum_{r=0}^{n} \sum_{s=0}^{\left[\frac{n-r}{2}\right]} \frac{(-1)^r\ n!\ (1-\mu)^{r+s} (1+\mu)^s (2xyz)^r (x^2y^2+y^2z^2)^s}{r!\ s!\ (n-r-2s)!} \times H_{n-r-2s}(x, y, z) \quad (3.20)$$

$$H_n(x, y, \nu z) = \sum_{r=0}^{n} \sum_{s=0}^{\left[\frac{n-r}{2}\right]} \frac{(-1)^r\ n! (1-\nu)^{r+s} (1+\nu)^s (2xyz)^r (y^2z^2+z^2x^2)^s}{r!\ s!\ (n-r-2s)!} \times H_{n-r-2s}(x, y, z) \quad (3.21)$$

$$\begin{aligned} H_n(\lambda x, \mu y, \nu z) &= \sum_{r=0}^{n} \sum_{s=0}^{\left[\frac{n-r}{2}\right]} \frac{(-1)^r n! H_{n-r-2s}\left[(1-\lambda)x, (1-\mu)y, (1-\nu)z\right]}{r!} \\ &\times \frac{[(1-\lambda)(1-\mu)(1-\nu) - \lambda\mu\nu]^r (2xyz)^r \left\{[(1-2\lambda)(1-\mu)^2 + \lambda^2(1-2\mu)]x^2y^2\right.}{s!} \\ &\times \frac{\left.+[(1-2\mu)(1-\nu)^2 + \mu^2(1-2\nu)]y^2z^2 + [(1-2\nu)(1-\lambda)^2 + \nu^2(1-2\lambda)]z^2x^2\right\}^s}{(n-r-2s)!} \end{aligned} \quad (3.22)$$

4 Recurrence Relations

The following recurrence relations hold for $H_n(x,y,z)$:

$$\frac{\partial}{\partial x} H_n(x,y,z) = 2nyz\, H_{n-1}(x,y,z) - 2n(n-1)x(y^2+z^2)\, H_{n-2}(x,y,z) \tag{4.1}$$

$$\frac{\partial}{\partial y} H_n(x,y,z) = 2nxz\, H_{n-1}(x,y,z) - 2n(n-1)y(z^2+x^2)\, H_{n-2}(x,y,z) \tag{4.2}$$

$$\frac{\partial}{\partial z} H_n(x,y,z) = 2nxy\, H_{n-1}(x,y,z) - 2n(n-1)z(x^2+y^2)\, H_{n-2}(x,y,z) \tag{4.3}$$

$$H_n(x,y,z) = 2xyz\, H_{n-1}(x,y,z) - 2(n-1)(x^2y^2+y^2z^2+z^2x^2)\, H_{n-2}(x,y,z) \tag{4.4}$$

5 Relation Between $H_n(x,y,z)$ and $H_n(x)$

We obtain the following relation between $H_n(x,y,z)$ and $H_n(x)$:

$$H_n(x,y,z) = (x^2y^2+y^2z^2+z^2x^2)^{\frac{n}{2}}\, H_n\left(\frac{xyz}{\sqrt{x^2y^2+y^2z^2+z^2x^2}}\right). \tag{5.1}$$

Now

$$H_n(-x,y,z) = (-1)^n\, H_n(x,y,z). \tag{5.2}$$

For x=0, (5.1) reduces to

$$H_n(0,y,z) = (yz)^n\, H_n(0). \tag{5.3}$$

And for y=0, (5.1) reduces to

$$H_n(x,0,z) = (zx)^n\, H_n(0). \tag{5.4}$$

Also for z=0, (5.1) reduces to

$$H_n(x,y,0) = (xy)^n\, H_n(0). \tag{5.5}$$

But

$$\left.\begin{array}{l} H_{2n}(0) = (-1)^n\, 2^{2n}\left(\frac{1}{2}\right)_n \\ H_{2n+1}(0) = 0. \end{array}\right\} \tag{5.6}$$

