Thai Journal of Mathematics
Volume 9 (2011) Number 2 : 361–372
www.math.science.cmu.ac.th/thaijournal
Online ISSN 1686-0209

Nonderogatory of Sum and Product of Doubly Companion Matrices[1]

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Abstract : Butcher and Chartier in [1, pp. 274–276] first introduced the doubly companion matrices, after that Butcher and Wright [2] and Wright [3] used of doubly companion matrices as a tool to analyze numerical methods and some general linear methods property. In this paper, we prove that any doubly companion matrix, and the sum of two doubly companion matrices are nonderogatory, and obtain the explicit form of its minimal polynomials. Moreover, we construct some examples which show that those product of two doubly companion matrices may not be a nonderogatory matrix. As in [4], we gives some condition for which the product of (doubly) companion matrices is a nonderogatory matrix. In addition we assert that the product of two unreduced Hessenberg is not nonderogatory.

Keywords : Companion matrix; Doubly companion matrix; Nonderogatory; Smith normal form; Unreduced Hessenberg.
2010 Mathematics Subject Classification : 11C08; 15A18; 15A21; 65F15.

1 Introduction

Let $\mathbb{C}$ be the field of complex numbers and $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$. For a positive integer n, let M_n be the set of all $n \times n$ matrices over $\mathbb{C}$. Doubly companion matrices $C \in M_n$ were first introduced by Butcher and Chartier in [1, pp. 274–276], given

[1]This research was supported by Naresuan University.

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by

$$C = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & \dots & -\alpha_{n-1} & -\alpha_n - \beta_n \\ 1 & 0 & 0 & \dots & 0 & -\beta_{n-1} \\ 0 & 1 & 0 & \dots & 0 & -\beta_{n-2} \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & -\beta_2 \\ 0 & 0 & 0 & \dots & 1 & -\beta_1 \end{bmatrix},$$

that is, an $n \times n$ matrix C with $n > 1$ is called a *doubly companion matrix* if its entries c_{ij} satisfy $c_{ij} = 1$ for all entries in the sub-maindiagonal of C and else $c_{ij} = 0$ for $i \neq 1$ and $j \neq n$.

The characteristic polynomial $\det(xI - C)$ and also formulas for the eigenvectors of C was presented. Butcher and Wright in [2, pp. 363–364], and Wright in [3] used the doubly companion matrices as a tool for analyzing various extension of classical methods with inherent Runge-Kutta stability.

2 Preliminaries

Carl D. Meyer, [5, p. 644] asserted that matrices $A \in M_n$ for which the characteristic polynomial $c(x)$ equal to the minimum polynomial $m(x)$ are said to be *nonderogatory* matrices, and they are precisely the ones for which geometric multiplicity λ_j is equal to 1 for each eigenvalue λ_j of A.

We prefer to use the doubly companion matrix of the form

$$X(\alpha, \beta) = \begin{bmatrix} -b_{n-1} & -b_{n-2} & -b_{n-3} & \dots & -b_1 & -a_0 - b_0 \\ 1 & 0 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & 0 & \dots & 0 & -a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & -a_{n-2} \\ 0 & 0 & 0 & \dots & 1 & -a_{n-1} \end{bmatrix}. \tag{2.1}$$

and show that the matrix $X(\alpha, \beta)$ is a nonderogatory, that is, the characteristic polynomial $c_{X(\alpha,\beta)}$ is equal to the minimal polynomial $m_{X(\alpha,\beta)}$. In this paper we show that the sum of any two doubly companion matrices is also a nonderogatory, but in contrast the product of two doubly companion matrices need not be a nonderogatory matrix.

