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Iterative Approximations for Generalized Multivalued Mappings in Banach Spaces

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Abstract : In this paper, we introduce some condition on multivalued mappings, which can be a generalized multivalued nonexpansive mapping. Some convergence theorems for mappings satisfying the condition were established.

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1 Introduction

Let X be a Banach space and K a nonempty subset of X. We shall denote by 2^X the family of all subsets of X, $CB(X)$ the family of all nonempty closed bounded subsets of X and denote $C(X)$ by the family of nonempty compact subsets of X. A multivalued mapping $T : K \to 2^X$ is said to be nonexpansive (resp, contractive) if

$$H(Tx, Ty) \leq \|x - y\|, \quad x, y \in K,$$

$$(\text{resp}, H(Tx, Ty) \leq k\|x - y\|, \text{ for some } \; k \in (0, 1)).$$

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where $H(\cdot,\cdot)$ denotes the Hausdorff metric on $CB(X)$ defined by

$$H(A,B) := \max\left\{\sup_{x\in A}\inf_{y\in B}\|x-y\|,\ \sup_{y\in B}\inf_{x\in A}\|x-y\|\right\},\quad A,B\in CB(X).$$

Note that a multivalued mapping $T : K \to 2^X$ is said to be upper semicontinuous on K if $\{x \in K : Tx \subset V\}$ is open in K whenever $V \subset X$ is open; T is said to be lower semicontinuous if $T^{-1}(V) := \{x \in K : Tx \cap V \neq \emptyset\}$ is open in K whenever $V \subset X$ is open; and T is said to be continuous if it is both upper and lower semicontinuous. There exist another kind of continuity for set-valued operators: $T : X \to CB(X)$ is said to be continuous on X(with respect to the Hausdorff metric H) if $H(Tx_n, Tx) \to 0$ whenever $x_n \to x$. It is not hard to see (see [1])that both definitions of continuity are equivalent if Tx is compact for every $x \in X$. A point x is called a fixed point of T if $x \in Tx$. From now on, $F(T)$ stand for the fixed point set of a mapping T.

Since Banach's Contraction Mapping Principle was extended nicely to multivalued mappings by Nadler in 1969 (see [2]), many authors have studied the fixed point theory for multivalued mappings (e.g. see [3–9]). Very recently, in order to characterize the completeness of underlying metric spaces, Suzuki introduced a weaker notion of contractions and proved the following theorem.

Theorem 1.1 ([10]). *Define a nonincreasing function θ from $[0,1)$ onto $(1/2,1]$ by*

$$\theta(r)=\begin{cases}1 & \text{if } 0\le r\le(\sqrt{5}-1)/2,\\ (1-r)r^{-2} & \text{if } (\sqrt{5}-1)/2\le r\le 2^{-1/2},\\ (1+r)^{-1} & \text{if } 2^{-1/2}\le r<1.\end{cases}$$

Then for a metric space (X,d), the following are equivalent:

(i) X is complete.

(ii) There exists $r \in (0,1)$ such that every mapping T on X satisfying the following has a fixed point:

$$\theta(r)d(x,Tx)\le d(x,y) \text{ implies } d(Tx,Ty)\le rd(x,y) \text{ for all } x,y\in X.$$

Theorem 1.1 is meaningful because contractions do not characterize the metric completeness while Caristi and Kannan mappings do; see [11–13]. Since $\lim_{r\to 1^-} =$ 1/2, the author in [14] consider the following condition.

Definition 1.2. *Let T be a mapping on a subset K of a Banach space X. Then T is said to satisfy condition (C) if*

$$\frac{1}{2}\|x-Tx\|\le\|x-y\| \ \ \textit{implies}\ \ \|Tx-Ty\|\le\|x-y\| \ \ \textit{for all } x,y\in K.$$

The condition is weaker than nonexpansiveness and stronger than quasinonexpansiveness (see [14]). Furthermore, the authors present fixed point theorems and convergence theorems for mappings satisfying condition (C). It is a very natural question whether we can introduce some condition the same as condition (C) on multivalued mapping to obtain some convergence theorems for mappings satisfying the condition. In this paper, we introduce the following condition (D).

