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Convergence of Iterative Algorithm for Finding Common Solution of Fixed Points and General System of Variational Inequalities for Two Accretive Operators

Phayap Katchang[†,§] **and Poom Kumam**[‡,§,1]

[§]Centre of Excellence in Mathematics, CHE, Si Ayutthaya Road,
Bangkok 10400, Thailand
[†]Department of Mathematics and Statistics,
Faculty of Science and Agricultural Technology,
Rajamangala University of Technology Lanna Tak, Tak 63000, Thailand
e-mail : p.katchang@hotmail.com
[‡]Department of Mathematics, Faculty of Science,
King Mongkut's University of Technology Thonburi (KMUTT),
Bangmod, Bangkok 10140, Thailand
e-mail : poom.kum@kmutt.ac.th

Abstract : In this paper, we prove a strong convergence theorem for finding a common solutions of a general system of variational inequalities involving two different inverse-strongly accretive operators and solutions of fixed point problems involving the nenexpansive mapping in a Banach space by using a modified viscosity extragradient method. Moreover, using the above results, we can apply to finding solutions of zeros of accretive operators and the class of k-strictly pseudo-contractive mappings. The results presented in this paper extend and improve the corresponding results of Qin et al. [1], Aoyama et al. [2], Yao et al. [3] and many others.

Keywords : Inverse-strongly accretive operator; Fixed point; General system of

[1]Corresponding author email: poom.kum@kmutt.ac.th (P. Kumam)
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variational inequalities; Modified viscosity extragradient approximation method; sunny nonexpansive retraction.
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1 Introduction

Let E be a real Banach space with norm $\|\cdot\|$, C be a nonempty closed convex subset of E. Let E^* be the dual space of E and $\langle\cdot,\cdot\rangle$ denote the pairing between E and E^*. For $q > 1$, the *generalized duality mapping* $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^q, \|f\| = \|x\|^{q-1}\}$$

for all $x \in E$. In particular, if $q = 2$, the mapping J_2 is called the *normalized duality mapping* and, usually, write $J_2 = J$. Further, we have the following properties of the generalized duality mapping J_q: (i) $J_q(x) = \|x\|^{q-2}J_2(x)$ for all $x \in E$ with $x \neq 0$; (ii) $J_q(tx) = t^{q-1}J_q(x)$ for all $x \in E$ and $t \in [0,\infty)$; and (iii) $J_q(-x) = -J_q(x)$ for all $x \in E$. It is known that if X is smooth, then J is single-valued, which is denoted by j . Recall that the duality mapping j is said to be weakly sequentially continuous if for each $x_n \to x$ weakly, we have $j(x_n) \to j(x)$ weakly-*. We know that if X admits a weakly sequentially continuous duality mapping, then X is smooth. For the details, see [4, 5, 3].

Let $U = \{x \in E : \|x\| = 1\}$. A Banach space E is said to *uniformly convex* if, for any $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that, for any $x, y \in U$, $\|x - y\| \geq \epsilon$ implies $\|\frac{x+y}{2}\| \leq 1 - \delta$. It is known that a uniformly convex Banach space is reflexive and strictly convex. A Banach space E is said to be *smooth* if the limit $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for all $x, y \in U$. It is also said to be *uniformly smooth* if the limit is attained uniformly for $x, y \in U$. The *modulus of smoothness* of E is defined by

$$\rho(\tau) = \sup\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : x, y \in E, \|x\| = 1, \|y\| = \tau\},$$

where $\rho : [0,\infty) \to [0,\infty)$ is a function. It is known that E is uniformly smooth if and only if $\lim_{\tau\to 0} \frac{\rho(\tau)}{\tau} = 0$. Let q be a fixed real number with $1 < q \leq 2$. A Banach space E is said to be *q-uniformly smooth* if there exists a constant $c > 0$ such that $\rho(\tau) \leq c\tau^q$ for all $\tau > 0$: see, for instance, [6, 2].

We note that E is a uniformly smooth Banach space if and only if J_q is single-valued and uniformly continuous on any bounded subset of E. Typical examples of both uniformly convex and uniformly smooth Banach spaces are L^p, where $p > 1$. More precisely, L^p is $\min\{p, 2\}$-uniformly smooth for every $p > 1$. Note also that no Banach space is q-uniformly smooth for $q > 2$; see [4, 6, 7] for more details.

Next, we recall the following concepts (see also [4, 6] for). Let $S : C \to C$ a nonlinear mapping. We use $F(S)$ to denote the set of fixed points of S, that is, $F(S) = \{x \in C : Sx = x\}$. A mapping S is called *nonexpansive* if

$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C$. Recall that a mapping $f : C \to C$ is said to be contraction if there exists a constant $\alpha \in [0, 1)$ and $x, y \in C$ such that $\|f(x) - f(y)\| \leq \alpha\|x - y\|$. Let A be a monotone operator of C into Hilbert spaces H. The *variational inequality problem*, denote by $VI(C, A)$, is to find $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0,$$

for all $x \in C$. Recall that an operator A of C into E is said to be *accretive* if there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y) \rangle \geq 0$$

for all $x, y \in C$. A mapping $A : C \to E$ is said to be *β-strongly accretive* if there exists a constant $\beta > 0$ such that

$$\langle Ax - Ay, j(x - y) \rangle \geq \beta\|x - y\|^2 \quad \forall x, y \in C.$$

An operator A of C into E is said to be *β-inverse strongly accretive* if, for any $\beta > 0$,

$$\langle Ax - Ay, j(x - y) \rangle \geq \beta\|Ax - Ay\|^2$$

for all $x, y \in C$. Evidently, the definition of the inverse strongly accretive operator is based on that of the inverse strongly monotone operator.