So using (5.6) in (5.3), (5.4) and (5.5), we obtain

$$\left.\begin{array}{l} H_{2n}(0,y,z) = (-1)^n (2yz)^{2n}\left(\frac{1}{2}\right)_n \\ H_{2n+1}(0,y,z) = 0 \end{array}\right\} \tag{5.7}$$

and

$$\left.\begin{array}{l} H_{2n}(x,0,z) = (-1)^n (2xz)^{2n} \left(\frac{1}{2}\right)_n \\ H_{2n+1}(x,0,z) = 0 \end{array}\right\} \tag{5.8}$$

also

$$\left.\begin{array}{l} H_{2n}(x,y,0) = (-1)^n (2xy)^{2n} \left(\frac{1}{2}\right)_n \\ H_{2n+1}(x,y,0) = 0. \end{array}\right\} \tag{5.9}$$

For $x = y = z$=0, (5.1) reduces to

$$H_n(0,0,0) = 0 = H_n(0) \tag{5.10}$$

which in the light of (5.6), gives

$$H_{2n}(0,0,0) = (-1)^n 2^{2n} \left(\frac{1}{2}\right)_n = H_{2n}(0) \tag{5.11}$$

$$H_{2n+1}(0,0,0) = 0 = H_{2n+1}(0). \tag{5.12}$$

Now

$$H_n(x,y,z) = \sum_{r=0}^{\left[\frac{n}{2}\right]} \sum_{s=0}^{\left[\frac{n}{2}-r\right]} \frac{(-1)^{r+s} n!\ H_{n-2r-2s}(x)\ x^{2r+2s} y^{n-2s} z^{n-2r}}{r!\ s!\ (n-2r-2s)!}, \tag{5.13}$$

and $H'_n(x) = \frac{d}{dx} H_n(x)$. If we denote $\left[\frac{\partial}{\partial x} H_n(x,y,z)\right]_{x=0}$ by $\frac{\partial}{\partial x} H_n(0,y,z)$, then

$$\left.\begin{array}{l} \frac{\partial}{\partial x} H_{2n}(0,y,z) = 0 = H'_{2n}(0) \\ \frac{\partial}{\partial x} H_{2n+1}(0,y,z) = (-1)^n (2yz)^{2n+1} \left(\frac{3}{2}\right)_n = H'_{2n+1}(0) \end{array}\right\} \tag{5.14}$$

and

$$\left.\begin{array}{l} \frac{\partial}{\partial y} H_{2n}(x,0,z) = 0 = H'_{2n}(0) \\ \frac{\partial}{\partial y} H_{2n+1}(x,0,z) = (-1)^n (2xz)^{2n+1} \left(\frac{3}{2}\right)_n = H'_{2n+1}(0) \end{array}\right\} \tag{5.15}$$

also

$$\left.\begin{array}{l} \frac{\partial}{\partial z} H_{2n}(x,y,0) = 0 = H'_{2n}(0) \\ \frac{\partial}{\partial z} H_{2n+1}(x,y,0) = (-1)^n (2xy)^{2n+1} \left(\frac{3}{2}\right)_n = H'_{2n+1}(0) \end{array}\right\} \tag{5.16}$$

6 The Rodrigues Formula

The Rodrigue's formula for $H_n(x, y, z)$ is given by the following relation:

$$H_n(x,y,z) = (-1)^n(x^2y^2+y^2z^2+z^2x^2)^{\frac{n}{2}} e^{\frac{x^2y^2z^2}{x^2y^2+y^2z^2+z^2x^2}} \times \frac{d^n}{d\left(\frac{xyz}{\sqrt{x^2y^2+y^2z^2+z^2x^2}}\right)^n} e^{-\frac{x^2y^2z^2}{x^2y^2+y^2z^2+z^2x^2}} \qquad (6.1)$$

a formula of the same nature as Rodrigue's formula for Hermite polynomial of one variable $H_n(x)$.

