Thai Journal of Mathematics
Volume 9 (2011) Number 2 : 383–390

www.math.science.cmu.ac.th/thaijournal
Online ISSN 1686-0209

Congruences on E-Inversive Semigroups

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Abstract : A semigroup S is said to be E-inversive if for each $a \in S$ there exists $x \in S$ such that ax is an idempotent. In this paper we have obtained a characterization on E-inversive semigroup S in which $Reg(S)$ is completely simple sub semigroup of S. It is also proved that in an orthodox semigroup S if $a \leq b$ then $W(a) \subseteq W(b)$ and we have given an example for which the converse is not true. Finally in his paper, Certain congruences on E-inversive E-semigroups, the author has obtained regular congruences on an E-inversive semigroup S in which $W(a)$ has maximal element for all a in S. This motivates us to find the smallest regular congruence on an E-inversive semigroup in which each $W(a)$ has greatest element.

Keywords : E-inversive semigroup; E-semigroup; Regular congruence; Semilattice congruence.
2010 Mathematics Subject Classification : 20M10.

1 Introduction

A semigroup S is said to be an E-inversive semigroup, if for each $a \in S$ there exists $x \in S$ such that ax is an idempotent. A semigroup S is said to be an E-semigroup if $E(S)$ is a sub semigroup of S. In Lemma-1.2 of this paper it is proved that in an E-inversive semigroup S, $Reg(S)$ is a completely simple sub semigroup of S only when $W(a) = V(a)$ for all $a \in S$. In [1], it is observed that if S is an E-inversive semigroup then $E(S)$ is a rectangular band if and only if $W(a) \cap W(b) \neq \emptyset$ implies $W(a) = W(b)$. In this paper we have proved that if S

is an E-inversive semigroup such that for any $a, b \in S$, $W(a) \cap W(b) \neq \emptyset$ implies either $W(a) \subseteq W(b)$ or $W(b) \subseteq W(a)$ then $E(S)$ is a sub semigroup of S such that for any $e, f \in E(S)$, either $efe = e$ or $fef = f$. In an orthodox semigroup S for any $a, b \in S$ if $a \leq b$ then $W(a) \subseteq W(b)$ which is proved in Theorem 1.7 of this paper. Finally in [2], regular congruences on an E-inversive semigroup S in which each $W(a)$ has a maximal element are studied. This motivates us to find the smallest regular congruence on an E-inversive semigroup S in which each $W(a)$ has greatest element.

We start with the following lemma.

Lemma 1.1. *Let S be an E-inversive semigroup such that for any $a, x \in S$, $a = axa$ implies $x = xax$. Then S is completely simple and $W(a) = V(a)$, for all $a \in S$.*

Proof. Obviously $W(a) = V(a)$. Let $a \in S$. Since S is E-inversive there exists $x \in S$ such that $xax = x$ and hence by hypothesis $axa = a$. Thus S is regular. Therefore S is completely simple by Theorem IV 2.4 [3]. □

In general if S is an E-inversive semigroup such that $W(a) = V(a)$, for all $a \in Reg(S)$, then $a = axa$ need not imply $x = xax$ in S. The following is the example.

Example 1.2. *Let S be a zero semigroup with $|S| > 1$. Then 0 is the only regular element in S and $W(0) = V(0)$. For any $a \neq 0 \in S$, $0a0 = 0$ but $a0a \neq a$.*

The following lemma gives the characterization of an E-inversive semigroup S in which $Reg(S)$ is a completely simple sub semigroup of S.

Lemma 1.3. *Let S be an E-inversive semigroup then $W(a) = V(a)$, for all $a \in Reg(S)$ if and only if $Reg(S)$ is completely simple sub semigroup of S.*

Proof. Suppose that $W(a) = V(a)$, for all $a \in Reg(S)$ and let $a, b \in Reg(S)$ so there exists $x, y \in S$ such that $a = axa$ and $b = byb$. Choose $t \in W(ab)$ then $abtaxabt = abt$ so that $abt \in W(ax) = V(ax)$, since ax is idempotent and hence $axabtax = ax$ which implies $abtab = ab$. Therefore ab is regular. Thus $Reg(S)$ is a sub semigroup and hence by Lemma-1.1, $Res(S)$ is completely simple. Conversely assume that $Reg(S)$ is a completely simple sub semigroup of S. Let $a \in Reg(S)$ and $x \in W(a)$. Then $xax = x$ so that $x \in Reg(S)$ and hence by Theorem IV 2.4 [3], $axa = a$. Therefore $x \in V(a)$. Thus $W(a) = V(a)$, for all $a \in Reg(S)$. □

In [1], the author has observed that if S is an E-semigroup then $E(S)$ is a rectangular band if and only if $W(a) \cap W(b) \neq \emptyset$ implies $W(a) = W(b)$. This motivates us to find the following theorem.

