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Some Results for Finite Families of Uniformly L-Lipschitzian Mappings in Banach Spaces

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Abstract : The purpose of this paper is to prove a strong convergence theorem for two finite families of uniformly L-Lipschitzian mappings in Banach spaces. The results presented improve and extend some recent results in Chang [1–3], Cho et al. [4], Ofoedu [5], Schu [6] and Zeng [7, 8].

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1 Introduction and Preliminaries

Throughout this paper, we assume that E is a real Banach space, E^* is the dual space of E, K is a nonempty closed convex subset of E and $J : E \to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \left\{ f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2, \|x\| = \|f\| \right\}, \quad \forall x \in E,$$

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where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between E and E^*. The single-valued normalized duality mapping is denoted by j.

Definition 1.1. *Let $T : K \to K$ be a mapping.*

(1) T is said to be uniformly L-Lipschitzian if there exists $L > 0$ such that, for any $x, y \in K$,

$$\|T^n x - T^n y\| \leq L \|x - y\|, \forall n \geq 1;$$

(2) T is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that for any given $x, y \in K$,

$$\|T^n x - T^n y\| \leq k_n \|x - y\|, \forall n \geq 1;$$

(3) T is said to be asymptotically pseudo-contractive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that, for any $x, y \in K$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle T^n x - T^n y, j(x - y) \rangle \leq k_n \|x - y\|^2, \forall n \geq 1.$$

Remark 1.2. *It is easy to see that if T is an asymptotically nonexpansive mapping, then T is a uniformly L-Lipschitzian mapping, where $L = \sup_{n \geq 1} k_n$, and every asymptotically nonexpansive mapping is asymptotically pseudo-contractive, but the converse is not true, in general as shown by the following example.*

Example 1.3 ([9])**.** *Let $E = \mathbb{R}$ and $C = [0, 1]$ and let the mapping $T : C \to C$ be defined by*

$$Tx = \left(1 - x^{\frac{2}{3}}\right)^{\frac{3}{2}}$$

for all $x \in C$. It can be proved that T is not Lipschitzian, and so it is not asymptotically nonexpansive. Since T is monotonically decreasing and $T \circ T = I$, the identity mapping, we have

$$\begin{cases} \langle T^n x - T^n y, x - y \rangle = |x - y|^2 & \text{if } n \text{ is even,} \\ \langle T^n x - T^n y, x - y \rangle \leq |x - y|^2 & \text{if } n \text{ is odd.} \end{cases}$$

This implies that T is an asymptotically pseudo-contractive mapping with a constant sequence $\{1\}$.

Approximation problems using iterative methods for asymptotically nonexpansive mappings and asymptotically pseudo-contractive mappings have been studied by many authors. For example, Chang [1], Cho et al. [4, 10], Chidume [11], Goebel and Kirk [12], Khan et al. [13, 14], Ofoedu [5], Osilike and Aniagbosor [15], Rhoades [9] , Qin et al. [16], Schu [6] and Xu [17] in the setting of Hilbert or Banach spaces. Schu [6] proved the following theorem in the framework of Hilbert spaces.

Theorem 1.4 ([6]). *Let H be a Hilbert space, K be a nonempty bounded closed convex subset of H and $T : K \to K$ be a completely continuous, uniformly L-Lipschitzian and asymptotically pseudo-contractive mapping with a sequence $\{k_n\} \subset [1,\infty)$ satisfying the following conditions:*

(i) $k_n \to 1$ *as* $n \to \infty$;

(ii) $\sum_{n=1}^{\infty} \left(q_n^2 - 1\right) < \infty$, *where* $q_n = 2k_n - 1$.

Suppose further that $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0,1]$ such that $\varepsilon < \alpha_n < \beta_n \le b, \forall n \ge 1$, where $\varepsilon > 0$ and $b \in (0, L^{-2}[(1+L^2)^{\frac{1}{2}} - 1])$. For any $x_1 \in K$, let $\{x_n\}$ be the iterative sequence defined by

$$x_{n+1} = (1-\alpha_n)\, x_n + \alpha_n T^n x_n, \forall n \ge 1.$$

Then $\{x_n\}$ converges strongly to a fixed point of T in K.