Definition 1.3. *Let T be a multivalued mapping on a subset K of a Banach space X. Then T is said to satisfy condition (D) if*

$$\frac{1}{2}d(x,Tx) \le \|x-y\| \quad \textit{implies} \quad H(Tx,Ty) \le \|x-y\| \quad \textit{for all } x,y \in K.$$

where d is induced by the norm.

Obviously, the condition is weaker than the multivalued nonexpansive mapping (with respect to the Hausdorff metric H), in fact we can give a example to show the condition is strict. Meanwhile, we give some convergence theorems for mappings satisfying the condition.

2 Preliminaries

A Banach space X is said to be satisfy Opial's condition [15] if, for any sequence $\{x_n\}$ in X, $x_n \rightharpoonup x$ $(n \to \infty)$ implies the following inequality

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \in X$ with $y \ne x$. We know that Hilbert spaces and $l_p(1 < p < \infty)$ have the Opial's condition.

The following Lemmas will be useful in this paper.

Lemma 2.1 ([16])**.** *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X such that*

$$x_{n+1} = \gamma_n x_n + (1-\gamma_n)y_n, \qquad n \ge 0$$

where $\{\gamma_n\}$ is a sequence in $[0,1]$ such that

$$0 < \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty} \gamma_n < 1.$$

Assume $\limsup_{n\to\infty}(\|y_{n+1}-y_n\| - \|x_{n+1}-x_n\|) \le 0$, then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 2.2 ([2])**.** *Let X be a complete metric space and $A, B \in CB(X)$. Then, for any $a \in A$, there exists $b \in B$ such that*

$$d(a,b) \le H(A,B).$$

Let K be a nonempty closed convex subset of Banach space X and $T : K \to CB(K)$ be a multivalued mapping, $\alpha_n \in (0,1)$. Choose $x_0 \in K$ and $y_0 \in Tx_0$ such that

$$x_1 = \alpha_0 x_0 + (1-\alpha_0)y_0.$$

From Lemma 2.2, we can choose $y_1 \in Tx_1$ such that

$$\|y_0 - y_1\| \le H(Tx_0, Tx_1).$$

Inductively, we can get the sequence $\{x_n\}$ as follows:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)y_n, \quad \forall n \in N, \tag{2.1}$$

where, for each $n \in N$, $y_n \in Tx_n$ such that

$$\|y_n - y_{n-1}\| \le H(Tx_n, Tx_{n-1}).$$

A multivalued mapping $T : K \to CB(K)$ is said to satisfy Condition (E) if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for $r \in (0,\infty)$ such that

$$d(x, Tx) \ge f(d(x, F(T)) \quad for \quad all \ x \in K.$$

Examples of mappings that satisfy Condition (E) can be found in [17, 18].

3 Main Results

The following proposition is obvious.

Proposition 3.1. *Every multivalued nonexpansive mapping satisfies condition (D).*

Example 3.2. *Define a mapping T on $[0,3]$ by*

$$Tx = \begin{cases} \{0\} & if \ \ x \neq 3, \\ [0,1] & if \ x = 3. \end{cases}$$

Then T satisfies condition (D), but T is not multivalued nonexpansive.

Proof. Note that $F(T) = \{0\}$. If $x < y$ and $(x,y) \in ([0,3]\times[0,3]) \setminus ((2,3]\times\{3\})$, then $H(Tx,Ty) \le \|x-y\|$ holds. If $x \in (2,3)$ and $y = 3$, then

$$\frac{1}{2}d(x,Tx) = \frac{x}{2} > 1 > \|x-y\| \ \text{ and } \ \frac{1}{2}d(y,Ty) = 1 > \|x-y\| \ \text{ holds.}$$

If $x = 3$, $y = 3$, then

$$\frac{1}{2}d(x,Tx) = 1 > \|x-y\| = 0.$$

Thus T satisfies condition (D). However, since T is not continuous, then T is not multivalued nonexpansive. In fact, we can take a open set $V = (\frac{1}{2}, \frac{1}{3})$, $T^{-1}(V) := \{x \in K : Tx \cap V \neq \emptyset\} = \{3\}$ is not a open set, therefore T is not lower semicontinuous. □

Example 3.3. *Let $K = [0, \infty)$ and T be defined by $Tx = [x, 2x]$ for $x \in K$. We observe that*

$$\frac{1}{2}d(x, Tx) = 0 \leq \|x - y\| \text{ for all } x, y \in K,$$

however T is not multivalued nonexpansive (see [19]).