From (2.1), if $b_0 = b_1 = \cdots = b_{n-1} = 0$ then the doubly companion matrix is become a companion matrix of the form,

$$X(\alpha) = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & 0 & \dots & 0 & -a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & -a_{n-2} \\ 0 & 0 & 0 & \dots & 1 & -a_{n-1} \end{bmatrix},$$

and, if $a_0 = a_1 = \cdots = a_{n-1} = 0$ then the matrix in (2.1) is become a companion matrix of another form,

$$X(\beta) = \begin{bmatrix} -b_{n-1} & -b_{n-2} & -b_{n-3} & \dots & -b_1 & -b_0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}.$$

It is well known that these companion matrices are nonderogatory with the characteristic of $X(\alpha)$ and $X(\beta)$ are monic polynomials of the form

$$\alpha(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} + x^n$$

and

$$\beta(x) = b_0 + b_1x + b_2x^2 + \cdots + b_{n-1}x^{n-1} + x^n,$$

respectively.

We recall some well-known results from linear algebra.

Theorem 2.1 ([6, Theorem 3.3.15])**.** *A matrix $A \in M_n$ is similar to the companion matrix of its characteristic polynomial if and only if the minimal and characteristic polynomial of A are identical.*

Theorem 2.2 ([7, Theorem 7.12 (1)])**.** *A companion matrix $A = X(\alpha)$ is nonderogatory; in fact, $c_A(x) = m_A(x) = \alpha(x)$.*

Theorem 2.3 ([6, Theorem 3.3.14])**.** *Every monic polynomial is both the minimal polynomial and the characteristic polynomial of its companion matrix.*

If at least one of A or B is nonsingular, then AB and BA are similar; one may see [6, a part of Theorem 1.3.20]. Therefore the characteristic polynomial of BA is the same as that of AB, by [6, Theorem 1.3.3].

Theorem 2.4 ([8, Theorem 6.17])**.** *Let $m_A(x)$ be the minimum polynomial of $A \in M_n$ and let*

$$S(x) = \mathrm{Diag}[f_1(x), f_2(x), \dots, f_n(x)]$$

be the Smith canonical matrix equivalent of $xI - A$. Then $f_n(x) = m_A(x)$.

Lemma 2.5 ([9, Lemma 2])**.** *Let A be the companion matrix of the polynomial $\lambda^n - c_1\lambda^{n-1} - \cdots - c_{n-1}\lambda - c_n$ and $E = \mathrm{Diag}(d_1, d_2, \dots, d_n)$ with $\det E \neq 0$. Let*

$$E_{k+1} = d_n d_{n-1} \cdots d_{n-k} \quad \textit{for} \quad k = 0, 1, \dots, n-1.$$

Then AE is similar to the companion matrix of the polynomial: $\lambda^n - c_1E_1\lambda^{n-1} - \cdots - c_nE_n$.

Note: *$E_n = \det(E)$ and $a_0 = (-1)^n \det(A)$.*

3 Main Results

According to any companion matrix $C(\alpha)$ is a nonderogatory with the characteristic and minimal polynomial both equal to $\alpha(x)$, by Theorem 2.3. First, we show that the sum of two companion matrices of the same size are also nonderogatory. In general, we prove that the sum of two doubly companion matrices of the same size is also nonderogatory, but the product of two doubly companion matrices may not be nonderogatory matrix.

Theorem 3.1. *Let A and B be companion matrices of the same (type and same) size. Then $A+B$ is a nonderogatory matrix.*

Proof. Let $\alpha(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$ and $\beta(x) = x^n + b_{n-1}x^{n-1} + \cdots + b_2x^2 + b_1x + b_0$ are in $\mathbb{C}[x]$. We denote

$$A := X(\alpha) = \begin{bmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & \dots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -a_{n-1} \end{bmatrix}$$

(is companion matrix of $\alpha(x)$, A is called a *first companion* $n \times n$ matrix,) and

$$B := X(\beta) = \begin{bmatrix} 0 & 0 & \dots & 0 & -b_0 \\ 1 & 0 & \dots & 0 & -b_1 \\ 0 & 1 & \dots & 0 & -b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -b_{n-1} \end{bmatrix}$$