Chang [1] extended Theorem 1.4 to a real uniformly smooth Banach spaces. To be more precise, he proved the following theorem:

Theorem 1.5 ([1]). *Let E be a uniformly smooth Banach space, K be a nonempty bounded closed convex subset of E, $T : K \to K$ be an asymptotically pseudo-contractive mapping with a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ and $F(T) \ne \emptyset$, where $F(T)$ is the set of fixed points of T in K. Let $\{\alpha_n\}$ be a sequence in $[0,1]$ satisfying the following conditions:*

(i) $\alpha_n \to 0$;

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

For any $x_0 \in K$, let $\{x_n\}$ be the iterative sequence defined by

$$x_{n+1} = (1-\alpha_n)\, x_n + \alpha_n T^n x_n, \forall n \ge 0.$$

If there exists a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that

$$\langle T^n x_n - x^*, j(x_n - x^*)\rangle \le k_n \|x_n - x^*\|^2 - \phi(\|x_n - x^*\|), \forall n \ge 0,$$

where $x^* \in F(T)$ is some fixed point of T in K, then $x_n \to x^*$ as $n \to \infty$.

In 2006, Ofoedu [5] proved the following theorem:

Theorem 1.6 ([5]). *Let E be a real Banach space, K be a nonempty closed convex subset of E, $T : K \to K$ be a uniformly L -Lipschitzian asymptotically pseudo-contractive mapping with a sequence $\{k_n\} \subset [1,\infty)$, $k_n \to 1$ such that $x^* \in F(T)$, where $F(T)$ is the set of fixed points of T in K. Let $\{\alpha_n\}$ be a sequence in $[0,1]$ satisfying the following conditions:*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$;

(iii) $\sum_{n=0}^{\infty} \alpha_n (k_n - 1) < \infty$.

For any $x_0 \in K$, *let* $\{x_n\}$ *be the iterative process defined by*

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T^n x_n, \forall n \geq 0.$$

If there exists a strict increasing function $\phi : [0, \infty) \to [0, \infty)$ *with* $\phi(0) = 0$ *such that*

$$\langle T^n x - x^*, j(x - x^*) \rangle \leq k_n \|x - x^*\|^2 - \phi(\|x - x^*\|), \forall x \in K,$$

then $\{x_n\}$ *converges strongly to* x^*.

Remark 1.7. *It may be noted that Theorem* 1.6 *extends Theorem* 1.5 *from a real uniformly smooth Banach space to an arbitrary real Banach space and removes the boundedness condition imposed on* K. *For a correction and further improvement of this result, see Chang et al. [2].*

Xu and Ori [18] introduced the following implicit iterative process for a finite family of nonexpansive mappings $\{T_i : i \in I\}$ where $I = \{1, 2, ..., N\}$, with $\{\alpha_n\}$ a real sequence in $(0, 1)$, and an initial point $x_0 \in K$:

$$x_n = (1 - \alpha_n) x_{n-1} + \alpha_n T_n x_n, \quad \forall n \geq 1, \tag{1.1}$$

where $T_n = T_{n(modN)}$ and $modN$ function takes values in I. Xu and Ori proved the weak convergence of this process to a common fixed point for a finite family defined in a Hilbert space.

Chidume-Shahzad [19] and Zhou-Chang [20] studied the weak and strong convergence of this implicit process to a common fixed point for finite families of nonexpansive and asymptotically nonexpansive mappings respectively.

In 2004, Sun [21] improved the results of Xu and Ori [18] from nonexpansive mappings to asymptotically quasi-nonexpansive mappings in Banach spaces. In doing so, he considered the following implicit iterative process for a finite family of asymptotically quasi-nonexpansive mappings $\{T_i : i \in I\}$ with $\{\alpha_n\}$ a real sequence in $(0, 1)$, and an initial point $x_0 \in K$:

$$x_n = (1 - \alpha_n) x_{n-1} + \alpha_n T_{i(n)}^{k(n)} x_n, \forall n \geq 1, \tag{1.2}$$

where $T_n = T_{n(modN)}$, $n = (k - 1) N + i, i \in I$.