Theorem 1.4. *If S is an E-inversive semigroup such that for any $a, b \in S$, $W(a) \cap W(b) \neq \emptyset$ implies either $W(a) \subseteq W(b)$ or $W(b) \subseteq W(a)$ then $E(S)$ is a sub semigroup of S such that for any $e, f \in S$ either $efe = e$ or $fef = f$.*

Proof. Let $e, f \in E(S)$. Then $W(e) \cap W(f) \neq \emptyset$ so that $W(e) \subseteq W(f)$ or $W(f) \subseteq W(e)$. Therefore either $efe = e$ or $fef = f$ and hence $E(S)$ is a subsemigroup of S. $\square$

The above condition is only sufficient but not necessary because of the following example.

Example 1.5. *Let G be a group with $|G| > 2$ and let $S = G \cup \{0\}$. Then $E(S)$ is a sub semigroup of S such that for any $e, f \in E(S)$ either $efe = e$ or $fef = f$. Further for any $a \neq b \in G$, $0 \in W(a) \cap W(b)$ but neither $W(a) \subseteq W(b)$ nor $W(b) \subseteq W(a)$.*

Corollary 1.6. *If S is an E-inversive semigroup such that for any $a, b \in S$, $W(a) \cap W(b) \neq \emptyset$ implies either $W(a) \subseteq W(b)$ or $W(b) \subseteq W(a)$ then $E(S)$ is a rectangular band if and only if $W(a) = V(a)$, for all $a \in Reg(S)$.*

In the following theorem we have proved that if S is an orthodox semigroup then $a \leq b$ in S implies $W(a) \subseteq W(b)$.

Theorem 1.7. *Let S be an orthodox semigroup. For any $a, b \in S$, if $a \leq b$ then $W(a) \subseteq W(b)$.*

Proof. Let $a, b \in S$ such that $a \leq b$ then $a = eb$, for some idempotent $e \in Ra \leq Rb$. Let $c \in W(a)$ then $c = cac$ so that $cebc = c$. Hence $ce \in W(b)$ and since S is E-semigroup we have $ceac \in W(b)$. Therefore $c \in W(b)$. Thus $W(a) \subseteq W(b)$. $\square$

If S is a regular semigroup such that for any $a, b \in S$, $a \leq b$ implies $W(a) \subseteq W(b)$ then S need not be an orthodox semigroup because of the following example.

Example 1.8. *Consider the semigroup $S = \{a, b, c, d, e, f, g, h\}$ with the following multiplication table*

a	f	c	h	c	f	a	h
e	b	g	d	e	d	g	b
c	h	a	f	a	h	c	f
g	d	e	b	g	b	e	d
e	d	g	b	g	d	e	b
e	f	a	h	c	h	a	f
g	b	e	d	e	b	g	d
a	h	c	f	a	f	c	h

Clearly S is a regular semigroup. There exist no $x \neq y \in S$ such that $x \leq y$. Therefore S satisfies the condition that for any $a, b \in S$, $a \leq b$ implies $W(a) \subseteq W(b)$ but S is not orthodox.

Theorem 1.9. *Let S be an E-inversive E-semigroup then the following are equivalent.*

(1) $W(a) = V(a)$, for all $a \in Res(S)$.

(2) $W(e) = V(e)$, *for all* $e \in E(S)$.

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1) : Assume (2). Let $e, f \in E(S)$ then $efe \in W(e) = V(e)$ so that $eefee = e$ which implies $efe = e$. Hence $E(S)$ is a rectangular band. Therefore by Theorem 4.6 in [4], $W(a) = V(a)$, for all $a \in Reg(S)$. □

In the following theorem we have obtained characterizations of a semigroup S in which $E(S)$ is a chain of rectangular groups.

