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A Note on a Fixed Point Theorem in Hilbert Spaces

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Abstract : In this note we observe that a fixed point theorem (due to Nigam et al. [1]) proved in Hilbert spaces remains true in metric spaces.

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1 Introduction

There exists an extensive literature on fixed point theorems for various contractive conditions whose comprehensive survey can be found in Rhoades [3, 4]. Recently, Nigam et al. [1] proved the following fixed point theorem in Hilbert spaces using the parallelogram identity.

Theorem 1.1([1]) *Let S and T be non-commuting self-mappings of a closed subset C of a Hilbert space H satisfying*

$$\|STx-TSy\|^2 \leq a\|x-y\|^2+b[\|x-STx\|^2+\|y-TSy\|^2]+c[\|x-TSy\|^2+\|STx-y\|^2]$$

for all $x, y \in C$, and a, b, c being positive reals with $0 < a+2b+4c < 1$, $0 < b+c < 1$ and $0 < a + 2c < 1$. Then ST and TS have a unique common fixed point.

It is straightforward to note the following :

(a) Theorem 1.1 is in fact a result for a pair of mappings $T_1 = ST$ and $T_2 = TS$.

(b) Theorem 1.1 reduces to a result for a single map in case S and T commute.

(c) If the pair (S, T) commutes at z and Tz, then z remains the unique fixed point of S and T separately provided z is the unique common fixed point of ST and TS.

To see this, consider

$$ST(Sz) = S(TSz) = Sz,\ TS(Sz) = ST(Sz) = S(TSz) = Sz$$

$$ST(Tz) = TS(Tz) = T(STz) = Tz,\ TS(Tz) = T(STz) = Tz$$

which show that Sz and Tz are common fixed points of ST and TS. Now using the uniqueness of the common fixed point of ST and TS, one gets $z = Sz = Tz$.

The purpose of this note is to observe that even the following general form of Theorem 1.1 remains true in metric spaces. In doing this, we never need the specific properties of an inner product norm.

In fact, a slightly improved form of Theorem 1.1 remains true in metric spaces which is as under :

Theorem 1.2 *Let T_1 and T_2 be self-mappings of a closed subset C of a complete metric space X satisfying*

$$\|T_1x - T_2y\|^2 \leq a\|x-y\|^2 + b\|x - T_1x\|\|y - T_2y\| + c\|x - T_2y\|\|T_1x - y\| \quad (1)$$

for all $x, y \in C$ and a, b, c non negative reals, with max$\{a+b, a+c\} < 1$. Then T_1 and T_2 have a unique common fixed point.

Unfortunately, Theorem 1.2 is not a new result and is in fact a corollary to Theorem 3.1 of Park [2]. Here we state relevant portion of Theorem 3.1 of Park [2] which runs as follows :

Theorem 1.3([2]) *Let S and T be self-mappings of a metric space (X, d). If there exists a sequence $\{x_i \mid i = 0, 1, 2, 3, \ldots\} \subset X$, where $x_{2n+1} = Sx_{2n}, x_{2n+2} = Tx_{2n+1}$ such that $\overline{\{x_i\}}$ is complete, and if there exists a $\lambda \in [0, 1)$ such that*

$$d(Sx, Ty) \leq \lambda d(x, y) \quad (2)$$

for each distinct $x, y \in \overline{\{x_i\}}$ satisfying either $x = Ty$ or $y = Sx$, then either:

(i) S or T has a fixed point in $\{x_i\}$, or

(ii) $\{x_i\}$ converges to some $z \in X$ and $d(x_i, z) \leq \lambda^i d(x_0, x_1)/(1-\lambda)$ for $i > 0$.

To substantiate our claim, let us set $y = T_1x$ in (1) so as to get

$$\begin{aligned}
\|T_1x - T_2T_1x\|^2 &\leq a\|x - T_1x\|^2 + b\|x - T_1x\|\|T_1x - T_2T_1x\| + 0 \\
&\leq (a+b)\max\{\|x - T_1x\|^2, \|x - T_1x\|\|T_1x - T_2T_1x\|\} \\
&\leq (a+b)\max\{\|x - T_1x\|^2, \max\{\|x - T_1x\|^2, \|T_1x - T_2T_1x\|^2\}\} \\
&= (a+b)\max\{\|x - T_1x\|^2, \|T_1x - T_2T_1x\|^2\} \\
&= (a+b)\|x - T_1x\|^2,
\end{aligned}$$

yielding thereby

$$\|T_1x - T_2T_1x\| \leq k\|x - T_1x\|, \quad (3)$$

where $k = (a+b)^{1/2}$.

Thus inequality (3) implies (2). If Case (i) holds, then (1) implies that any fixed point of T_1 is a fixed point of T_2 and vice-versa. Otherwise, if (ii) is satisfied, then $\{x_n\}$ converges to $u \in C$ (as C is closed). Using (1),

$$\begin{aligned}\|T_1u - x_{2n+2}\|^2 \leq a\|u - x_{2n+1}\|^2 + b\|u - T_1u\|\|x_{2n+1} - x_{2n+2}\| \\ + c\|u - x_{2n+2}\|\|T_1u - x_{2n+1}\|.\end{aligned}$$

Taking as $n \to \infty$ yields

$$\|T_1u - u\|^2 \leq 0,$$

and $u = T_1u$. Now in view of (1) one gets $u = T_2u$ and hence u is a common fixed point of T_1 and T_2.

The uniqueness of a common fixed point is a consequence of the condition (1).

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