7 Expansion of Polynomials

Since

$$e^{2xyzt-(x^2y^2+y^2z^2+z^2x^2)t^2} = \sum_{n=0}^{\infty} \frac{H_n(x,y,z)t^n}{n!}$$

it follows that

$$e^{2xyzt} = e^{(x^2y^2+y^2z^2+z^2x^2)t^2} \sum_{n=0}^{\infty} \frac{H_n(x,y,z)t^n}{n!}$$

or,

$$\sum_{n=0}^{\infty} \frac{(2xyzt)^n}{n!} = \sum_{n=0}^{\infty} \sum_{p=0}^{\left[\frac{n}{2}\right]} \sum_{q=0}^{\left[\frac{n}{2}-p\right]} \sum_{r=0}^{\left[\frac{n}{2}-p-q\right]} \frac{x^{2p+2r}y^{2p+2q}z^{2q+2r}t^n}{p!\ q!\ r!\ (n-2p-2q-2r)!} \times H_{n-2p-2q-2r}(x,y,z).$$

Equating coefficients of t^n, we get

$$\frac{(2xyzt)^n}{n!} = \sum_{p=0}^{\left[\frac{n}{2}\right]} \sum_{q=0}^{\left[\frac{n}{2}-p\right]} \sum_{r=0}^{\left[\frac{n}{2}-p-q\right]} \frac{H_{n-2p-2q-2r}(x,y,z)\ x^{2p+2r}y^{2p+2q}z^{2q+2r}}{p!\ q!\ r!\ (n-2p-2q-2r)!}. \qquad (7.1)$$

Let we employ (7.1) to expand the Legendre polynomial in a series of Hermite polynomials. Consider the series

$$\sum_{n=0}^{\infty} P_n(xyz)t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k \left(\frac{1}{2}\right)_{n-k} (2xyz)^{n-2k} t^n}{k!(n-2k)!}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{1}{2}\right)_{n+k} (2xyz)^n t^{n+2k}}{k!n!}. \qquad (7.2)$$

Hence by using (7.1), we may write (7.2) as

$$\sum_{n=0}^{\infty} P_n(xyz)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty} \frac{(-1)^k \left(\frac{1}{2}\right)_{n+k+2p+2q+2r} t^{n+2k+2p+2q+2r}}{k!\ p!\ q!\ r!\ n!} \times H_n(x,y,z)\ x^{2p+2r}\ y^{2p+2q}\ z^{2q+2r}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \left\{ \sum_{p=0}^{k}\sum_{q=0}^{k}\sum_{r=0}^{k} \frac{(-k)_{p+q+r} \left(\frac{1}{2}+n+k\right)_{p+q+r} x^{2p+2r} y^{2p+2q} z^{2q+2r}}{p!\ q!\ r!} \right\} \times \frac{(-1)^k \left(\frac{1}{2}\right)_{n+k} H_n(x,y,z) t^{n+2k}}{k!\ n!}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{2}]} F^{(3)} \left[\begin{matrix} -k, \frac{1}{2}+n-k :: -; -; - : -; -; -; \\ - :: -; -; - : -; -; -; \end{matrix} \ x^2y^2, y^2z^2, z^2x^2 \right] \times \frac{(-1)^k \left(\frac{1}{2}\right)_{n-k} H_{n-2k}(x,y,z) t^n}{k!\ (n-2k)!}.$$

Equating the coefficients of t^n, we get

$$P_n(xyz) = \sum_{k=0}^{[\frac{n}{2}]} F^{(3)} \left[\begin{matrix} -k, \frac{1}{2}+n-k :: -; -; - : -; -; -; \\ - :: -; -; - : -; -; -; \end{matrix} \ x^2y^2, y^2z^2, z^2x^2 \right] \times \frac{(-1)^k \left(\frac{1}{2}\right)_{n-k} H_{n-2k}(x,y,z)}{k!\ (n-2k)!}. \tag{7.3}$$

We now employ (7.1) to expand the Legendre polynomial of two variables (defined and studied by Khan and Shakeel [5]) in a series of Hermite polynomials of two variables. We have

$$P_n(x,y) = \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k \left(\frac{1}{2}\right)_{n-k} (2x)^{n-2k} (1+w)^k}{k!\ (n-2k)!}.$$