Recently, Aoyama et al. [2] first considered the following generalized variational inequality problem in a smooth Banach space. Let A be an accretive operator of C into E. Find a point $x \in C$ such that

$$\langle Ax, j(y - x) \rangle \geq 0, \tag{1.1}$$

for all $y \in C$. This problem is connected with the fixed point problem for nonlinear mappings, the problem of finding a zero point of an accretive operator and so on. For the problem of finding a zero point of an accretive operator by the proximal point algorithm, see Kamimura and Takahashi [8, 9] . In order to find a solution of the variational inequality (1.1), Aoyama et al. [2] proved the strong convergence theorem in the framework of Banach spaces which is generalized Iiduka et al. [10] from Hilbert spaces.

In 2006, Aoyama, Iiduka and Takahashi [2] proved the following weak convergence theorem.

Theorem AIT.[2, Theorem 3.1] *Let E be a uniformly convex and 2-uniformly smooth Banach space and C a nonempty closed convex subset of E. Let Q_C be a sunny nonexpansive retraction from E onto C, $\alpha > 0$, and A be an α-inverse strongly accretive operator of C into E with $S(C, A) \neq \emptyset$, where*

$$S(C, A) = \{x^* \in C : \langle Ax^*, j(x - x^*) \rangle \geq 0, \quad x \in C\}.$$

If $\{\lambda_n\}$ and $\{\alpha_n\}$ are chosen such that $\lambda_n \in [a, \frac{\alpha}{K^2}]$, for some $a > 0$ and $\alpha_n \in [b, c]$, for some b, c with $0 < b < c < 1$, then the sequence $\{x_n\}$ defined by the following manners: $x_1 - x \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Q_C(x_n - \lambda_n Ax_n),$$

converges weakly to some element z of $S(C, A)$, where K is the 2-uniformly smoothness constant of E and Q_C is a sunny nonexpansive retraction.

Motivated by Aoyama et al. [2] and also Ceng et al. [11], Qin et al. [1] and Yao et al. [3] first considered the following system of general variational inequalities in Banach spaces:

Let $A : C \to E$ be an β-inverse strongly accretive mapping. Find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, j(x - x^*)\rangle \geq 0 \quad \forall x \in C, \\ \langle \mu Ax^* + y^* - x^*, j(x - y^*)\rangle \geq 0 \quad \forall x \in C. \end{cases} \tag{1.2}$$

Let C be nonempty closed convex subset of a real Banach space E. For given two operators $A, B : C \to E$, we consider the problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, j(x - x^*)\rangle \geq 0 \quad \forall x \in C, \\ \langle \mu Bx^* + y^* - x^*, j(x - y^*)\rangle \geq 0 \quad \forall x \in C, \end{cases} \tag{1.3}$$

where λ and μ are two positive real numbers. This system is called the *system of general variational inequalities* in a real Banach spaces. If we add up the requirement that $A = B$, then the problem (1.3) is reduced to the system (1.2).

An interesting problem to extend the above results to find a solution of a general system of variational inequalities. In 2008, Ceng et al. [11] introduced a relaxed extragradient method for finding solutions of a general system of variational inequalities with inverse-strongly monotone mappings in a real Hilbert space. Suppose $x_1 = u \in C$ and x_n is generated by

$$\begin{cases} y_n = P_C(x_n - \mu Bx_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n SP_C(y_n - \lambda Ay_n), \end{cases} \tag{1.4}$$

for all $n \geq 1$ where $\lambda \in (0, 2\alpha)$, $\mu \in (0, 2\beta)$, S is a nonexpansive mapping and A and B are α and β-inverse-strongly monotone, respectively. They proved the strong convergence theorem under quite mild conditions. Recently, Yao et al. [3] introduce the following iteration scheme for solving a general system of variational inequality problem (1.3) and some fixed point problem involving the nonexpansive mapping in Banach spaces. For arbitrarily given $x_0 = u \in C$ and $\{x_n\}$ is given by

$$\begin{cases} y_n = Q_C(x_n - Bx_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n Q_C(y_n - Ay_n), \end{cases} \tag{1.5}$$

for all $n \geq 0$ where $C \subset E$, Q_C is a sunny nonexpansive retraction from E onto C and A and B are inverse-strongly accretive mappings. They obtained a strong convergence theorem in Banach spaces.

In this paper, motivated and inspired by the idea of Ceng et al. [11], Yao et al. [3], Iiduka, Takahashi and Toyoda [10], and Iiduka and Takahashi [27] we introduce an iterative scheme for finding solutions of a general system of variational inequalities (1.3) involving two different inverse-strongly accretive operators and solutions of fixed point problems involving the nonexpansive mapping in a Banach space by using a modified viscosity extragradient method. Consequently, we obtain new strong convergence theorems for fixed point problems which solves the system of general variational inequalities (1.2) and (1.3). Moreover, using the above theorem, we can apply to finding solutions of zeros of accretive operators and the class of k-strictly pseudocontractive mappings. The results presented in this paper extend and improve the corresponding results of Yao et al. [3], Ceng et al. [11], Qin et al. [1] and many others.

2 Preliminaries

Let D be a subset of C and $Q : C \to D$. Then Q is said to *sunny* if

$$Q(Qx + t(x - Qx)) = Qx,$$

whenever $Qx+t(x-Qx) \in C$ for $x \in C$ and $t \geq 0$. A subset D of C is said to be a *sunny nonexpansive retract* of C if there exists a sunny nonexpansive retraction Q of C onto D. A mapping $Q : C \to C$ is called a *retraction* if $Q^2 = Q$. If a mapping $Q : C \to C$ is a retraction, then $Qz = z$ for all z is in the range of Q. For example, see [2, 12] for more details. The following result describes a characterization of sunny nonexpansive retractions on a smooth Banach space.

Proposition 2.1. ([13]) *Let E be a smooth Banach space and let C be a nonempty subset of E. Let $Q : E \to C$ be a retraction and let J be the normalized duality mapping on E. Then the following are equivalent:*
(i) Q is sunny and nonexpansive;
(ii) $\|Qx - Qy\|^2 \leq \langle x - y, J(Qx - Qy)\rangle, \forall x, y \in E$;
(iii) $\langle x - Qx, J(y - Qx)\rangle \leq 0, \forall x \in E, y \in C$.