Lemma 3.4. *Let (X, d) be a complete metric space and T be a mapping from X into $CB(X)$. Assume that T satisfies condition (D), then for $x, y \in X$, the following conclusions holds:*

(i) $H(Tx, T^2x) \leq d(x, Tx)$.

(ii) *Either* $\frac{1}{2}d(x, Tx) \leq d(x, y)$ *or* $\frac{1}{2}H(Tx, T^2x) \leq d(Tx, y)$.

(iii) *Either* $H(Tx, Ty) \leq d(x, y)$ *or* $H(T^2x, Ty) \leq d(Tx, y)$.

Proof. (i) Since $\frac{1}{2}d(x, Tx) \leq \|x - y\|$ for $y \in Tx$, it then follows from Definition 1.3 that

$$\inf_{y \in Tx} H(Tx, Ty) \leq d(x, Tx)$$

for $x \in K$. From the definition of Hausdorff metric, we can get the following inequality

$$H(Tx, T^2x) \leq \inf_{y \in Tx} H(Tx, Ty) \;\; \textit{holds}.$$

Therefore we obtain the desired result.
(iii) follows from (ii). Let us prove (ii). Suppose that

$$\frac{1}{2}d(x, Tx) > d(x, y) \;\text{ and }\; \frac{1}{2}H(Tx, T^2x) > d(Tx, y).$$

Then we have by (i)

$$\begin{aligned} d(x, Tx) &\leq d(x, y) + d(y, Tx) \\ &< \frac{1}{2}d(x, Tx) + \frac{1}{2}H(Tx, T^2x) \\ &\leq d(x, Tx). \end{aligned}$$

This is a contraction. Therefore we obtain the desired result. □

Lemma 3.5. *Let (X, d) be a complete metric space and T be a mapping from X into $CB(X)$. Assume that T satisfies condition (D), then*

$$d(x, Ty) \leq 3d(x, Tx) + d(x, y)$$

holds for all $x, y \in X$.

Proof. By Lemma 3.4, we have either

$$H(Tx,Ty)\le d(x,y)\quad\text{or}\quad H(T^2x,Ty)\le d(Tx,y)$$

holds. In the first case, we have

$$d(x,Ty)\le d(x,Tx)+H(Tx,Ty)\le d(x,Tx)+d(x,y).$$

In the second case, we get by Lemma 3.4,

$$\begin{aligned} d(x,Ty) &\le d(x,Tx)+H(Tx,T^2x)+H(T^2x,Ty)\\ &\le 2d(x,Tx)+d(Tx,y)\\ &\le 3d(x,Tx)+d(x,y).\end{aligned}$$

Therefore we get the desired result. □

Lemma 3.6. *Let X be a Banach space and T be a mapping from X into $CB(X)$. Assume that T satisfies condition (D). The sequence $\{x_n\}$ is defined by (2.1), and $\alpha_n\in[1/2,1)$. Then we have*

$$\lim_{n\to\infty} d(x_n,Tx_n)=0$$

holds, where d is distance induced by the norm.

Proof. From the assumption, we have that

$$\frac{1}{2}d(x_n,Tx_n)\le \alpha_n d(x_n,y_n)=d(x_n,x_{n+1})$$

for $n\in\mathbb{N}$. From condition (D), we have

$$H(Tx_n,Tx_{n+1})\le d(x_n,x_{n+1}).$$

Therefore

$$\|y_{n+1}-y_n\|\le H(Tx_n,Tx_{n+1}))\le \|x_{n+1}-x_n\|$$

Then we have

$$\lim_{n\to\infty}(\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|)\le 0$$

By the Lemma 2.1, we get

$$\lim_{n\to\infty}\|x_n-y_n\|=0.$$

Since $y_n\in Tx_n$, then $0\le \lim_{n\to\infty} d(x_n,Tx_n)\le \lim_{n\to\infty}\|x_n-y_n\|=0$. We get the desired result. □

