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On *Rad*-$\oplus$-Supplemented Modules

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Abstract : Let R be a ring and M a right R-module. M is called *Rad-$\oplus$-s-module* if every submodule of M has a *Rad*-supplement that is a direct summand of M, and M is called *completely Rad-$\oplus$-s-module* if every direct summand of M is *Rad*-$\oplus$-s-module. In this paper various properties of such modules are developed. It is shown that any finite direct sum of *Rad*-$\oplus$-s-modules is *Rad*-$\oplus$-s-module. We also show that if M is *Rad*-$\oplus$-s-module with (D_3), then M is completely *Rad*-$\oplus$-s-module.

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1 Introduction

In this paper all rings are associative with identity and all modules are unital right modules. Let R be a ring and M be an R-module. $N \leq M$ will mean N is a submodule of M. $E(M)$, $Rad(M)$, $Z(M)$ will indicate *injective hull*, *Jacobson radical* and *singular submodule* of M, respectively. We set $Z^*(M) = \{m \in M : \text{mR} \text{ is small } in \text{ E(mR)}\}$, which is a submodule of M. A submodule E of M is called *essential* in M (notation $E \leq_e M$) if $E \cap A \neq 0$ for any non-zero submodule A of M. Dually, a submodule S of M is called *small* in M (notation $S \ll M$) if $M \neq S + T$ for any proper submodule T of M. Let $A \subseteq B \subseteq M$, submodule B is said to be a *closure* of A in M if A is a essential submodule of B and B a closed submodule in M. Let N and L be submodules of M, N is called a *supplement* of L

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in M if $N+L=M$ and N is minimal with respect to this property, or equivalenty, $M=N+L$ and $N\cap L\ll N$. M is called an *amply supplemented* module if for any two submodules A and B of M with $A+B=M$, B contains a supplement of A. M is called a *supplemented* module if every submodule M has a supplement in M. A non-zero module M is called *hollow* if every proper submodule of M is small in M and M is called *local* if the sum of all proper submodules of M is also a proper submodule of M. Every local module is hollow. M has property (p^*) (see [1]) if for any submodule N of M, there exists a direct summand K of M such that $K\leq N$ and $N/K\leq Rad(M/K)$. The notions which are not explained here will be found in [2].

Lemma 1.1 ([2])**.** *Let M be a module and K supplement submodule of M. Then $K\cap Rad(M)=Rad(K)$.*

Let M be a module. We consider the following conditions.

(D_1) For every submodule N of M, M has a decoposition with $M=M_1\oplus M_2$, $M_1\leq N$ and $M_2\cap N$ is small in M_2.

(D_3) If M_1 and M_2 are direct summands of M with $M=M_1+M_2$, then $M_1\cap M_2$ is also a direct summand of M.

By [3, Lemma 4.6 and Proposition 4.38], every quasi-projective module has (D_3).

2 *Rad*-⊕-s-modules

Let M be a module. If $U,U'\leq M$ and $M=U+U'$, then U' is called a Rad-*supplement* of U in case $U\cap U'\leq Rad(U')$. Clearly, each supplement submodule is a *Rad*-supplement submodule. M is called a Rad-⊕-*supplemented* module if every submodule of M has a *Rad*-supplement that is a direct summand of M, denoted by *Rad*-⊕-s-module. For example, hollow modules and modules with (p^*) are *Rad*-⊕-s-module.

Let M be a module. Then by [3, Proposition 4.8], M has (D_1) if and only if M is amply supplemented and every supplement submodule of M is a direct summand. Therefore every (D_1)-module is *Rad*-⊕-s-module. But in general the converse is not true as the following example shows.

Example 2.1. *Let R be a discrete valuation ring with field of fractions K. Let P be the unique maximal ideal of R such that $P=Ra$ for some element $a\in P$. Let $M=(K/R)\oplus(R/P)$. By [3, Proposition A.7], M is* Rad-⊕*-s-module.*

Recall that a projective module M is *semiperfect* if every homomorphic image of M has a projective cover. Then we have the following lemma.

Lemma 2.2. *Let M be a projective module. Consider the following conditions.*

(i) M is a semiperfect module.

(ii) M *is a* Rad-⊕-*s-module.*

Then $(i) \Rightarrow (ii)$ *and if* M *is a finitely generated module then* $(ii) \Rightarrow (i)$.