(is companion matrix of $\beta(x)$, B is called a *second companion* $n \times n$ matrix,) are companion matrices of $\alpha(x)$ and $\beta(x)$ respectively. Then

$$A + B = \begin{bmatrix} 0 & 0 & \dots & 0 & -(a_0 + b_0) \\ 2 & 0 & \dots & 0 & -(a_1 + b_1) \\ 0 & 2 & \dots & 0 & -(a_2 + b_2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 2 & -(a_{n-1} + b_{n-1}) \end{bmatrix}.$$

Apply Lemma 2.5, by let $M = \mathrm{Diag}(1/2^{n-1}, 1/2^{n-2}, \dots, 1/2, 1) \in M_n$. To show that $M^{-1}(A+B)M$ is a companion matrix. Now, consider

$$\begin{aligned} M^{-1}(A+B)M &= \begin{bmatrix} 2^{n-1} & 0 & 0 & \dots & 0 \\ 0 & 2^{n-2} & 0 & \dots & 0 \\ 0 & 0 & 2^{n-3} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & 1 \end{bmatrix} \\ &\times \begin{bmatrix} 0 & 0 & \dots & 0 & -(a_0+b_0) \\ 2 & 0 & \dots & 0 & -(a_1+b_1) \\ 0 & 2 & \dots & 0 & -(a_2+b_2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 2 & -(a_{n-1}+b_{n-1}) \end{bmatrix} \\ &\times \begin{bmatrix} 1/2^{n-1} & 0 & 0 & \dots & 0 \\ 0 & 1/2^{n-2} & 0 & \dots & 0 \\ 0 & 0 & 1/2^{n-3} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & \dots & 0 & -2^{n-1}(a_0+b_0) \\ 1 & 0 & \dots & 0 & -2^{n-2}(a_1+b_1) \\ 0 & 1 & \dots & 0 & -2^{n-3}(a_2+b_2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -(a_{n-1}+b_{n-1}) \end{bmatrix}. \end{aligned}$$

Therefore $A+B$ is similar to a companion matrix, so that $A+B$ is a nonderogatory matrix, by Theorem 2.1. □

Corollary 3.2. *Let $A_1, A_2, \dots, A_m$ be companion matrices in M_n. Then $\sum_{i=1}^m A_i$ is a nonderogatory matrix.*

Proof. Let $\alpha_i(x) = x^n + a_{i,n-1}x^{n-1} + a_{i,n-2}x^{n-2} + \dots + a_{i,1}x + a_{i,0}$; $i = 1, \dots, m$, are in $\mathbb{C}[x]$, and

$$A_i = \begin{bmatrix} 0 & 0 & \dots & 0 & -a_{i,0} \\ 1 & 0 & \dots & 0 & -a_{i,1} \\ 0 & 1 & \dots & 0 & -a_{i,2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -a_{i,n-1} \end{bmatrix}; \quad i = 1, 2, \dots, m,$$

are companion matrices of $\alpha_i(x)$; $i = 1, 2, \dots, m$, respectively. Then

$$\sum_{i=1}^m A_i = \begin{bmatrix} 0 & 0 & \dots & 0 & -\sum_{i=1}^m a_{i,0} \\ m & 0 & \dots & 0 & -\sum_{i=1}^m a_{i,1} \\ 0 & m & \dots & 0 & -\sum_{i=1}^m a_{i,2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & m & -\sum_{i=1}^m a_{i,n-1} \end{bmatrix}.$$

Let $M = \mathrm{Diag}(1/m^{n-1}, 1/m^{n-2}, \ldots, 1/m, 1) \in M_n$. Then, M is a nonsingular matrix, as in Theorem 3.1, it is easy to verify that $M^{-1}\sum_{i=1}^{m} A_i M$ is the desired companion matrix, so that $\sum_{i=1}^{m} A_i$ is a nonderogatory matrix. □

Butcher and Charier [1, Lemma 1] asserted that, the characteristic polynomial of $C(\alpha,\beta)$ given by omitting the negative powers of x in $x^{-n}\alpha(x)\beta(x)$. Now, we wish to show that any doubly companion matrix $X(\alpha,\beta)$ in (2.1) is similar to a companion matrix.