Proposition 2.2. ([14]) *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E and let T be a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. Then the set $F(T)$ is a sunny nonexpansive retract of C.*

We need the following lemmas for proving our main results.

Lemma 2.3. ([7]) *Let E be a real 2-uniformly smooth Banach space with the best smooth constant K. Then the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, Jx\rangle + 2\|Ky\|^2, \qquad \forall x, y \in E.$$

Lemma 2.4. ([15]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2.5. ([16]) *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \delta_n, \;\; n \geq 0$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(1) $\sum_{n=1}^{\infty}\alpha_n=\infty$

(2) $\limsup_{n\to\infty}\frac{\delta_n}{\alpha_n}\leq 0$ *or* $\sum_{n=1}^{\infty}|\delta_n|<\infty.$

Then $\lim_{n\to\infty}a_n=0$.

Lemma 2.6. ([17]) *Let $(E,\langle .,.\rangle)$ be an inner product space. Then for all $x,y,z\in E$ and $\alpha,\beta,\gamma\in[0,1]$ with $\alpha+\beta+\gamma=1$, we have*

$$\|\alpha x+\beta y+\gamma z\|^2=\alpha\|x\|^2+\beta\|y\|^2+\gamma\|z\|^2-\alpha\beta\|x-y\|^2-\alpha\gamma\|x-z\|^2-\beta\gamma\|y-z\|^2.$$

Lemma 2.7. ([18]) *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E and let T be nonexpansive mapping of C into itself. If $\{x_n\}$ is a sequence of C such that $x_n\to x$ weakly and $x_n-Tx_n\to 0$ strongly, then x is s fixed point of T.*

Lemma 2.8. (Yao et al. [3, Lemma 3.1]; see also [2, Lemma 2.8]) *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space E. Let the mapping $A:C\to E$ be β-inverse-strongly accretive. Then, we have*

$$\|(I-\lambda A)x-(I-\lambda A)y\|^2\leq\|x-y\|^2+2\lambda(\lambda K^2-\beta)\|Ax-Ay\|^2.$$

If $\beta\geq\lambda K^2$, then $I-\lambda A$ is nonexpansive.

Proof. For any $x,y\in C$, from Lemma 2.3, we have

$$\begin{aligned}\|(I-\lambda A)x-(I-\lambda A)y\|^2 &= \|(x-y)-\lambda(Ax-Ay)\|^2\\ &\leq \|x-y\|^2-2\lambda\langle Ax-Ay,j(x-y)\rangle\\ &\quad +2\lambda^2K^2\|Ax-Ay\|^2\\ &\leq \|x-y\|^2-2\lambda\beta\|Ax-Ay\|^2+2\lambda^2K^2\|Ax-Ay\|^2\\ &= \|x-y\|^2+2\lambda(\lambda K^2-\beta)\|Ax-Ay\|^2.\end{aligned}$$

If $\beta\geq\lambda K^2$, then $I-\lambda A$ is nonexpansive. □

3 Main Results

In this section, we prove a strong convergence theorem. In order to prove our main results, we need the following two lemmas which is proved along the proof of Yao et al.'s lemmas as it appears in [3].

Lemma 3.1. *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space E. Let Q_C be the sunny nonexpansive retraction from E onto C. Let the mapping $A,B:C\to E$ be β-inverse-strongly accretive and γ-inverse-strongly accretive, respectively. Let $G:C\to C$ be a mapping defined by*

$$G(x)=Q_C(Q_C(x-\mu Bx)-\lambda AQ_C(x-\mu Bx))\quad\forall x\in C.$$

If $\beta\geq\lambda K^2$ and $\gamma\geq\mu K^2$, then G is nonexpansive.

Proof. For any $x, y \in C$, from Lemma 2.8 and Q_C is nonexpansive, we have

$$\begin{aligned}
\|G(x)-G(y)\| &= \|Q_C[Q_C(I-\mu B)x-\lambda AQ_C(I-\mu B)x] \\
&\quad -Q_C[Q_C(I-\mu B)y-\lambda AQ_C(I-\mu B)y]\| \\
&\leq \|[Q_C(I-\mu B)x-\lambda AQ_C(I-\mu B)x] \\
&\quad -[Q_C(I-\mu B)y-\lambda AQ_C(I-\mu B)y]\| \\
&= \|(I-\lambda A)Q_C(I-\mu B)x-(I-\lambda A)Q_C(I-\mu B)y\| \\
&\leq \|Q_C(I-\mu B)x-Q_C(I-\mu B)y\| \\
&\leq \|(I-\mu B)x-(I-\mu B)y\| \\
&\leq \|x-y\|.
\end{aligned}$$

Therefore G is nonexpansive. $\square$

Lemma 3.2. *Let C be a nonempty closed convex subset of a real smooth Banach space E. Let Q_C be the sunny nonexpansive retraction from E onto C. Let $A, B : C \to E$ be two possibly nonlinear mappings. For given $x^*, y^* \in C$, (x^*, y^*) is a solution of problem* (1.3) *if and only if $x^* = Q_C(y^* - \lambda Ay^*)$ where $y^* = Q_C(x^* - \mu Bx^*)$.*

Proof. From (1.3), we rewrite as

$$\begin{cases} \langle x^*-(y^*-\lambda Ay^*), j(x-x^*)\rangle \geq 0 & \forall x \in C, \\ \langle y^*-(x^*-\mu Bx^*), j(x-y^*)\rangle \geq 0 & \forall x \in C. \end{cases} \tag{3.1}$$

From Proposition 2.1 (iii), the system (3.1) equivalent to

$$\begin{cases} x^* = Q_C(y^*-\lambda Ay^*), \\ y^* = Q_C(x^*-\mu Bx^*). \end{cases} \tag{3.2}$$

$\square$

Remark 3.3. *From Lemma 3.2, we note that*

$$x^* = Q_C(Q_C(x^*-\mu Bx^*)-\lambda AQ_C(x^*-\mu Bx^*)),$$

which implies that x^* is a fixed point of the mapping G.