Theorem 1.10. *Let* S *be a semigroup in which* $E(S)$ *is non-empty. Then the following are equivalent.*

(1) $E(S)$ *is a chain of rectangular groups.*

(2) $E(S)$ *is a band and for any* $e, f \in E(S)$ *either* $efe = e$ *or* $fef = f$.

(3) $\{W(e)|e \in E(S)\}$ *is a chain.*

Proof. (1) $\Leftrightarrow$ (2) follows by Problem 4 in page no. 120 in [3].

(2) $\Rightarrow$ (3) : Let $e, f \in E(S)$ then by hypothesis either $efe = e$ or $fef = f$ and since $E(S)$ is a band we have $W(e) \subseteq W(f)$ or $W(f) \subseteq W(e)$. Therefore $\{W(e)|e \in E(S)\}$ is a chain.

(3) $\Rightarrow$ (2) : Let $e, f \in E(S)$ then either $W(e) \subseteq W(f)$ or $W(f) \subseteq W(e)$ so that $efe = e$ or $fef = f$ and hence $efef = ef$ therefore $E(S)$ is a band. □

In Lemmas 3.6 and 3.7 in [1], it is proved that if S is an E-inversive E-semigroup then it satisfies the conditions (2) and (3) in the following theorem where as we have proved the equivalence.

Proposition 1.11. *Let* S *be a semigroup such that* $E(S) \neq \emptyset$. *Then the following are equivalent.*

(1) $E(S)$ *is a band.*

(2) For any $a \in S$, $a' \in W(a)$ *and* $e \in E(S)$, $ea', a'e \in W(a)$.

(3) For any $a \in S$ *and* $a' \in V(a)$, $W(a) = E(S).a'.E(S)$.

Proof. (2) $\Rightarrow$ (1) : Let $e, f \in E(S)$ since $e \in W(e)$, by hypothesis $ef \in W(e)$ so that $efeef = ef$ that is $efef = ef$ and hence $E(S)$ is a band.

(3) $\Rightarrow$ (1) : Let $e, f \in E(S)$. Since $e \in V(e)$, we have $W(e) = E(S)eE(S)$. Now $eef \in E(S)eE(S) = W(e)$ so that $efef = ef$ Hence $E(S)$ is a band. □

The following is an example in which S is an E-inversive E-semigroup and $a \in W(b)$ such that $W(b) = E(S)aE(S)$ but b is not regular.

Example 1.12. *Let $S = \{a, b, c, d\}$ be a semigroup with the following multiplication table*

$$\begin{array}{cccc} a & a & a & a \\ a & a & b & a \\ a & a & c & a \\ a & b & a & d \end{array}$$

Proposition 1.13. *If S an E-inversive semigroup such that $E(S)$ is a semilattice and $\{W(a)\}_{a\in S}$ is a chain then $E(S)$ is a chain.*

Proof. Let $e, f \in E(S)$. Then either $W(e) \subseteq W(f)$ or $W(f) \subseteq W(e)$ so that either $efe = e$ or $fef = f$. Since $E(S)$ is a semilattice either $ef = fe = e$ or $ef = fe = f$. Hence either $e \leq f$ or $f \leq e$. Thus $E(S)$ is a chain. □

In the following example it is observed that the converse of the above proposition is not true.

Example 1.14. *Let S be a group with identity e and $|S| > 2$ then $E(S) = \{e\}$ so that $E(S)$ is a chain and $W(a) = V(a)$, for all $a \in S$ but $\{W(a)\}_{a\in S}$ is not chain.*

Theorem 1.15. *Let S be an E-semigroup. Then the following are equivalent.*

(1) $E(S)$ is a semilattice.

(2) $W(e)$ has greatest element, for all $e \in E(S)$.

(3) For every $a \in S$, $V(a)$ contains at most one element.

Proof. (1) ⇒ (2) : Suppose $E(S)$ is a semilattice and let $e \in E(S)$. Then for any $f \in W(e)$, $fef = f$ and $E(S)$ is semilattice so that we have $ef = fe = f$. Therefore $f \leq e$ and $e \in W(e)$. Hence $W(e)$ has greatest element for all $e \in W(e)$.