Theorem 3.3. *The doubly companion matrix $X(\alpha,\beta)$ is nonderogatory.*

Proof. Let

$$X(\alpha,\beta) = \begin{bmatrix} -b_{n-1} & -b_{n-2} & -b_{n-3} & \ldots & -b_1 & -a_0-b_0 \\ 1 & 0 & 0 & \ldots & 0 & -a_1 \\ 0 & 1 & 0 & \ldots & 0 & -a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & -a_{n-2} \\ 0 & 0 & 0 & \ldots & 1 & -a_{n-1} \end{bmatrix}.$$

To show that $X := X(\alpha,\beta)$ is similar to a companion matrix. We shall prove by explicit construction the existence of an invertible matrix T such that TXT^{-1} is a companion matrix. Now, choose an upper triangular matrix of size $n \times n$,

$$T = \begin{bmatrix} 1 & b_{n-1} & b_{n-2} & \ldots & b_1 \\ 0 & 1 & b_{n-1} & \ddots & \vdots \\ 0 & 0 & 1 & \ddots & b_{n-2} \\ \vdots & \vdots & \ddots & \ddots & b_{n-1} \\ 0 & 0 & \ldots & 0 & 1 \end{bmatrix}.$$

The matrix T is an upper triangular Toeplitz matrix with diagonal-constant 1 (so-called symmetrizer T of X.) Then T is nonsingular matrix, it is obtained that

$$T^{-1} = \begin{bmatrix} \mathbf{e}_1 & X\mathbf{e}_1 & X^2\mathbf{e}_1 & \ldots & X^{n-1}\mathbf{e}_1 \end{bmatrix},$$

where $\mathbf{e}_1 = [1, 0, \ldots, 0]^T$. Then

$$TXT^{-1} = \begin{bmatrix} 0 & 0 & \ldots & 0 & 0 & -(\sum\limits_{i+j=n} a_i b_j + a_0 + b_0) \\ 1 & 0 & \ldots & 0 & 0 & -(\sum\limits_{i+j=n+1} a_i b_j + a_1 + b_1) \\ 0 & 1 & \ldots & 0 & 0 & -(\sum\limits_{i+j=n+2} a_i b_j + a_2 + b_2) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 & -(\sum\limits_{i+j=n+n-2} a_i b_j + a_{n-2} + b_{n-2}) \\ 0 & 0 & \ldots & 0 & 1 & -(a_{n-1} + b_{n-1}) \end{bmatrix}.$$

The matrix $X(\sigma) := TXT^{-1}$ is the desired companion matrix, where

$$\begin{aligned}\sigma(x) &= x^n + (a_{n-1}+b_{n-1})x^{n-1} + (a_{n-1}b_{n-1}+a_{n-2}+b_{n-2})x^{n-2} + \cdots \\ &\quad + \left(\sum_{i+j=n+1} a_ib_j + a_1 + b_1\right)x + \left(\sum_{i+j=n} a_ib_j + a_0 + b_0\right).\end{aligned}$$

Therefore, the doubly companion matrix $X(\alpha,\beta)$ is nonderogatory matrix, by Theorem 2.1. $\square$

There is another method to show that the matrix $X(\alpha,\beta)$ in (2.1) is a nonderogatory by computing the minimal polynomial in the Smith normal form. Let us begin with the case in which X is the doubly companion matrix (2.1) of the monic polynomials

$$\begin{aligned}\alpha(x) &= a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} + x^n, \\ \beta(x) &= b_0 + b_1x + b_2x^2 + \cdots + b_{n-1}x^{n-1} + x^n \in \mathbb{C}[x].\end{aligned}$$