Now consider the series

$$\sum_{n=0}^{\infty} P_n(xyz,w)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k \left(\frac{1}{2}\right)_{n-k} (2xyz)^{n-2k} (1+w)^k\ t^n}{k!\ (n-2k)!}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty} \frac{(-1)^k \left(\frac{1}{2}\right)_{n+k+2p+2q+2r} (1+w)^k\ t^{n+2k+2p+2q+2r}}{k!\ p!\ q!\ r!\ n!} \times H_n(x,y,z)\ x^{2p+2r} y^{2p+2q} z^{2q+2r}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \left\{ \sum_{p=0}^{k}\sum_{q=0}^{k}\sum_{r=0}^{k} \frac{(-k)_{p+q+r}\left(\frac{1}{2}+n+k\right)_{p+q+r}}{p!\ q!\ r!} \left(\frac{x^2y^2}{1+w}\right)^p \left(\frac{y^2z^2}{1+w}\right)^q \right.$$

$$\left. \times \left(\frac{z^2x^2}{1+w}\right)^r \right\} \frac{(-1)^k \left(\frac{1}{2}\right)_{n+k} H_n(x,y,z)\ (1+w)^k\ t^{n+2k}}{k!n!}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{2}\right]} F^{(3)} \left[\begin{array}{l} -k, \frac{1}{2}+n-k :: -;-;- : -;-;-; \\ - :: -;-;- : -;-;-; \end{array} \frac{x^2y^2}{1+w}, \frac{y^2z^2}{1+w}, \frac{z^2x^2}{1+w} \right]$$

$$\times \frac{(-1)^k \left(\frac{1}{2}\right)_{n-k} H_{n-2k}(x,y,z)\ (1+w)^k\ t^n}{k!(n-2k)!}.$$

The final result is

$P_n(xyz, w)$

$$= \sum_{k=0}^{\left[\frac{n}{2}\right]} F^{(3)} \left[\begin{array}{l} -k, \frac{1}{2}+n-k :: -;-;- : -;-;-; \\ - :: -;-;- : -;-;-; \end{array} \frac{x^2y^2}{1+w}, \frac{y^2z^2}{1+w}, \frac{z^2x^2}{1+w} \right]$$

$$\times \frac{(-1)^k \left(\frac{1}{2}\right)_{n-k} H_{n-2k}(x,y,z)\ (1+w)^k}{k!\ (n-2k)!} \tag{7.4}$$

8 Binomial and Trinomial Operator Representations

In a recent paper in 2008, Khan and Shukla [6] obtained binomial and trinomial operator representations of certain polynomials. Using their technique we have obtained certain results of binomial and trinomial operator representation type for three variables Hermite polynomials $H_n(x,y,z)$ by using their Rodrigues formula. Here we need the following results of [6]:

$$(D_x + D_y)^n \{f(x)\ g(y)\} = \sum_{r=0}^{n} \binom{n}{r} D_x^{n-r}\ f(x)\ D_y^r\ g(y) \tag{8.1}$$

$(D_x+D_y+D_z)^n \{f(x)\ g(y)\ h(z)\}$

$$= \sum_{r=0}^{n}\sum_{s=0}^{n-r} \frac{(-n)_{r+s}(-1)^{r+s}}{r!s!} D_x^{n-r-s}\ f(x)\ D_y^r\ g(y)\ D_z^s\ h(z) \tag{8.2}$$

and

$(D_xD_y + D_xD_z + D_yD_z)^n \{f(x)\ g(y)\ h(z)\}$

$$= \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s}(-1)^{r+s}}{r!s!} D_x^{n-s}\ f(x)\ D_y^{n-r}\ g(y)\ D_z^{r+s}\ h(z). \tag{8.3}$$