Throughout this paper, the set of fixed points of the mapping G is denoted by $F(G)$.

The next result states the main result of this work.

Theorem 3.4. *Let E be a uniformly convex and 2-uniformly smooth Banach space which admits a weakly sequentially continuous duality mapping and C be a nonempty closed convex subset of E. Let $S : C \to C$ be a nonexpansive mapping and Q_C be a sunny nonexpansive retraction from E onto C. Let $A, B : C \to E$ be β-inverse-strongly accretive with $\beta \geq \lambda K^2$ and γ-inverse-strongly accretive with*

$\gamma \geq \mu K^2$, respectively and K be the best smooth constant. Let f be a contraction of C into itself with coefficient $\alpha \in [0,1)$. Suppose $\mathcal{F} := F(G) \cap F(S) \neq \emptyset$ where G defined by Lemma 3.1. For arbitrary given $x_0 = x \in C$, the sequence $\{x_n\}$ generated by

$$\begin{cases} y_n = Q_C(x_n - \mu B x_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S Q_C(y_n - \lambda A y_n). \end{cases} \tag{3.3}$$

where the sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ in $(0,1)$ satisfy $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 1$ and λ, μ are positive real numbers. The following conditions are satisfied:
(C1). $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(C2). $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,
Then $\{x_n\}$ converges strongly to $\bar{x} = Q_{\mathcal{F}} f(\bar{x})$ and $(\bar{x}, \bar{y})$ is a solution of the problem (1.3), where $\bar{y} = Q_C(\bar{x} - \mu B \bar{x})$ and $Q_{\mathcal{F}}$ is the sunny nonexpansive retraction of C onto $\mathcal{F}$.

Proof. First, we prove that $\{x_n\}$ is bounded. Let $x^* \in \mathcal{F}$, from Lemma 3.2, we see that

$$x^* = Q_C(Q_C(x^* - \mu B x^*) - \lambda A Q_C(x^* - \mu B x^*)).$$

Put $y^* = Q_C(x^* - \mu B x^*)$ and $v_n = Q_C(y_n - \lambda A y_n)$. Then $x^* = Q_C(y^* - \lambda A y^*)$. By nonexpansiveness of $I - \lambda A$, $I - \mu B$ and Q_C, we have

$$\begin{aligned} \|v_n - x^*\| &= \|Q_C(y_n - \lambda A y_n) - Q_C(y^* - \lambda A y^*)\| \\ &\leq \|(y_n - \lambda A y_n) - (y^* - \lambda A y^*)\| \\ &= \|(I - \lambda A) y_n - (I - \lambda A) y^*\| \\ &\leq \|y_n - y^*\| \\ &= \|Q_C(x_n - \mu B x_n) - Q_C(x^* - \mu B x^*)\| \\ &\leq \|(x_n - \mu B x_n) - (x^* - \mu B x^*)\| \\ &= \|(I - \mu B) x_n - (I - \mu B) x^*\| \\ &\leq \|x_n - x^*\|. \end{aligned} \tag{3.4}$$

It follows that

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n S v_n - x^*\| \\ &\leq \alpha_n \|f(x_n) - x^*\| + \beta_n \|x_n - x^*\| + \gamma_n \|S v_n - x^*\| \\ &\leq \alpha \alpha_n \|x_n - x^*\| + \alpha_n \|f(x^*) - x^*\| + \beta_n \|x_n - x^*\| + \gamma_n \|v_n - x^*\| \\ &\leq \alpha \alpha_n \|x_n - x^*\| + \alpha_n \|f(x^*) - x^*\| + \beta_n \|x_n - x^*\| + \gamma_n \|x_n - x^*\| \\ &= (1 - \alpha_n + \alpha \alpha_n) \|x_n - x^*\| + \alpha_n \|f(x^*) - x^*\| \\ &= (1 - \alpha_n(1 - \alpha)) \|x_n - x^*\| + \alpha_n (1 - \alpha) \frac{\|f(x^*) - x^*\|}{1 - \alpha} \\ &\leq \max\{\|x_1 - x^*\|, \frac{\|f(x^*) - x^*\|}{1 - \alpha}\}. \end{aligned}$$

This implies that $\{x_n\}$ is bounded, so are $\{f(x_n)\}$, $\{y_n\}$, $\{v_n\}$, $\{Sv_n\}$, $\{Ay_n\}$ and $\{Bx_n\}$.

Next, we show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. Notice that

$$\begin{aligned}
\|v_{n+1}-v_n\| &= \|Q_C(y_{n+1}-\lambda Ay_{n+1})-Q_C(y_n-\lambda Ay_n)\|\\
&\leq \|(y_{n+1}-\lambda Ay_{n+1})-(y_n-\lambda Ay_n)\|\\
&= \|(I-\lambda A)y_{n+1}-(I-\lambda A)y_n\|\\
&\leq \|y_{n+1}-y_n\|\\
&= \|Q_C(x_{n+1}-\mu Bx_{n+1})-Q_C(x_n-\mu Bx_n)\|\\
&\leq \|(x_{n+1}-\mu Bx_{n+1})-(x_n-\mu Bx_n)\|\\
&= \|(I-\mu B)x_{n+1}-(I-\mu B)x_n\|\\
&\leq \|x_{n+1}-x_n\|.
\end{aligned}$$