Theorem 3.7. *Let K be a nonempty compact convex subset of a Banach space X. Suppose that $T : K \to CB(K)$ is a multivalued mapping satisfies condition (D) for which $F(T) \neq \emptyset$ and $T(y) = \{y\}$ for each $y \in F(T)$. Let $\{x_n\}$ be defined by (2.1). Assume that*

$$\frac{1}{2} \leq \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1.$$

Then the sequence $\{x_n\}$ strongly converges to a fixed point of T.

Proof. Take $p \in F(T)$, noting $Tp = \{p\}$ and $\|y_n - p\| = d(y_n, Tp)$. We have

$$\frac{1}{2}d(p, Tp) = 0 \leq \|p - x_n\|,$$

then $H(Tp, Tx_n) \leq \|x_n - p\|$ follows from T satisfies condition (D). Therefore we have

$$\begin{aligned} \|x_{n+1} - p\| &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\|y_n - p\| \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n(H(Tx_n, Tp)) \\ &\leq \|x_n - p\|. \end{aligned}$$

Then $\{\|x_n - p\|\}$ is a decreasing sequence and $\lim_{n\to\infty} \|x_n - p\|$ exists for each $p \in F(T)$. From the compactness of K, there exist a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converging to q. By Lemma 3.5, we have

$$d(x_{n_j}, Tq) \leq 3d(Tx_{n_j}, x_{n_j}) + d(x_{n_j}, q)$$

for all $j \in \mathbb{N}$. From Lemma 3.6 this implies $q \in Tq$. That is, q is a fixed point of T. Now on taking q in place of p, we have that $\lim_{n\to\infty} \|x_n - q\|$ exists. So we get the desired result. □

Theorem 3.8. *Let X be a Banach space satisfying Opial's condition and K be a nonempty weakly compact convex subset of X. Suppose that $T : K \to C(K)$ is a multivalued mapping satisfies condition (D). Let $\{x_n\}$ be defined by (2.1). Assume that $F(T) \neq \emptyset$ and satisfies $T(y) = \{y\}$ for each $y \in F(T)$ and*

$$\frac{1}{2} \leq \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1.$$

Then the sequence $\{x_n\}$ weakly converges to a fixed point of T.

Proof. From the proof of Theorem 3.7 that $\lim_{n\to\infty} \|x_n - p\|$ exists for each $p \in F(T)$. Since K is weakly compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^*$ for some $x^* \in K$. Suppose x^* does not belong to Tx^*. By the compactness of Tx^*, for any given x_{n_k}, we can choose a convergent subsequence $p_k \in Tx^*$ such that $\|x_{n_k} - p_k\| = d(x_{n_k}, Tx^*)$ and $p_k \to p \in Tx^*$. Then $x^* \neq p$.

From Lemma 3.6 we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, then we can take N_k such that when $k > N_k$,

$$\frac{1}{2}d(x_{n_k}, Tx_{n_k}) \leq d(x_{n_k}, x^*),$$

since T satisfies condition (D), then $H(Tx_{n_k}, Tx^*) \leq \|x_{n_k} - x^*\|$. Since X satisfies Opial's condition, then we get

$$\begin{aligned}
\limsup_{k\to\infty} \|x_{n_k} - p\| &\leq \limsup_{k\to\infty}[\|x_{n_k} - p_k\| + \|p_k - p\|] = \limsup_{k\to\infty} \|x_{n_k} - p_k\| \\
&\leq \limsup_{k\to\infty}[d(x_{n_k}, Tx_{n_k}) + H(Tx_{n_k}, Tx^*)] \\
&\leq \limsup_{k\to\infty} \|x_{n_k} - x^*\| < \limsup_{k\to\infty} \|x_{n_k} - p\|.
\end{aligned}$$

This is a contraction. Hence $x^* \in Tx^*$.
Next, we will show $x_n \rightharpoonup x^*$. Arguing by contraction, assume that there exists another subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup x \neq x^*$. Then, we also have $x \in Tx$. From Opial's property,