Let

$$X = \begin{bmatrix} -b_{n-1} & -b_{n-2} & -b_{n-3} & \dots & -b_1 & -a_0-b_0 \\ 1 & 0 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & 0 & \dots & 0 & -a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & -a_{n-2} \\ 0 & 0 & 0 & \dots & 1 & -a_{n-1} \end{bmatrix}.$$

We want to give a direct calculation which find the characteristic polynomial for X, as in [10, pp. 251–252]. In this case,

$$xI - X = \begin{bmatrix} x+b_{n-1} & b_{n-2} & b_{n-3} & \dots & b_1 & a_0-b_0 \\ -1 & x & 0 & \dots & 0 & a_1 \\ 0 & -1 & x & \dots & 0 & a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & x & a_{n-2} \\ 0 & 0 & 0 & \dots & -1 & x+a_{n-1} \end{bmatrix}.$$

Add x times row n to row $(n-1)$. This will remove the x in the $(n-1,n-1)$ place and it will not change the determinant. Then, add x times the new row $(n-1)$ to row $(n-2)$. Continue successively until all of the x's on the main diagonal have been removed by that process. The result is the matrix

$$\begin{bmatrix} b_{n-1} & b_{n-2} & b_{n-3} & \dots & b_1 & x^n + \cdots + a_1x + a_0 + b_0 \\ -1 & 0 & 0 & \dots & 0 & x^{n-1} + \cdots + a_2x + a_1 \\ 0 & -1 & 0 & \dots & 0 & x^{n-3} + \cdots + a_3x + a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & x^2 + a_{n-1}x + a_{n-2} \\ 0 & 0 & 0 & \dots & -1 & x + a_{n-1} \end{bmatrix},$$

which has the same determinant as $xI - X(\alpha, \beta)$. The upper right-hand entry of this matrix is the polynomial $\alpha + b_0$, and denoted the successive downward entries of the last column by $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$, that is

$$\begin{bmatrix} b_{n-1} & b_{n-2} & b_{n-3} & \ldots & b_1 & \alpha + b_0 \\ -1 & 0 & 0 & \ldots & 0 & \alpha_1 \\ 0 & -1 & 0 & \ldots & 0 & \alpha_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & \alpha_{n-2} \\ 0 & 0 & 0 & \ldots & -1 & \alpha_{n-1} \end{bmatrix}.$$

We clean up the first row by adding to it appropriate multiples of the other rows:

$$\begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & \alpha + b_0 + b_{n-1}\alpha_1 + b_{n-2}\alpha_2 + \cdots + b_2\alpha_{n-2} + b_1\alpha_{n-1} \\ -1 & 0 & 0 & \ldots & 0 & \alpha_1 \\ 0 & -1 & 0 & \ldots & 0 & \alpha_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & \alpha_{n-2} \\ 0 & 0 & 0 & \ldots & -1 & \alpha_{n-1} \end{bmatrix},$$

and clean up the last column by adding to it appropriate multiples of the other columns:

$$\begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & \alpha + b_{n-1}\alpha_1 + b_{n-2}\alpha_2 + \cdots + b_2\alpha_{n-2} + b_1\alpha_{n-1} + b_0 \\ -1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & -1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & -1 & 0 \end{bmatrix}.$$

Multiply each of the first $(n-1)$ columns by -1 and then perform $(n-1)$ interchanges of adjacent columns to bring the present column n to the first position. The total effect of the $2n-2$ sign changes is to leave the determinant unaltered. We obtain the matrix

$$\begin{bmatrix} \alpha + b_{n-1}\alpha_1 + b_{n-2}\alpha_2 + \cdots + b_2\alpha_{n-2} + b_1\alpha_{n-1} + b_0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{bmatrix}. \tag{3.1}$$

It is then clear that

$$\sigma(x) := \det(xI - X) = \alpha + b_{n-1}\alpha_1 + b_{n-2}\alpha_2 + \cdots + b_2\alpha_{n-2} + b_1\alpha_{n-1} + b_0.$$

Any $n \times n$ matrix X, there is a succession of row and column operations which will transform $xI - X$ into a matrix much like (3.1), in which the invariant factors of X appear down the main diagonal. From Theorem 2.4, it follows that $\sigma(x)$ is the minimum polynomial of X. Therefore X is nonderogatory.