Recently, Khan et al. [22] introduced the following implicit iteration process for common fixed points of two finite families of Lipschitzian pseudocontractive mappings $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$ in Banach spaces. For arbitrarily chosen $x_0 \in K$, $\{x_n\}$ is defined as follows:

$$x_n = (1 - \alpha_n - \beta_n) x_{n-1} + \alpha_n S_n x_{n-1} + \beta_n T_n x_n, \ \forall n \geq 1, \tag{1.3}$$

where $T_n = T_{n(modN)}, S_n = S_{n(modN)}$ and $\{\alpha_n\}$, $\{\beta_n\}$ are two real sequences in $[0, 1]$ satisfying $\alpha_n + \beta_n \leq 1$ for all $n \geq 1$.

Inspired by above works, the iterative process (1.3) for two finite families of uniformly L-Lipschitzian mappings $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$, is introduced and studied in this paper. This process can be viewed as an extension for $(1.1), (1.2)$ and (1.3). This scheme reads as:

$$x_n = (1-\alpha_n-\beta_n)\,x_{n-1} + \alpha_n S_{i(n)}^{k(n)} x_{n-1} + \beta_n T_{i(n)}^{k(n)} x_n,\ \forall n \geq 1, \tag{1.4}$$

where $T_n = T_{n(modN)}, S_n = S_{n(modN)}, n = (k-1)\,N + i, i \in I$ and $\{\alpha_n\}$, $\{\beta_n\}$ are two real sequences in $[0,1]$ satisfying $\alpha_n + \beta_n \leq 1$ for all $n \geq 1$.

Now, we show that implicit iterative process (1.4) can be employed for approximating common fixed points of two finite families of uniformly L-Lipschitzian mappings.

Let E be a Banach space, K a nonempty closed convex subset of E and $\{T_i\}_{i=1}^N$, $\{S_i\}_{i=1}^N : K \to K$ be N uniformly L-Lipschitzian mappings where $L = \max\{L_1, L_2, ..., L_N\}$. Let $\{x_n\}$ be defined by (1.4). Define a mapping $W_n : K \to K$ by $W_n x = (1-\alpha_n-\beta_n)\,x_0 + \alpha_n S_{i(n)}^{k(n)} x_0 + \beta_n T_{i(n)}^{k(n)} x$ for all $x \in K$ and $\forall n \geq 1$. Now for any $x, y \in K$ and $\forall n \geq 1$, we have

$$\begin{aligned} \|W_n x - W_n y\| &= \left\|\beta_n T_{i(n)}^{k(n)} x - \beta_n T_{i(n)}^{k(n)} y\right\| \\ &\leq \beta_n L\,\|x-y\|\,. \end{aligned}$$

If $\beta_n L < 1$, W_n is a contraction mapping. By Banach Contraction Principle, $W_n, \forall n \geq 1$ has a unique fixed point. Thus the implicit iterative processes (1.4) is well-defined.

The purpose of this paper is, by using a simple and quite different method, to study the convergence of implicit iterative sequence $\{x_n\}$ defined by (1.4) to a common fixed points for two finite families of uniformly L-Lipschitzian mappings instead of the assumption that T is a uniformly L-Lipschitzian and asymptotically pseudo-contractive mapping in a Banach space. Our results extend and improve some recent results in [1–8].

In order to prove our main results, we need the following lemmas.

Lemma 1.8 ([23])**.** *Let E be a real Banach space and $J : E \to 2^{E^*}$ be the normalized duality mapping. Then, for any $x, y \in E$,*

$$\|x+y\|^2 \leq \|x\|^2 + 2\,\langle y, j\,(x+y)\rangle\,, \forall j\,(x+y) \in J\,(x+y)\,.$$

Lemma 1.9 ([24])**.** *Let $\{\theta_n\}$ be a sequence of nonnegative real numbers and $\{\lambda_n\}$ be a real sequence satisfying the following conditions:*

$$0 \leq \lambda_n \leq 1, \quad \sum_{n=0}^{\infty} \lambda_n = \infty.$$

If there exists a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ such that

$$\theta_{n+1}^2 \leq \theta_n^2 - \lambda_n \phi\,(\theta_{n+1}) + \sigma_n, \quad \forall n \geq n_0,$$

where n_0 is some nonnegative integer and $\{\sigma_n\}$ is a sequence of nonnegative numbers such that $\sigma_n = o\,(\lambda_n)$, then $\theta_n \to 0$ as $n \to \infty$.

Basically the following lemma is due to [25] when $a_{n+1} \leq (1+\lambda_n)\, a_n + b_n$, $\forall n \geq 1$ is satisfied. However, the following also holds.