(2) ⇒ (3) : Suppose $W(e)$ has greatest element for all $e \in E(S)$. Then e is the greatest element in $W(e)$, for any $e \in E(S)$. Let $a \in S$ and $a', a'' \in V(a)$ then $aa'a = a$,$a'aa' = a'$ and $aa''a = a$, $a''aa'' = a''$. Then $aa'aa''aa' = aa'$ so that we have $aa' \leq aa''$. Similarly we can prove that $aa'' \leq aa'$. Therefore $aa' = aa''$. By a similar argument we have $a'a = a''a$. Thus $a' = a''$ as required.

(3) ⇒ (1) follows by Theorem 3.12 [1]. □

Definition 1.16. A congruence ρ on an E-inversive semigroup S is said to be *regular congruence* if for each $a \in S$, there exists $a' \in W(a)$ such that $a\rho aa'a$.

Let S be an E-inversive semigroup in which each $W(a)$ has maximal element. Let ρ be the relation defined by $a\rho b \Leftrightarrow$ there exists $z \in W(a) \cap W(b)$ such that for all $a' \in W(a)$, $a' \geq z$ there exists $b' \in W(b)$ such that $a'a = b'b$, $aa' = bb'$ and for all $b' \in W(b)$, $b \geq z$ there exists $a' \in W(a)$ such that $a'a = b'b$, $aa' = bb'$.

In Theorem 5.9 [2] it is observed that the congruence generated by ρ is a regular congruence. Now it is interesting to find the smallest regular congruence on E-inversive semigroup. In this connection we have obtained the smallest regular congruence on E-inversive semigroups when ever $W(a)$ has greatest element for all $a \in S$.

Theorem 1.17. *Let S be an E-inversive semigroup such that $W(a)$ has greatest element for all $a \in S$. Then the congruence generated by the relation $R = \{(a, aa'a) | a \in S$ and $a' \in W(a)$ is the greatest element of $W(a)\}$ is the smallest regular congruence on S. If $E(S)$ is a sub semigroup of S then the congruence generated by R is the smallest inverse congruence on S.*

Proof. Let ρ be a regular congruence on S and $a \in S$. Let $a' \in W(a)$ be the greatest element of $W(a)$. Since ρ is a regular congruence there exists $a^* \in W(a)$ such that $a\rho aa^*a$. Then $a^* \leq a'$ so that $aa^* \leq aa'$ and $a^*a \leq a'a$ (see [1]). We have $a\rho aa^*a = aa^*aa'a\rho aa'a$. Therefore $R \subseteq \rho$. Thus the congruence generated by R is the smallest regular congruence on S. If $E(S)$ is a sub semigroup S then $E(S)$ becomes a semilattice by the above theorem-1.15 and hence the congruence generated by R is the smallest inverse congruence on S. □

The following example shows that the above defined congruence is in general not trivial.

Example 1.18. *Let S be the set of all non-negative integers. Then S is a semigroup with respect to the usual multiplication. Here $R = \{(1,1)\} \cup \{(a, 0) | a \in S - \{1\}\}$ and hence the congruence generated by R is $\{(1,1)\} \cup \{(a, b) | a, b \in S - \{1\}\}$.*

Let S be an E-inversive semigroup and ρ be the relation defined by $\rho = \{(a, b) \in E(S) \times E(S)$ such that $eaf = ebf$, for all $e, f \in E(S)\}$. In Proposition 2.1 [5], it is proved that if $E(S)$ is a rectangular band then ρ is rectangular band and it is also observed that if $E(S)$ is a normal band then ρ is a semilattice congruence on $E(S)$. In fact if $E(S)$ is a rectangular band then ρ becomes the trivial congruence $E(S) \times E(S)$ and the result can be effectively modified.

Theorem 1.19. *Let S be an E-inversive semigroup then the relation ρ defined above is a congruence when ever $E(S)$ is a band and more over ρ is the smallest semilattice congruence if and only $E(S)$ is a normal band.*