Setting $x_{n+1}=(1-\beta_n)z_n+\beta_n x_n$ for all $n\geq 0$, we see that $z_n=\frac{x_{n+1}-\beta_n x_n}{1-\beta_n}$, then we have

$$\begin{aligned}
\|z_{n+1}-z_n\| &= \|\frac{x_{n+2}-\beta_{n+1}x_{n+1}}{1-\beta_{n+1}}-\frac{x_{n+1}-\beta_n x_n}{1-\beta_n}\|\\
&= \|\frac{\alpha_{n+1}f(x_{n+1})+\gamma_{n+1}Sv_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n f(x_n)+\gamma_n Sv_n}{1-\beta_n}\|\\
&= \|\frac{\alpha_{n+1}f(x_{n+1})+\gamma_{n+1}Sv_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_{n+1}f(x_n)}{1-\beta_{n+1}}+\frac{\alpha_{n+1}f(x_n)}{1-\beta_{n+1}}\\
&\quad -\frac{\gamma_{n+1}Sv_n}{1-\beta_{n+1}}+\frac{\gamma_{n+1}Sv_n}{1-\beta_{n+1}}-\frac{\alpha_n f(x_n)+\gamma_n Sv_n}{1-\beta_n}\|\\
&= \|\frac{\alpha_{n+1}}{1-\beta_{n+1}}(f(x_{n+1})-f(x_n))+\frac{\gamma_{n+1}}{1-\beta_{n+1}}(Sv_{n+1}-Sv_n)\\
&\quad +(\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n})f(x_n)+(\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n})Sv_n\|\\
&\leq \frac{\alpha\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\|+\frac{\gamma_{n+1}}{1-\beta_{n+1}}\|v_{n+1}-v_n\|\\
&\quad +|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}|\|f(x_n)\|\\
&\quad +|\frac{1-\beta_{n+1}-\alpha_{n+1}}{1-\beta_{n+1}}-\frac{1-\beta_n-\alpha_n}{1-\beta_n}|\|Sv_n\|\\
&= \frac{\alpha\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\|+\frac{\gamma_{n+1}}{1-\beta_{n+1}}\|v_{n+1}-v_n\|\\
&\quad +|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}|(\|f(x_n)\|+\|Sv_n\|)\\
&\leq \frac{\alpha\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\|+|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}|(\|f(x_n)\|+\|Sv_n\|)\\
&\quad +\|v_{n+1}-v_n\|
\end{aligned}$$

$$\leq \frac{\alpha\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\| + |\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}|(\|f(x_n)\| + \|Sv_n\|) + \|x_{n+1}-x_n\|.$$

Therefore

$$\begin{aligned}\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\| \quad \leq \quad & \frac{\alpha\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\| \\ & +|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}|(\|f(x_n)\| + \|Sv_n\|).\end{aligned}$$

It follow from the condition (C1) and (C2), which implies that

$$\limsup_{n\to\infty}(\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\|) \leq 0.$$

Applying Lemma 2.4, we obtain $\lim_{n\to\infty}\|z_n - x_n\| = 0$ and also

$$\|x_{n+1}-x_n\| = (1-\beta_n)\|z_n - x_n\| \to 0$$

as $n \to \infty$. Therefore, we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \tag{3.5}$$

Next, we show that $\lim_{n\to\infty}\|Sv_n - v_n\| = 0$. Since $x^* \in \mathcal{F}$, from Lemma 2.6, we obtain

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n Sv_n - x^*\|^2 \\ &\leq \alpha_n\|f(x_n)-x^*\|^2 + (1-\alpha_n-\gamma_n)\|x_n - x^*\|^2 + \gamma_n\|v_n - x^*\|^2 \\ &= \alpha_n\|f(x_n)-x^*\|^2 + (1-\alpha_n)\|x_n - x^*\|^2 \\ &\quad -\gamma_n(\|x_n - x^*\|^2 - \|v_n - x^*\|^2) \\ &\leq \alpha_n\|f(x_n)-x^*\|^2 + \|x_n - x^*\|^2 \\ &\quad -\gamma_n\|x_n - v_n\|(\|x_n - x^*\| + \|v_n - x^*\|).\end{aligned}$$

Therefore, we have
$\gamma_n\|x_n - v_n\|(\|x_n - x^*\| + \|v_n - x^*\|)$

$$\begin{aligned}&\leq \alpha_n\|f(x_n)-x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1}-x^*\|^2 \\ &\leq \alpha_n\|f(x_n)-x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1}-x^*\|)\|x_n - x_{n+1}\|.\end{aligned}$$

From the condition (C1) and (3.5), this implies that $\|x_n - v_n\| \to 0$ as $n \to \infty$. Now, we have

$$\begin{aligned}\|x_n - Sv_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - Sv_n\| \\ &= \|x_n - x_{n+1}\| + \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n Sv_n - Sv_n\| \\ &= \|x_n - x_{n+1}\| + \|\alpha_n(f(x_n) - Sv_n) + \beta_n(x_n - Sv_n)\| \\ &\leq \|x_n - x_{n+1}\| + \alpha_n\|f(x_n) - Sv_n\| + \beta_n\|x_n - Sv_n\|.\end{aligned}$$

Therefore, we get

$$\|x_n - Sv_n\| \quad \leq \quad \frac{1}{1-\beta_n}\|x_n - x_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|f(x_n) - Sv_n\|.$$

From the condition (C1), (C2) and (3.5), this implies that $\|x_n - Sv_n\| \to 0$ as $n \to \infty$. Also, observe that

$$\|Sv_n - v_n\| \quad \leq \quad \|Sv_n - x_n\| + \|x_n - v_n\|,$$

and hence it follows that $\lim_{n\to\infty} \|Sv_n - v_n\| = 0$.