3.1 The Sum of Two Doubly Companion Matrices

Now, to prove that the sum of two doubly companion matrices is also a nonderogatory.

Theorem 3.4. *Let $X(\alpha, \beta)$ and $Y(\gamma, \delta)$ be two companion matrices of the same size. Then $X(\alpha, \beta) + Y(\gamma, \delta)$ is a nonderogatory matrix.*

Proof. Let $\alpha(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$, $\beta(x) = x^n + b_{n-1}x^{n-1} + \cdots + b_2x^2 + b_1x + b_0$, $\gamma(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_2x^2 + c_1x + c_0$, and $\delta(x) = x^n + d_{n-1}x^{n-1} + \cdots + d_2x^2 + d_1x + d_0$ are in $\mathbb{C}[x]$. Then

$$
\begin{aligned}
Z &:= X(\alpha, \beta) + Y(\gamma, \delta) \\
&= \begin{bmatrix} -b_{n-1} - d_{n-1} & -b_{n-2} - d_{n-2} & \dots & -b_1 - d_1 & -a_0 - b_0 - c_0 - d_0 \\ 2 & 0 & \dots & 0 & -a_1 - c_1 \\ 0 & 2 & \dots & 0 & -a_2 - c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & -a_{n-2} - c_{n-2} \\ 0 & 0 & \dots & 2 & -a_{n-1} - c_{n-1} \end{bmatrix}.
\end{aligned}
$$

Let $D = \mathrm{Diag}(1, 2, 2^2, \dots, 2^{n-1})$. Then $D^{-1} = \mathrm{Diag}(1, \frac{1}{2}, \frac{1}{2^2}, \dots, \frac{1}{2^{n-1}})$. Consider

$$
\begin{aligned}
& D^{-1}ZD \\
&= \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{2} & 0 & \dots & 0 \\ 0 & 0 & \frac{1}{2^2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{2^{n-1}} \end{bmatrix} \\
&\times \begin{bmatrix} -b_{n-1} - d_{n-1} & -b_{n-2} - d_{n-2} & \dots & -b_1 - d_1 & -a_0 - b_0 - c_0 - d_0 \\ 2 & 0 & \dots & 0 & -a_1 - c_1 \\ 0 & 2 & \dots & 0 & -a_2 - c_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ddots & 0 & -a_{n-2} - c_{n-2} \\ 0 & 0 & \dots & 2 & -a_{n-1} - c_{n-1} \end{bmatrix} \\
&\times \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 2 & 0 & \dots & 0 \\ 0 & 0 & 2^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 2^{n-1} \end{bmatrix},
\end{aligned}
$$

that is

$$D^{-1}ZD$$
$$= \begin{bmatrix} -(b_{n-1}+d_{n-1}) & -2(b_{n-2}+d_{n-2}) & \dots & -2^{n-2}(b_1+d_1) & -2^{n-1}(a_0+b_0+c_0+d_0) \\ 1 & 0 & \dots & 0 & -2^{n-2}(a_1+c_1) \\ 0 & 1 & \dots & 0 & -2^{n-3}(a_2+c_2) \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ddots & 0 & -2(a_{n-2}+c_{n-2}) \\ 0 & 0 & \dots & 1 & -(a_{n-1}+c_{n-1}) \end{bmatrix}.$$