The results obtained are as follows: Let $\frac{d}{d\left(\frac{xyz}{\sqrt{x^2y^2+y^2z^2+z^2x^2}}\right)} \equiv D_1$ and $\frac{d}{d\left(\frac{uvw}{\sqrt{u^2v^2+v^2w^2+w^2u^2}}\right)} \equiv D_2$, then

$$(D_1+D_2)^n \left\{ e^{-\frac{x^2y^2z^2}{x^2y^2+y^2z^2+z^2x^2}} .\ e^{-\frac{u^2v^2w^2}{u^2v^2+v^2w^2+w^2u^2}} \right\}$$

$$= (-1)^n (x^2y^2+y^2z^2+z^2x^2)^{-\frac{n}{2}} e^{-\frac{x^2y^2z^2}{x^2y^2+y^2z^2+z^2x^2}} e^{-\frac{u^2v^2w^2}{u^2v^2+v^2w^2+w^2u^2}}$$

$$\times \sum_{r=0}^{n} \binom{n}{r} \left(\sqrt{\frac{x^2y^2+y^2z^2+z^2x^2}{u^2v^2+v^2w^2+w^2u^2}}\right)^r H_{n-r}(x,y,z) H_r(u,v,w). \tag{8.4}$$

Again let $\frac{d}{d\left(\frac{xyz}{\sqrt{x^2y^2+y^2z^2+z^2x^2}}\right)} \equiv D_1$, $\frac{d}{d\left(\frac{uvw}{\sqrt{u^2v^2+v^2w^2+w^2u^2}}\right)} \equiv D_2$

and $\frac{d}{d\left(\frac{fgh}{\sqrt{f^2g^2+g^2h^2+h^2f^2}}\right)} \equiv D_3$, then

$$(D_1+D_2+D_3)^n \left\{ e^{-\frac{x^2y^2z^2}{x^2y^2+y^2z^2+z^2x^2}} .e^{-\frac{u^2v^2w^2}{u^2v^2+v^2w^2+w^2u^2}} .e^{-\frac{f^2g^2h^2}{f^2g^2+g^2h^2+h^2f^2}} \right\}$$

$$= (-1)^n (x^2y^2+y^2z^2+z^2x^2)^{-\frac{n}{2}} e^{-\frac{x^2y^2z^2}{x^2y^2+y^2z^2+z^2x^2}} e^{-\frac{u^2v^2w^2}{u^2v^2+v^2w^2+w^2u^2}}$$

$$\times e^{-\frac{f^2g^2h^2}{f^2g^2+g^2h^2+h^2f^2}} \sum_{r=0}^{n} \sum_{s=0}^{n-r} \binom{n}{r} \binom{n-r}{s} \left(\sqrt{\frac{x^2y^2+y^2z^2+z^2x^2}{u^2v^2+v^2w^2+w^2u^2}}\right)^r$$

$$\times \left(\sqrt{\frac{x^2y^2+y^2z^2+z^2x^2}{f^2g^2+g^2h^2+h^2f^2}}\right)^s H_{n-r-s}(x,y,z)\ H_r(u,v,w)\ H_s(f,g,h) \tag{8.5}$$

and

$$(D_1D_2+D_1D_3+D_2D_3)^n \left\{ e^{-\frac{x^2y^2z^2}{x^2y^2+y^2z^2+z^2x^2}} .e^{-\frac{u^2v^2w^2}{u^2v^2+v^2w^2+w^2u^2}} .e^{-\frac{f^2g^2h^2}{f^2g^2+g^2h^2+h^2f^2}} \right\}$$

$$= (x^2y^2+y^2z^2+z^2x^2)^{-\frac{n}{2}} (u^2v^2+v^2w^2+w^2u^2)^{-\frac{n}{2}} e^{-\frac{x^2y^2z^2}{x^2y^2+y^2z^2+z^2x^2}}$$

$$\times e^{-\frac{u^2v^2w^2}{u^2v^2+v^2w^2+w^2u^2}} e^{-\frac{f^2g^2h^2}{f^2g^2+g^2h^2+h^2f^2}} \sum_{r=0}^{n} \sum_{s=0}^{n-r} \binom{n}{r} \binom{n-r}{s}$$

$$\times \left(\sqrt{\frac{u^2v^2+v^2w^2+w^2u^2}{f^2g^2+g^2h^2+h^2f^2}}\right)^r \left(\sqrt{\frac{x^2y^2+y^2z^2+z^2x^2}{f^2g^2+g^2h^2+h^2f^2}}\right)^s H_{n-s}(x,y,z)$$

$$\times H_{n-r}(u,v,w) H_{r+s}(f,g,h). \tag{8.6}$$

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