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - x\| &= \limsup_{i\to\infty} \|x_{n_i} - x\| \\
&< \limsup_{i\to\infty} \|x_{n_i} - x^*\| = \limsup_{k\to\infty} \|x_{n_k} - x^*\| \\
&< \limsup_{k\to\infty} \|x_{n_k} - x\| = \lim_{n\to\infty} \|x_n - x\|.
\end{aligned}$$

Which is a contraction. So the conclusion of Theorem follows. □

Theorem 3.1. *Let K be a nonempty closed convex subset of a Banach space X. Suppose that $T : K \to CB(K)$ is a multivalued mapping that satisfies condition (E) and condition (D). Let $\{x_n\}$ be the sequence of defined by (2.1). Assume that $F(T) \neq \emptyset$ and satisfies $T(y) = \{y\}$ for each $y \in F(T)$ and*

$$\frac{1}{2} \leq \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1.$$

Then the sequence $\{x_n\}$ strongly converges to a fixed point of T.

Proof. From the proof of Theorem 3.5 that $\lim_{n\to\infty} \|x_n - p\|$ exists for each $p \in F(T)$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ from Lemma 3.4. Then condition (E) implies

$$\lim_{n\to\infty} d(x_n, F(T)) = 0.$$

Thus, for arbitrary given $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that

$$d(x_n, F(T)) < \epsilon \quad \text{for all} \quad n \geq N_\epsilon.$$

We can take $\epsilon_k = \frac{1}{2^k}$, then there exist $N_k \in \mathbb{N}$ and $p_k^n \in F(T)$ such that $N_k \leq N_{k+1}$ and

$$\|x_n - p_k^n\| < \frac{\epsilon_k}{4} \quad \text{for all} \quad n \geq N_k.$$

Thus, we have

$$\|p_{k+1}^n - p_k^n\| \le \|p_{k+1}^n - x_{N_{k+1}}\| + \|x_{N_{k+1}} - p_k^n\| \le \frac{\epsilon_{k+1}}{4} + \frac{\epsilon_k}{4} = \frac{3\epsilon_{k+1}}{4}.$$

We denote $S(p,r) = \{x \in X; \|x-p\| \le r\}$. For $x \in S(p_{k+1}^n, \epsilon_{k+1})$, we have

$$\begin{aligned} \|p_k^n - x\| &\le \|p_k^n - p_{k+1}^n\| + \|p_{k+1}^n - x\| \\ &\le \frac{3\epsilon_{k+1}}{4} + \epsilon_{k+1} < 2\epsilon_{k+1} = \epsilon_k. \end{aligned}$$

Therefore we have $S(p_{k+1}^n, \epsilon_{k+1}) \subset S(p_k^n, \epsilon_k)$ for all $n \ge N_{k+1}$. By Cantor intersection theorem, there exists a single point x^* such that

$$\bigcap_{k=1}^{\infty} S(p_k^n, \epsilon_k) = \{x^*\}.$$

Thus we get

$$\|p_k^n - x^*\| \le \epsilon_k \quad \text{for all} \quad k \in \mathbb{N} \quad \text{and} \quad n \ge N_{k+1}.$$

That is, $\lim_{k\to\infty} \|p_k^n - x^*\| = 0$. Which assures $\lim_{n\to\infty} \|x_n - x^*\| = 0$ since $\lim_{k\to\infty} N_k = \infty$ implies $n \to \infty$. For any $x \in Tx^*$, noting $Tp_k^n = \{p_k^n\}$, we have

$$0 = \frac{1}{2} d(p_k^n, Tp_k^n) \le d(p_k^n, x^*),$$

since T satisfies condition (D), we have $H(Tp_k^n, Tx^*) \le \|p_k^n - x^*\|$. Thus we get

$$\begin{aligned} \|x^* - x\| &\le \|x^* - p_k^n\| + d(x, Tp_k^n) \le \|x^* - p_k^n\| + H(Tp_k^n, Tx^*) \\ &\le 2\|x^* - p_k^n\| \to 0. \end{aligned}$$

So x^* is a fixed point of T and $\{x_n\}$ strongly converges to x^*. □

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