Next, we show that $\limsup_{n\to\infty}\langle (f-I)\bar{x}, J(x_n - \bar{x})\rangle \leq 0$, where $\bar{x} = Q_{\mathcal{F}} f(\bar{x})$. Since $\{x_n\}$ is bounded, we can choose a sequence $\{x_{n_i}\}$ of $\{x_n\}$ which $x_{n_i} \rightharpoonup x^*$ such that

$$\limsup_{n\to\infty}\langle (f-I)\bar{x}, J(x_n - \bar{x})\rangle = \lim_{i\to\infty}\langle (f-I)\bar{x}, J(x_{n_i} - \bar{x})\rangle. \tag{3.6}$$

Next, we prove that $x^* \in \mathcal{F} := F(G) \cap F(S)$.

(a) First, we show that $x^* \in F(S)$. To show this, we choose a subsequence $\{v_{n_i}\}$ of $\{v_n\}$. Since $\{v_{n_i}\}$ is bounded, we have that a subsequence $\{v_{n_{i_j}}\}$ of $\{v_{n_i}\}$ converges weakly to x^*. We may assume without loss of generality that $v_{n_i} \rightharpoonup x^*$. Since $\|Sv_n - v_n\| \to 0$, we obtain $Sv_{n_i} \rightharpoonup x^*$. Then we can obtain $x^* \in \mathcal{F}$. Assume that $x^* \notin F(S)$. Since $v_{n_i} \rightharpoonup x^*$ and $Sx^* \neq x^*$, from Opial's condition, we have

$$\begin{aligned}
\liminf_{i\to\infty}\|v_{n_i} - x^*\| &< \liminf_{i\to\infty}\|v_{n_i} - Sx^*\| \\
&\leq \liminf_{i\to\infty}(\|v_{n_i} - Sv_{n_i}\| + \|Sv_{n_i} - Sx^*\|) \\
&\leq \liminf_{i\to\infty}\|v_{n_i} - x^*\|.
\end{aligned}$$

This is a contradiction. Thus, we obtain $x^* \in F(S)$.

(b) Next, we show that $x^* \in F(G)$. From Lemma 3.1, we know that G is nonexpansive, it follows that

$$\begin{aligned}
\|v_n - G(v_n)\| &= \|Q_C(Q_C(x_n - \mu Bx_n) - \lambda AQ_C(x_n - \mu Bx_n)) - G(v_n)\| \\
&= \|G(x_n) - G(v_n)\| \\
&\leq \|x_n - v_n\| \to 0, \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Thus $\lim_{n\to\infty} \|v_n - G(v_n)\| = 0$. Since G is nonexpansive, we get

$$\begin{aligned}
\|x_n - G(x_n)\| &\leq \|x_n - v_n\| + \|v_n - G(v_n)\| + \|G(v_n) - G(x_n)\| \\
&\leq 2\|x_n - v_n\| + \|v_n - G(v_n)\|,
\end{aligned}$$

and so

$$\lim_{n\to\infty}\|x_n - G(x_n)\| = 0. \tag{3.7}$$

According to Lemma 2.7 and (3.7), we have $x^* \in F(G)$. Therefore $x^* \in \mathcal{F}$.

Now, from (3.6), Proposition 2.1 (iii) and the weakly sequential continuity of the duality mapping J, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle (f-I)\bar{x}, J(x_n-\bar{x})\rangle &= \lim_{i\to\infty}\langle (f-I)\bar{x}, J(x_{n_i}-\bar{x})\rangle \\ &= \langle (f-I)\bar{x}, J(x^*-\bar{x})\rangle \le 0. \qquad (3.8)\end{aligned}$$

From (3.5), it follows that

$$\limsup_{n\to\infty}\langle (f-I)\bar{x}, J(x_{n+1}-\bar{x})\rangle \le 0. \qquad (3.9)$$

Finally, we show that $\{x_n\}$ converges strongly to $\bar{x} = Q_{\mathcal{F}}f(\bar{x})$. Observe that

$$\begin{aligned}\|x_{n+1}-\bar{x}\|^2 &= \langle x_{n+1}-\bar{x}, J(x_{n+1}-\bar{x})\rangle \\ &= \langle \alpha_n f(x_n)+\beta_n x_n+\gamma_n Sv_n-\bar{x}, J(x_{n+1}-\bar{x})\rangle \\ &= \langle \alpha_n(f(x_n)-\bar{x})+\beta_n(x_n-\bar{x})+\gamma_n(Sv_n-\bar{x}), J(x_{n+1}-\bar{x})\rangle \\ &= \alpha_n\langle f(x_n)-f(\bar{x}), J(x_{n+1}-\bar{x})\rangle+\alpha_n\langle f(\bar{x})-\bar{x}, J(x_{n+1}-\bar{x})\rangle \\ &\quad +\beta_n\langle x_n-\bar{x}, J(x_{n+1}-\bar{x})\rangle+\gamma_n\langle Sv_n-\bar{x}, J(x_{n+1}-\bar{x})\rangle \\ &\le \alpha\alpha_n\|x_n-\bar{x}\|\|x_{n+1}-\bar{x}\|+\alpha_n\langle f(\bar{x})-\bar{x}, J(x_{n+1}-\bar{x})\rangle \\ &\quad +\beta_n\|x_n-\bar{x}\|\|x_{n+1}-\bar{x}\|+\gamma_n\|v_n-\bar{x}\|\|x_{n+1}-\bar{x}\| \\ &\le \alpha\alpha_n\|x_n-\bar{x}\|\|x_{n+1}-\bar{x}\|+\alpha_n\langle f(\bar{x})-\bar{x}, J(x_{n+1}-\bar{x})\rangle \\ &\quad +\beta_n\|x_n-\bar{x}\|\|x_{n+1}-\bar{x}\|+\gamma_n\|x_n-\bar{x}\|\|x_{n+1}-\bar{x}\| \\ &= \frac{\alpha\alpha_n+\beta_n+\gamma_n}{2}(\|x_n-\bar{x}\|^2+\|x_{n+1}-\bar{x}\|^2) \\ &\quad +\alpha_n\langle f(\bar{x})-\bar{x}, J(x_{n+1}-\bar{x})\rangle \\ &= \frac{\alpha\alpha_n+1-\alpha_n}{2}(\|x_n-\bar{x}\|^2+\|x_{n+1}-\bar{x}\|^2) \\ &\quad +\alpha_n\langle f(\bar{x})-\bar{x}, J(x_{n+1}-\bar{x})\rangle \\ &= \frac{1-\alpha_n(1-\alpha)}{2}(\|x_n-\bar{x}\|^2+\|x_{n+1}-\bar{x}\|^2) \\ &\quad +\alpha_n\langle f(\bar{x})-\bar{x}, J(x_{n+1}-\bar{x})\rangle \\ &\le \frac{1-\alpha_n(1-\alpha)}{2}\|x_n-\bar{x}\|^2+\frac{1}{2}\|x_{n+1}-\bar{x}\|^2 \\ &\quad +\alpha_n\langle f(\bar{x})-\bar{x}, J(x_{n+1}-\bar{x})\rangle,\end{aligned}$$