Lemma 1.10. *Let $\{a_n\}$, $\{b_n\}$ and $\{\lambda_n\}$ be nonnegative real sequences satisfying*

$$a_{n+1} \leq (1+\lambda_n)\, a_n + b_n, \quad \forall n \geq n_0.$$

If $\sum_{n=0}^{\infty} \lambda_n < \infty$ and $\sum_{n=0}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

2 Main Results

In this section, we shall prove our main theorems in this paper:

Theorem 2.1. *Let E be a real Banach space, K be a nonempty closed convex subset of E, $T_i, S_i : K \to K$, $i = 1, 2, ..., N$ be two finite families of uniformly L-Lipschitzian mappings where $L = \max\{L_1, L_2, ..., L_N\}$. Let $F = \left(\cap_{i=1}^{N} F(T_i)\right) \cap \left(\cap_{i=1}^{N} F(S_i)\right)$, the set of the common fixed points of T_i and S_i, be nonempty. Let x^* be a point in F. Let $\{k_n\} \subset [1, \infty)$ be a sequence with $k_n \to 1$ and $\{\alpha_n\}$,$\{\beta_n\}$ be two sequences in $[0,1]$ satisfying the following conditions:*

(i) $\alpha_n + \beta_n \leq 1, \forall n \geq 1$,

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(iii) $\sum_{n=0}^{\infty} (\alpha_n + \beta_n)^2 < \infty$,

(iv) $\sum_{n=0}^{\infty} (\alpha_n + \beta_n)(k_n - 1) < \infty$,

(v) $L\beta_n < 1, \forall n \geq 1$.

For any $x_0 \in K$, let $\{x_n\}$ be the iterative sequence defined by

$$x_n = (1-\alpha_n-\beta_n)\, x_{n-1} + \alpha_n S_{i(n)}^{k(n)} x_{n-1} + \beta_n T_{i(n)}^{k(n)} x_n, \ \forall n \geq 1.$$

where $T_n = T_{n(modN)}, S_n = S_{n(modN)}, n = (k-1)N + i, i \in I = \{1, 2, ..., N\}$. If there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$\left\langle T_{i(n)}^{k(n)} x - x^*, j(x - x^*) \right\rangle \leq k_n \|x - x^*\|^2 - \phi(\|x - x^*\|)$$

and

$$\left\langle S_{i(n)}^{k(n)} x - x^*, j(x - x^*) \right\rangle \leq k_n \|x - x^*\|^2 - \phi(\|x - x^*\|)$$

for all $j(x - x^*) \in J(x - x^*)$ and $x \in K$, then $\{x_n\}$ converges strongly to x^*.

Proof. The proof is divided into two steps.

(I) First, we prove that the sequence $\{x_n\}$ defined by (1.4) is bounded.

In fact, it follows from (1.4) and Lemma 1.8 that

$$\begin{aligned}
\|x_n - x^*\|^2 &= \left\|(1-\alpha_n-\beta_n)(x_{n-1}-x^*) + \alpha_n\left(S_{i(n)}^{k(n)}x_{n-1}-x^*\right)\right.\\
&\quad \left.+\beta_n\left(T_{i(n)}^{k(n)}x_n - x^*\right)\right\|^2\\
&\le (1-\alpha_n-\beta_n)^2\|x_{n-1}-x^*\|^2 + 2\alpha_n\left\langle S_{i(n)}^{k(n)}x_{n-1}-x^*, j(x_n-x^*)\right\rangle\\
&\quad +2\beta_n\left\langle T_{i(n)}^{k(n)}x_n - x^*, j(x_n-x^*)\right\rangle
\end{aligned}$$