Therefore $Z = X(\alpha, \beta) + Y(\gamma, \delta)Z$ is similar to a doubly companion matrix. The proof of Theorem 3.3 assert that for each doubly companion matrix is similar to a companion matrix. Since similarity is an equivalence relation, so that $X(\alpha, \beta) + Y(\gamma, \delta)Z$ must be similar to some companion matrix. Therefore $X(\alpha, \beta) + Y(\gamma, \delta)$ is a nonderogatory matrix. □

3.2 The Product of Two Doubly Companion Matrices

We wish to give some examples of a product of two doubly companion matrices which is a derogatory matrix. In particular every companion matrix is also a doubly companion matrix, Key and Volkmer in [4] gave some conditions such that an eigenvalue of a product of companion matrices has geometric multiplicity equal to one, equivalently under some suitable conditions a product of two doubly companion matrices may be a nonderogatory matrix.

Key and Volkmer [4, p. 112] assert that the product of two companion matrices of the form,

$$B_k = \begin{bmatrix} b_k & -1 & b_k \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

for $k = 1, 2$, is derogatory matrix. Consider

$$\begin{bmatrix} b_1 & 1 & -b_1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} b_2 & 1 & -b_2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} b_1b_2+1 & 0 & -b_1b_2 \\ b_2 & 1 & -b_2 \\ 1 & 0 & 0 \end{bmatrix}.$$

The characteristic polynomial is

$$x^3 + (-b_1b_2 - 2)x^2 + (2b_1b_2 + 1)x + (b_1b_2(b_1b_2 + 1) - b_1b_2(b_1b_2 + 2)),$$

but the minimum polynomial is $x^2 + (-b_1b_2 - 1t)x + b_1b_2$.

Similarly, product of two doubly companion matrices may not be a nonderogatory matrix, for example,

$$\begin{bmatrix} -b & 1 & b \\ 1 & 0 & k \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -d & 1 & d \\ 1 & 0 & k \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} bd+1 & 0 & k-bd \\ -d & k+1 & d \\ 1 & 0 & k \end{bmatrix},$$

since the characteristic polynomial is

$$(x - bd)(k - x + 1)^2$$

and the minimum polynomial is

$$-(x - bd)(k - x + 1).$$

The characteristic polynomial of the product matrix is not equal to its minimal polynomial. In general, we have proved the following theorem:

Theorem 3.5. *Let $X(\alpha, \beta)$ and $Y(\gamma, \delta)$ be two companion matrices of the same size. Then the product $X(\alpha, \beta)Y(\gamma, \delta)$ is not a nonderogatory matrix.*

3.3 The Product of two Unreduced Hessenberg Matrices

Definition 3.6 ([11, p. 43-3])**.** A matrix $A \in M_n$ is called *upper Hessenberg* if $a_{ij} = 0$ whenever $i > j+1$. This means that every entry below the first subdiagonal of A is zero. An upper Hessenberg matrix is called unreduced upper Hessenberg if $a_{j+1,j} \neq 0$ for $j = 1, \ldots, n-1$.

A Hessenberg matrix H is *unreduced* if all subdiagonals are nonzero. It is well known that an unreduced Hessenberg matrix is nonderogatory; one may see [12, Lemma, p. 805], (see for example [13, Lemma 2.2, p. 13], or [14, p. 559]).

Theorem 3.7 ([12, Lemma, p. 805])**.** *An unreduced Hessenberg matrix is not derogatory.*

Proof. The minor of the $(1, n)$ element of $H - zI$ is nonzero and independent of z. Thus the null space of $H - zI$ has dimension ≤ 1 for all z. □

Since some unreduced Hessenberg matrices are doubly companion matrices, from Theorem 3.5 we obtain the following corollary.

Corollary 3.8. *Product of two unreduced Hessenberg is not nonderogatory.*

Acknowledgements : The author are grateful to referees for their helpful comments and valuable suggestions.

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(Received 5 July 2010)
(Accepted 5 April 2011)