which implies that

$$\begin{aligned}\|x_{n+1}-\bar{x}\|^2 &\le (1-\alpha_n(1-\alpha))\|x_n-\bar{x}\|^2 \\ &\quad +2\alpha_n\langle f(\bar{x})-\bar{x}, J(x_{n+1}-\bar{x})\rangle. \qquad (3.10)\end{aligned}$$

Now, from (C1), (3.9) and applying Lemma 2.5 to (3.10), we get $\|x_n-\bar{x}\| \to 0$ as $n\to\infty$. This completes the proof. □

Corollary 3.5. *Let E be a uniformly convex and 2-uniformly smooth Banach space which admits a weakly sequentially continuous duality mapping and C be a nonempty closed convex subset of E. Let $S : C \to C$ be a nonepansive mapping and Q_C be a sunny nonexpansive retraction from E onto C. Let $A, B : C \to E$ be β-inverse-strongly accretive with $\beta \geq \lambda K^2$ and γ-inverse-strongly accretive with $\gamma \geq \mu K^2$, respectively and K be the best smooth constant. Let the sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ in $(0, 1)$ satisfy $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 1$ and satisfy the condition (C1) and (C2) in Theorem 3.4. Suppose $\mathcal{F} := F(G) \cap F(S) \neq \emptyset$ where G defined by Lemma 3.1 and let λ, μ are positive real numbers. For arbitrary given $x_0 = x \in C$, the sequences $\{x_n\}$ generated by*

$$\begin{cases} y_n = Q_C(x_n - \mu B x_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S Q_C(y_n - \lambda A y_n). \end{cases} \tag{3.11}$$

Then $\{x_n\}$ converges strongly to $Q_{\mathcal{F}}u$, where $Q_{\mathcal{F}}$ is the sunny nonexpansive retraction of C onto $\mathcal{F}$.

***Proof*.** Taking $f(x_n) = u$ for all $n \in \mathbb{N}$ for any fixed $u \in C$ in (3.3). So, by Theorem 3.4, we can conclude the desired conclusion easily. This completes the proof. □

Corollary 3.6. *[3, Theorem 3.1,] Let E be a uniformly convex and 2-uniformly smooth Banach space which admits a weakly sequentially continuous duality mapping and C be a nonempty closed convex subset of E. Let Q_C be a sunny nonexpansive retraction from E onto C. Let $A, B : C \to E$ be β-inverse-strongly accretive with $\beta \geq K^2$ and γ-inverse-strongly accretive with $\gamma \geq K^2$, respectively and K be the best smooth constant. Suppose the sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ in $(0, 1)$ satisfy $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 1$ and satisfy the condition (C1) and (C2) in Theorem 3.4. Assume $F(G) \neq \emptyset$ where G defined by Lemma 3.1. For arbitrary given $x_1 = u \in C$, the sequences $\{x_n\}$ generated by (1.5). Then $\{x_n\}$ converges strongly to $Q_{F(G)}u$, where $Q_{F(G)}$ is the sunny nonexpansive retraction of C onto $F(G)$.*

***Proof*.** Taking $f(x) = u$ for all $x \in C$, $S = I$ and $\lambda = \mu = 1$ in (3.3). Then, from Theorem 3.4, we can conclude the desired conclusion easily. □

4 Applications

(I) Application to finding zeros of accretive operators.

In Banach space E, we always assume that E is a uniformly convex and 2-uniformly smooth. Recall that an accretive operator T is m-accretive if $R(I+rT) = E$ for each $r > 0$. We assume that T is m-accretive and has a zero (i.e., the inclusion $0 \in T(z)$ is solvable) [19, 20, 21]. The set of zeros of T is denoted by $T^{-1}(0)$, that

$$T^{-1}(0) = \{z \in D(T) : 0 \in T(z)\}.$$

The resolvent of T, i.e., $J_r^T = (I + rT)^{-1}$, for each $r > 0$. If T is m-accretive, then $J_r^T : E \to E$ is nonexpansive and $F(J_r^T) = T^{-1}(0), \forall r > 0$. For example, see Rockafellar [22] and [13, 23, 24, 25, 26] for more details.

From the main result Theorem 3.4, we can conclude the following result immediately.

Theorem 4.1. *Let E be a uniformly convex and 2-uniformly smooth Banach space and C a nonempty closed convex subset of E. Let $A, B : C \to E$ be β-inverse-strongly accretive with $\beta \geq \lambda K^2$ and γ-inverse-strongly accretive with $\gamma \geq \mu K^2$, respectively, K is the 2-uniformly smoothness constant of E and let T be an m-accretive mapping. Let f be a contraction of C into itself with coefficient $\alpha \in [0, 1)$ and suppose the sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ in $(0, 1)$ satisfy $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 1$. Suppose $\Omega := T^{-1}(0) \cap A^{-1}(0) \cap B^{-1}(0) \neq \emptyset$ and let λ, μ are positive real numbers. The following conditions are satisfied:*
(i). $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii). $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.
The sequences $\{x_n\}$ generated by $x_0 = x \in C$ and

$$\begin{cases} y_n = x_n - \mu B x_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n J_r^T (y_n - \lambda A y_n). \end{cases} \tag{4.1}$$

Then $\{x_n\}$ converges strongly to $\bar{x} = Q_\Omega f(\bar{x})$, where Q_Ω is the sunny nonexpansive retraction of E onto Ω.