Proof. $(i) \Rightarrow (ii)$. Let N be a submodule of M. Then by assumption, there exists a projective cover $\pi : P \to M/N$. For the canonical epimorphism $\sigma : M \to M/N$, since M is projective, there exists a homomorphism $f : M \to P$ such that $\pi \circ f = \sigma$. Since π is small, f is epic by [2] and so f splits (P is projective). Then, by [2], there exists some homomorphism $g : P \to M$ such that $f \circ g = 1_P$, and hence $\pi = \pi \circ f \circ g = \sigma \circ g$. Note that $M = Kerf \oplus g(P)$ and $Kerf \leq N$; therefore, $M = N + g(p)$. Let μ be the restriction of σ to $g(p)$. Then $\pi = \mu \circ g$ and so μ is epic. Therefore since π is small, μ is small by [2]. That is, $Ker\mu = N \cap g(p) \ll g(p)$. Hence, $g(p)$ is a supplement of N.

$(ii) \Rightarrow (i)$. Let M be a finitely generated module and N be a submodule of M. Since M is *Rad*-⊕-s-module, there exist submodules K and K' of M such that $M = N + K$, $N \cap K \leq Rad(K)$, and $K \oplus K' = M$. Clearly, K is projective. For the inclusion homomorphism $i : K \to M$ and the canonical epimorphism $\sigma : M \to M/N$, $\sigma \circ i : K \to M/N$ is an epimorphism, and by hypothesis $Rad(M) \ll M$, this implies that $Rad(K) \ll K$ and hence $Ker\sigma \circ i = N \cap K \ll K$. □

Lemma 2.3. *Let* N, L *be submodules of a module* M *such that* $N + L$ *has a* Rad-*supplement* H *in* M *and* $N \cap (H + L)$ *has a* Rad-*supplement* G *in* N. *Then* $H + G$ *is a* Rad-*supplement of* L *in* M.

Proof. Let H be a *Rad*-supplement of $N + L$ in M and G be a *Rad*-supplement of $N \cap (H + L)$ in N. Then $M = (N + L) + H$ such that $(N + L) \cap H \leq Rad(H)$ and $N = [N \cap (H + L)] + G$ such that $(H + L) \cap G \leq Rad(G)$. Since $(H + G) \cap L \leq [(G + L) \cap H] + [(H + L) \cap G] \leq Rad(H) + Rad(G) \leq$ Rad(H+G), $H + G$ is a *Rad*-supplement of L in M. □

Theorem 2.4. *Let* M_1 *and* M_2 *be* Rad-⊕-*s-modules. If* $M = M_1 \oplus M_2$, *then* M *is a* Rad-⊕-*s-module.*

Proof. Let L be any submodule of M. Then $M = M_1 + M_2 + L$ so that $M_1 + M_2 + L$ has a *Rad*-supplement 0 in M. Let H be a *Rad*-supplement of $M_2 \cap (M_1 + L)$ in M_2 such that H is a direct summand of M_2. By Lemma 2.3, H is a *Rad*-supplement of $M_1 + L$ in M. Let K be a *Rad*-supplement of $M_1 \cap (L + H)$ in M_1 such that K is a direct summand of M_1. Again by applying Lemma 2.3, we have that $H + K$ is a *Rad*-supplement of L in M. Since H is a direct summand of M_2 and K is a direct summand of M_1, it follows that $H + K = H \oplus K$ is a direct summand of M. Thus $M = M_1 \oplus M_2$ is *Rad*-⊕-s-module. □

Corollary 2.5. *Any finite direct sum of* Rad-⊕-*s-modules is a* Rad-⊕-*s-module.*

Corollary 2.6. *Any finite direct sum of modules with* (p^*) *is* Rad-⊕-*s-module.*

Corollary 2.7. *Any finite direct sum of hollow (or local) modules is* Rad-⊕-*s-module.*

Let M be a module. A Submodule X of M is called *fully invariant* if, for every $h \in End_R(M)$, $h(X) \subseteq X$. The module M is called *duo module*, if every submodule of M is fully invariant.

Lemma 2.8. *Let M be a duo module. If $M = M_1 \oplus M_2$, then $A = (A \cap M_1) \oplus (A \cap M_2)$ for any submodule A of M.*

Proof. See [4]. □

Now we investigate conditions which ensure that a factor module of a *Rad*-$\oplus$-s-module will be a *Rad*-$\oplus$-s-module.