Proof. Suppose $E(S)$ is a band. Obviously ρ is an equivalence relation. Let $(a, b) \in \rho$ and $c \in E(S)$ then $eaf = ebf$, for all $e, f \in E(S)$. Since ec, cf are also idempotents we have $eacf = ebcf$ and $eacf = ebcf$, for all $e, f \in E(S)$. Therefore $(ac, bc), (ca, cb) \in \rho$. Therefore ρ is a congruence on $E(S)$ and hence a band congruence. Now let us assume that $E(S)$ is a normal band. For any $a, b \in E(S)$, we have $eabf = ebaf$, for all $e, f \in E(S)$. Therefore ρ is a semilattice congruence on $E(S)$. Since $\mathcal{D}$ is the smallest semilattice congruence on $E(S)$, $\mathcal{D} \subseteq \rho$. Suppose $(a, b) \in \rho$ then $eaf = ebf$, for all $e, f \in E(S)$ so that $SaS = SbS$ and hence $(a, b) \in \mathcal{J} = \mathcal{D}$. Thus $\mathcal{D} = \rho$. Therefore ρ is the smallest semilattice congruence on $E(S)$. Conversely suppose ρ is the smallest semilattice congruence on $E(S)$ and hence a semilattice congruence on $E(S)$ so that $(ab, ba) \in \rho$, for all $a, b \in E(S)$. Therefore $eabf = ebaf$, for all $e, f \in E(S)$. Thus $E(S)$ is a normal band. □

In Theorem 2.6 [5] it is proved that if S is an E-inversive semigroup then the relation ν defined by $\nu = \{(a, b) \in S \times S | \exists a' \in W(a), b' \in W(b)$ such that $W(aea') = W(beb')$, for all $e \in E(S)\}$ is an inverse congruence on S when ever $E(S)$ is a rectangular band . In fact this congruence ν is trivial congruence $S \times S$ on S when ever $E(S)$ is a rectangular band this is because for any $a \in S$, $a' \in W(a)$, $e \in E(S)$, we have aea' is an idempotent and $W(f) = E(S)$, for all $f \in E(S)$.

Definition 1.20. A semi group S is said to satisfy the *condition* $(*)$ if for any $x, y \in S$, $xy, yx \in E(S)$ implies $xy = yx$.

For any $a \in S$, $E(a) = \{e \in E(S) | a \geq e\}$.

Theorem 1.21. *If S an E-inversive semigroup then the following are equivalent.*

(1) S satisfies the condition $(*)$.

(2) The relation $\eta = \{(a, b) \in S \times S | E(a) = E(b)\}$ is a semilattice congruence on S.

(3) For any $a, b \in S$ if $ab = e \in E(S)$ then $bea = e$.

Proof. (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) follow by Theorem 2.5 [5] and Lemma 2.3 [5] respectively.

(2) $\Rightarrow$ (1) : Suppose η is a semilattice congruence on S and $a, b \in S$ such that $ab, ba \in E(S)$. Put $e = ab$ and $f = ba$. Since η is a semilattice congruence $(ab, ba) \in \eta$ so that $(e, f) \in \eta$. Therefore $E(e) = E(f)$. Since $e \in E(e) = E(f)$ we have $f \geq e$. Similarly we can prove that $e \geq f$. Thus $e = f$ as required.

(3) $\Rightarrow$ (1) : Assume the condition (3). Let $a, b \in S$ such that $ab, ba \in E(S)$. Put $ab = e \in E(S)$ then by assumption $bea = e$ so that $baba = ab$ and ba is an idempotent therefore $ba = ab$. Hence S satisfies the condition $(*)$. □

Proposition 1.22. *Let S be an E-inversive semigroup satisfying the condition $(*)$. If for any $a \in S$, $W(a) \subseteq E(S)$ implies a is an idempotent then $E(S)$ is a semilattice.*

Proof. Let $e, f \in E(S)$. Choose $x \in W(ef)$ then $fxe \in W(e) \cap W(f)$ so that $efxe = fxef = fxe$ and hence $efxef = fxe \in E(S)$. Put $g = efxef$. Then $x = xgx = xggx = xg \in E(S)$, as $xg, gx \in E(S)$. Therefore $W(ef) \subseteq E(S)$ and hence ef is an idempotent. Thus $E(S)$ is a semilattice. □

If S is an E-inversive semigroup satisfying $(*)$ and $E(S)$ is a semilattice then $W(a) \subseteq E(S)$ need not imply that a is an idempotent.

Example 1.23. *Let S be the set of all non-negative integers. Then S is an E-inversive semigroup satisfying $(*)$ and $E(S)$ is a semilattice but for any $a \neq 1$, $W(a) \subseteq E(S)$ and a is not an idempotent.*

Acknowledgements : The author would like to thank the referees for his comments and suggestions on the manuscript. The author would like to express her gratitude to Prof. P.V. Ramana Murty for his guidance and valuable suggestions through out the preparation of this paper.

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(Received 8 May 2010)
(Accepted 31 March 2011)