(II) Application to strictly pseudocontractive mappings

Let E be a Banach space and let C be a subset of E. Recall that a mapping $T : C \to C$ is said to be *k-stricly pseudocontractive* if there exist $k \in [0, 1)$ and $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \leq \|x - y\|^2 - \frac{1-k}{2}\|(I - T)x - (I - T)y\|^2 \tag{4.2}$$

for all $x, y \in C$. Then (4.2) can be written in the following form

$$\langle (I - T)x - (I - T)y, j(x - y)\rangle \geq \frac{1-k}{2}\|(I - T)x - (I - T)y\|^2. \tag{4.3}$$

We know that, A is $\frac{1-k}{2}-$ inverse strongly monotone and $A^{-1}0 = F(T)$ (see [27]).

Theorem 4.2. *Let E be a uniformly convex and 2-uniformly smooth Banach space and C a nonempty closed convex subset of E. Let $S : C \to C$ be a nonepansive mapping and a sunny nonexpansive retraction of E. Let $T, U : C \to C$ be k-stricly pseudocontractive and l-stricly pseudocontractive with $\lambda \leq \frac{(1-k)}{2K^2}$ and $\mu \leq \frac{(1-l)}{2K^2}$, respectively. Let f be a contraction of C into itself with coefficient $\alpha \in [0, 1)$ and suppose the sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ in $(0, 1)$ satisfy $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 1$. Suppose $\mathcal{F} := F(S) \cap F(T) \cap F(U) \neq \emptyset$ and let λ, μ are positive real numbers. The following conditions are satisfied:*
(i). $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii). $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.
The sequences $\{x_n\}$ generated by $x_0 = x \in C$ and

$$\begin{cases} y_n = (1 - \mu)x_n + \mu U x_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S((1 - \lambda)y_n + \lambda T y_n). \end{cases} \tag{4.4}$$

Then $\{x_n\}$ *converges strongly to* $Q_{\mathcal{F}}$*, where* $Q_{\mathcal{F}}$ *is the sunny nonexpansive retraction of* E *onto* $\mathcal{F}$*.*

Proof. Put $A = I - T$ and $B = I - U$. Form (4.3), we get A, B are $\frac{1-k}{2}$ and $\frac{1-l}{2}-$ inverse strongly accretive operators, respectively. It follows that $VI(C, A) = VI(C, I - T) = F(T) \neq \emptyset$, $CI(C, B) = VI(C, I - U) = F(U) \neq \emptyset$ and $CI(C, I - T) \cap VI(C, I - U) = F(U) = F(G) \Leftrightarrow$ is the solution of problems (1.2) $\Leftrightarrow$ problems (1.3) (see also Ceng et al. [11, Theorem 4.1 pp. 388–389]) and also have (see Aoyama et al.[2, Theorem 4.1 pp. 10.])

$$(1-\lambda)y_n + \lambda T y_n = Q_C((1-\lambda)y_n - \lambda T y_n) \text{ and } (1-\lambda)x_n + \lambda U x_n = Q_C((1-\lambda)x_n - \lambda U x_n).$$

Therefore, by Theorem 3.4, $\{x_n\}$ converges strongly to some element x^* of $\mathcal{F}$. □

(III) Application to Hilbert spaces.

In real Hilbert spaces, by Lemma 3.2 and Remark 3.3 it follow from Lemma 4.1 of [1], we obtain the following Lemma:

Lemma 4.3. *For given* $(x^*, y^*) \in C$*, where* $y^* = P_C(x^* - \mu B x^*)$*,* (x^*, y^*) *is a solution of problem* (1.3) *if and only if* x^* *is a fixed point of the mapping* $G' : C \to C$ *defined by*

$$G'(x) = P_C[P_C(x - \mu Bx) - \lambda A P_C(x - \mu Bx)],$$

where P_C *is a metric projection* H *onto* C*.*

It is well known that the smooth constant $K = \frac{\sqrt{2}}{2}$ in Hilbert spaces. From Theorem 3.4, we can obtain the following result immediately.

Theorem 4.4. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H*. Let* $A, B : C \to H$ *are an* β*-inverse-strongly monotone mapping with* $\lambda \in (0, 2\beta)$ *and* γ*-inverse-strongly monotone mapping with* $\mu \in (0, 2\gamma)$*, respectively, and let* f *be a contraction of* C *into itself with coefficient* $\alpha \in [0, 1)$*. Suppose the sequences* $\{\alpha_n\}, \{\beta_n\}$ *and* $\{\gamma_n\}$ *in* $(0, 1)$ *satisfy* $\alpha_n + \beta_n + \gamma_n = 1$*,* $n \geq 1$*. Assume that* $F(G') \cap F(S) \neq \emptyset$ *where* G' *defined by Lemma 4.3 and let* λ, μ *are positive real numbers. The following conditions are satisfied:*
(i). $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$*;*
(ii). $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$*.*
For arbitrary given $x_0 = x \in C$*, the sequences* $\{x_n\}$ *is generated by*

$$\begin{cases} y_n = P_C(x_n - \mu B x_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S P_C(y_n - \lambda A y_n). \end{cases} \tag{4.5}$$

Then $\{x_n\}$ *converges strongly to* $\bar{x} = P_{F(G') \cap F(S)} f(\bar{x})$ *and* $(\bar{x}, \bar{y})$ *is a solution of the problem* (1.3)*, where* $\bar{y} = P_C(\bar{x} - \mu B \bar{x})$*.*

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