Proposition 2.9. *Assume that M is a* Rad-$\oplus$-*s-duo module and $N \leq M$. Then M/N is a* Rad-$\oplus$-*s-module.*

Proof. For any submodule K of M containing N, since M is a *Rad*-$\oplus$-s-module, there exist submodules L and L' of M such that $M = K + L = L \oplus L'$, and $K \cap L \leq Rad(L)$. Note that $M/N = K/N + (L + N)/N$, and $K \cap (L + N) = (K \cap L) + N$. Since $K \cap L \leq Rad(L)$, we have $K/N \cap (L+N)/N = [(K \cap L)+N]/N \leq Rad((L + N)/N)$. This implies that $(L + N)/N$ is a *Rad*-supplemented of K/N in M/N. Now by Lemma 2.8, $N = (N \cap L) \oplus (N \cap L')$, implies that $(L+N) \cap (L'+N) \leq N + (L + N \cap L + N \cap L') \cap L'$. It follows that $(L + N) \cap (L' + N) \leq N$ and $M/N = ((L + N)/N) \oplus ((L' + N)/N)$. Then $(L + N)/N$ is a direct summand of M/N. Consequently, M/N is a *Rad*-$\oplus$-s-module. □

A module M is called *distributive* if its lattice of submodules is a distributive lattice, equivalently for submodules K,L,N of M, $N + (K \cap L) = (N + K) \cap (N + L)$ or $N \cap (K + L) = (N \cap K) + (N \cap L)$.

Theorem 2.10.

(1) Let M be a Rad-$\oplus$-*s-module and N a submodule of M. If for every direct summand K of M, $(N + K)/N$ is a direct summand of M/N then M/N is a* Rad-$\oplus$-*s-module.*

(2) Let M be a distributive Rad-$\oplus$-*s-module. Then M/N is a* Rad-$\oplus$-*s-module for every submodule N of M.*

Proof. (1) For any submodule X of M containing N, since M is a *Rad*-$\oplus$-s-module, there exists a direct summand D of M such that $M = X + D = D \oplus D'$ and $X \cap D \leq Rad(D)$ for some submodule D' of M. Now $M/N = X/N + (D + N)/N$. By hypothesis, $(D + N)/N$ is a direct summand of M/N. Note that $(X/N) \cap ((D + N)/N) = [X \cap (D + N)]/N = [N + (D \cap X)]/N$. Since $X \cap D \leq Rad(D)$, we have $[(D \cap X) + N]/N \leq Rad((D + N)/N)$. This implies that $(D + N)/N$ is a *Rad*-supplement submodule of X/N in M/N. Hence M/N is a *Rad*-$\oplus$-s-module.

(2) Let D be a direct summand of M. Then $M = D \oplus D'$ for some submodule D' of M. Now $M/N = [(D + N)/N] + [(D' + N)/N]$ and $N = N + (D \cap D') = (N + D) \cap (N + D')$ by distributivity of M. This implies that $M/N = [(D + N)/N] \oplus [(D' + N)/N]$. By (1), M/N is a *Rad*-$\oplus$-s-module. □

3 Completely *Rad*-⊕-s-modules

While the properties lifting, amply supplemented and supplemented are inherited by summands, it is unknown (and unlikely) that the same is true for the property *Rad*-⊕-s-module. In this vein we call a module M completely *Rad*-⊕-s-module if every direct summand of M is *Rad*-⊕-s-module.

Given a positive integer n, the modules M_i $(1 \leq i \leq n)$ are called relatively projective if M_i is M_j-projective for all $(1 \leq i \neq j \leq n)$.

Theorem 3.1. *Let M_i $(1 \leq i \leq n)$ be any finite collection of relatively projective modules. Then the module $M = M_1 \oplus \cdots \oplus M_n$ is* Rad-⊕-*s-module if and only if M_i is* Rad-⊕-*s-module for each $1 \leq i \leq n$.*

Proof. The sufficiency is proved in Theorem 2.4. Conversely, we only prove M_1 to be *Rad*-⊕-s-module. Let $A \leq M_1$. Then there exists $B \leq M$ such that $M = A+B$, B is a direct summand of M and $A \cap B \leq Rad(B)$. Since $M = A + B = M_1 + B$, By [3, Lemma 4.47], there exists $B_1 \leq B$ such that $M = M_1 \oplus B_1$. Then $B = B_1 \oplus (M_1 \cap B)$. Note that $M_1 = A + (M_1 \cap B)$ and $M_1 \cap B$ is a direct summand of M_1. Therefore $A \cap B = A \cap (M_1 \cap B)$ and $A \cap B \leq Rad(M)$, $A \cap B \leq M_1 \cap B$, then $A \cap B \leq (M_1 \cap B) \cap Rad(M) = Rad(M_1 \cap B)$ by Lemma 1.1. Hence M_1 is *Rad*-⊕-s-module. □

Proposition 3.2. *Let M be a* Rad-⊕-*s-module with (D_3). Then M is completely* Rad-⊕-*s-module.*

Proof. Let N be a direct summand of M and A a submodule of N. We show that A has a *Rad*-supplement in N that is direct summand of N. Since M is *Rad*-⊕-s-module, there exists a direct summand B of M such that $M = A + B$ and $A \cap B \leq Rad(B)$. Hence $N = A + (N \cap B)$. Furthermore $N \cap B$ is a direct summand of M because M has (D_3). Then $A \cap (N \cap B) = A \cap B$ and $A \cap B \leq Rad(M)$, $A \cap B \leq N \cap B$, then have $A \cap B \leq (N \cap B) \cap Rad(M) = Rad(N \cap B)$. □

A module M is said to have the summand sum property (SSP) if the sum of any pair of direct summands of M is a direct summands of M, i.e., if N and K are direct summands of M then $N + K$ is also a direct summand of M.

Theorem 3.3. *Let M be a* Rad-⊕-*s-module with the SSP. Then M is completely* Rad-⊕-*s-module.*

Proof. Let N be a direct summand of M. Then $M = N \oplus N'$ for some $N' \leq M$. We want to show that M/N' is a *Rad*-⊕-s-module. Assume that L is a direct summand of M. Since M has the SSP, $L + N'$ is a direct summand of M. Let $M = (L + N') \oplus K$ for some $K \leq M$. Then $M/N' = (L + N')/N' \oplus (K + N')/N'$. Therefore M/N' is a *Rad*-⊕-s-module by Theorem 2.10(1). □

A module M is said to have the Summand Intersection Property (SIP) if the intersection of any pair of direct summands of M is a direct summand of M, i.e., if N and K are direct summands of M then $N \cap K$ is also a direct summand of M.

Lemma 3.4 ([4, Corollary 18]). *Let M be a duo module. Then M has the SIP and the SSP.*

As a result of Theorem 3.3 and Lemma 3.4, we can obtain the following Corollary;

Corollary 3.5. *Let M be a* Rad-$\oplus$-*s-duo module. Then M is completely* Rad-$\oplus$-*s-module.*

In [5], Smith calls a module M a (UC)-module if every submodule of M has a unique *closure* in M. M is called *extending* module if every closed submodule of M is a direct summand of M.

Theorem 3.6. *Let M be a UC extending module. Then M is* Rad-$\oplus$-*s-module if and only if M is completely* Rad-$\oplus$-*s-module.*

Proof. Sufficiency is clear. Conversly, assume that M is *Rad*-$\oplus$-s-module. By [6, Lemma 2.4], M has (D_3). Hence M is completely *Rad*-$\oplus$-s-module from Proposition 3.2. □

The module M has *finite Goldie dimension* if M does not contain an infinite direct sum of non-zero submodules. It is well-known that a module M has finite Goldie dimension if and only if there exists a positive integer n and uniform submodules U_i $(1 \leq i \leq n)$ of M such that $U_1 \oplus \cdots \oplus U_n$ is an essential submodule of M and in this case n is an invariant of the module M called the *Goldie dimension* of M (see, for example [7, p. 294 Ex. 2]).

Let M be a module. M is called *monoform* if each non-zero partial endomorphism of M is monomorphism. M is called *polyform* if each partial endomorphism has closed kernel. M is called locally finite dimensional if every finitely generated submodule has finite Goldie dimension, following [8], note that polyform extending modules have (D_3) [9, Lemma 1.11] and every monoform module is polyform.

Corollary 3.7. *Let M be a polyform (monoform) extending module. Then M is* Rad-$\oplus$-*s-module if and only if M is completely* Rad-$\oplus$-*s-module.*

Proof. By [8, Proposition 2.2], M is a (UC)-module. Then by Theorem 3.6, we have the result. □

Theorem 3.8. *Suppose that M is a locally finite dimensional polyform module. If M is quasi-injective, then for any index set I, $M^{(I)}$ is* Rad-$\oplus$-*s-module if and only if $M^{(I)}$ is completely* Rad-$\oplus$-*s-module.*

Proof. Suppose that $M^{(I)}$ is *Rad*-$\oplus$-s-module. Since M is polyform, $M^{(I)}$ is polyform from [10, Proposition 3.3] and $M^{(I)}$ is quasi-injective from [8, Corollary 3.4]. Hence $M^{(I)}$ is extending since every quasi-injective module is extending (see [3]). By Corollary 3.7, $M^{(I)}$ is completely *Rad*-$\oplus$-s-module. □

Lemma 3.9. *Let M be a supplemented module and N be a submodule of M such that $N \cap Rad(M) = 0$. Then N is semisimple.*

Proof. By [2], $M/Rad(M)$ is semisimple. Hence N is semisimple. □

Proposition 3.10. *Let M be a* Rad-$\oplus$-*s-module. Then $M = M_1 \oplus M_2$, where M_1 is a semisimple module and M_2 is a module whit $Rad(M_2)$ essential in M_2.*

Proof. For $Rad(M)$, there exists $M_1 \leq M$ such that $M_1 \cap Rad(M) = 0$ and $M_1 \oplus Rad(M) \leq_e M$. Since M is a Rad-$\oplus$-s-module, there exists a direct summand M_2 of M such that $M_1 + M_2 = M$ and $M_1 \cap M_2 \leq Rad(M_2)$. Since $M_1 \cap M_2 = M_1 \cap (M_1 \cap M_2) \leq M_1 \cap Rad(M_2) \leq M_1 \cap Rad(M) = 0$, $M = M_1 \oplus M_2$. By Lemma 3.9, M_1 is semisimple. Thus $Rad(M) = Rad(M_1) \oplus Rad(M_2) = Rad(M_2)$. Since $M_1 \oplus Rad(M) \leq_e M = M_1 \oplus M_2$, i.e., $M_1 \oplus Rad(M_2) \leq_e M = M_1 \oplus M_2$, $Rad(M_2) \leq_e M_2$ by [7, Proposition 5.20]. This completes the proof. □

Proposition 3.11. *Let M be a* Rad-$\oplus$-*s-module. Then $M = M_1 \oplus M_2$, where M_1 is a module with $Z^*(M_1) \leq Rad(M_1)$ and M_2 is a module with $Z^*(M_2) = M_2$.*

Proof. Since M is Rad-$\oplus$-s-module, there exists a direct summand M_1 of M such that $M = Z^*(M) + M_1$, $Z^*(M_1) = Z^*(M) \cap M_1 \leq Rad(M_1)$ and $M = M_1 \oplus M_2$ for some submodule M_2 of M. Since $Z^*(M) = Z^*(M_1) \oplus Z^*(M_2)$, then $Z^*(M_2) = M_2$. □

Theorem 3.12. *For a module M with (D_3) the following statements are equivalent.*

- *(i) M is completely* Rad-$\oplus$-*s-module.*
- *(ii) M is* Rad-$\oplus$-*s-module.*
- *(iii) $M = M_1 \oplus M_2$, where M_1 is a semisimple module and M_2 is a* Rad-$\oplus$-*s-module with $Rad(M_2)$ essential in M_2.*
- *(iv) $M = M_1 \oplus M_2$, where M_1 is a* Rad-$\oplus$-*s-module with $Z^*(M_1) \leq Rad(M_1)$ and M_2 is a* Rad-$\oplus$-*s-module with $Z^*(M_2) = M_2$.*

Proof. $(i) \Rightarrow (ii)$. Clear from definition.

$(ii) \Rightarrow (i)$. It follows from Proposition 3.2.

$(i) \Rightarrow (iii)$. By Proposition 3.10, $M = M_1 \oplus M_2$, where M_1 is semisimple and $Rad(M_2)$ is essential in M_2. By (i), M_2 is Rad-$\oplus$-s-module.

$(i) \Rightarrow (iv)$. By Proposition 3.11, we have $M = M_1 \oplus M_2$, where $Z^*(M_1) \leq Rad(M_1)$ and $Z^*(M_2) = M_2$ and hence M_1 and M_2 are Rad-$\oplus$-s-module by (i).

$(iii) \Rightarrow (ii)$, $(iv) \Rightarrow (ii)$. It follows by Theorem 2.4. □

Theorem 3.13. *The following statements are equivalent for a projective module M.*

- *(i) M is a direct sum of* Rad-$\oplus$-*s-modules and $Rad(M)$ has finite Goldie dimension.*
- *(ii) $M = M_1 \oplus M_2$ for some semisimple module M_1 and module M_2 such that M_2 has finite Goldie dimension and M_2 is a (finite) direct sum of local modules.*

Proof. $(ii) \Rightarrow (i)$. Clear. $(i) \Rightarrow (ii)$. Assume $M = \oplus_{i\in I} M_i$, M_i is *Rad*-$\oplus$-s-module and $Rad(M)$ has finite Goldie dimension. Since $Rad(M) = \oplus_{i\in I} Rad(M_i)$, then there is a finite subset J of I such that $Rad(M_i) = 0$ for all $i \in I - J$. Therefore M_i is semisimple for all $i \in I - J$. Hence there is a semisimple submodule M_1 of M such that $M = M_1 \oplus (\oplus_{j\in J} M_j)$. By Proposition 3.10, without loss of generality, we may assume $Rad(M_j)$ is essential in $M_j (j \in J)$. Then $M_j (j \in J)$ has finite Goldie dimension by [11, Proposition 3.20]. Next we prove each M_j is local or a finite direct sum of local modules, for $j \in J$. Set $H = M_j$ for any $j \in J$. First, note that $Rad(H) \neq H$ because H is projective [7, Proposition 17.14]. Assume H has Goldie dimension 1, and take some $x \in H - Rad(H)$. Since H is *Rad*-$\oplus$-s-module, there exists a submodule K of H such that $H = xR + K$, $xR \cap K \leq Rad(K)$ and $H = K \oplus K_1$ for some submodule K_1 of M. Then $K = 0$ or $K_1 = 0$. If $K_1 = 0$, then xR becomes a submodule of $Rad(H)$. This is a contradiction. Hence $K = 0$, thus $H = xR$. It follows that H is local. Let $n > 1$ be a positive integer and assume each M_j having Goldie dimension $k (1 \leq k < n)$ is local or a finite direct sum of local submodules. Let $j \in J$ and $H = M_j$ and assume H has Goldie dimension n. Suppose H is not local. Let $x \in H - Rad(H)$ such that $H \neq xR$. Then since H is *Rad*-$\oplus$-s-module there exists submodules K ,K_1 of H such that $H = xR + K = K \oplus K_1$ and $xR \cap K \leq Rad(K)$. It is clear that $K_1 \neq 0$. Also $K \neq 0$. Since projective modules satisfy (D_3) and then by Proposition 3.2, any direct summand of M is *Rad*-$\oplus$-s-module. Thus K and K_1 are *Rad*-$\oplus$-s-modules by induction, K and K_1 are local or finite direct sums of local submodules. This completes the proof of $(i) \Rightarrow (ii)$. □

Lemma 3.14. *Let M be an indecomposable module. Then M is a hollow module if and only if M is a completely* Rad-$\oplus$-*s-module.*

Proof. Clear from definitions. □

Proposition 3.15. *Let $M = U \oplus V$ such that U and V have local endomorphism rings. Then M is completely* Rad-$\oplus$-*s-module if and only if U and V are hollow modules.*

Proof. The necessity is clear from Lemma 3.14. Conversly, let K be a direct summand of M. If $K = M$ then by Corollary 2.7, K is *Rad*-$\oplus$-s-module. Assume $K \neq M$. Then either $K \cong U$ or $K \cong V$ by Krull-Schmidt-Azumaya Theorem [7, Corollary 12.7]. In either case K is *Rad*-$\oplus$-s-module. Thus M is completely *Rad*-$\oplus$-s-module. □

Theorem 3.16. *Let M be a non-zero module with finite Goldie dimension. Then the following statements are equivalent.*

(i) Every direct summand of M is a finite direct sum of hollow modules.

(ii) M is a completely Rad-$\oplus$-*s-module.*

Proof. $(i) \Rightarrow (ii)$. Clear by Corollary 2.7. $(ii) \Rightarrow (i)$. Let N be a direct summand of M. Since N has finite Goldie dimension, N has a decomposition $N = L_1 \oplus \cdots \oplus L_n$, where each L_i is indecomposable for $1 \leq i \leq n$ for some finite integer $1 \leq n$. Hence each L_i $(1 \leq i \leq n)$ is hollow from Lemma 3.14. □

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