$$\begin{aligned}
&\le (1-\alpha_n-\beta_n)^2\|x_{n-1}-x^*\|^2\\
&\quad +2\alpha_n\left(\left\langle S_{i(n)}^{k(n)}x_{n-1}-S_{i(n)}^{k(n)}x_n,\ j(x_n-x^*)\right\rangle\right.\\
&\quad \left.+\left\langle S_{i(n)}^{k(n)}x_n - x^*, j(x_n-x^*)\right\rangle\right) + 2\beta_n\left\langle T_{i(n)}^{k(n)}x_n-x^*, j(x_n-x^*)\right\rangle\\
&\le (1-\alpha_n-\beta_n)^2\|x_{n-1}-x^*\|^2 + 2\alpha_n\left(L\|x_n-x_{n-1}\|\|x_n-x^*\|\right.\\
&\quad \left.+k_n\|x_n-x^*\|^2 - \phi(\|x_n-x^*\|)\right)\\
&\quad +2\beta_n\left[k_n\|x_n-x^*\|^2-\phi(\|x_n-x^*\|)\right]\\
&= (1-\alpha_n-\beta_n)^2\|x_{n-1}-x^*\|^2 + 2\alpha_n L\|x_n-x_{n-1}\|\|x_n-x^*\|\\
&\quad +(2\alpha_n k_n + 2\beta_n k_n)\|x_n-x^*\|^2 - (2\alpha_n+2\beta_n)\phi(\|x_n-x^*\|) \qquad (2.1)
\end{aligned}$$

From (1.4), we have

$$\begin{aligned}
\|x_n - x_{n-1}\| &\le \alpha_n\left\|S_{i(n)}^{k(n)}x_{n-1}-x_{n-1}\right\| + \beta_n\left\|T_{i(n)}^{k(n)}x_n - x_{n-1}\right\|\\
&\le \alpha_n(L+1)\|x_{n-1}-x^*\| + \beta_n L\|x_n-x^*\|\\
&\quad +\beta_n\|x_{n-1}-x^*\|\\
&= (\alpha_n(L+1)+\beta_n)\|x_{n-1}-x^*\| + \beta_n L\|x_n-x^*\|. \qquad (2.2)
\end{aligned}$$

Substituting (2.2) into (2.1), we obtain

$$\begin{aligned}
\|x_n-x^*\|^2 &\le (1-\alpha_n-\beta_n)^2\|x_{n-1}-x^*\|^2 + 2\alpha_n L\|x_n-x^*\|\\
&\quad \times[(\alpha_n(L+1)+\beta_n)\|x_{n-1}-x^*\| + \beta_n L\|x_n-x^*\|]\\
&\quad +(2\alpha_n k_n+2\beta_n k_n)\|x_n-x^*\|^2 - (2\alpha_n+2\beta_n)\phi(\|x_n-x^*\|)\\
&\le (1-\alpha_n-\beta_n)^2\|x_{n-1}-x^*\|^2 + 2\alpha_n\beta_n L^2\|x_n-x^*\|^2\\
&\quad +\alpha_n L(\alpha_n(L+1)+\beta_n)\left\{\|x_n-x^*\|^2+\|x_{n-1}-x^*\|^2\right\}\\
&\quad +(2\alpha_n k_n+2\beta_n k_n)\|x_n-x^*\|^2 - (2\alpha_n+2\beta_n)\phi(\|x_n-x^*\|)\\
&= \left[(1-\alpha_n-\beta_n)^2+\alpha_n L(\alpha_n(L+1)+\beta_n)\right]\|x_{n-1}-x^*\|^2\\
&\quad +\left(2\alpha_n\beta_n L^2+\alpha_n L(\alpha_n(L+1)+\beta_n)+2\alpha_n k_n+2\beta_n k_n\right)\|x_n-x^*\|^2\\
&\quad -(2\alpha_n+2\beta_n)\phi(\|x_n-x^*\|). \qquad (2.3)
\end{aligned}$$

This implies

$$\begin{aligned}\|x_n - x^*\|^2 &\leq \frac{(1-\alpha_n-\beta_n)^2 + \alpha_n L(\alpha_n(L+1)+\beta_n)}{\delta_n}\|x_{n-1}-x^*\|^2 \\ &\quad -\frac{2\alpha_n+2\beta_n}{\delta_n}\phi(\|x_n-x^*\|) \\ &= \left[1+\frac{\begin{array}{c}2\alpha_n\beta_n L^2 + 2\alpha_n L(\alpha_n(L+1)+\beta_n) + 2\alpha_n k_n \\ +2\beta_n k_n - 2\alpha_n + \alpha_n^2 - 2\beta_n + 2\alpha_n\beta_n + \beta_n^2\end{array}}{\delta_n}\right]\|x_{n-1}-x^*\|^2 \\ &\quad -\frac{2\alpha_n+2\beta_n}{\delta_n}\phi(\|x_n-x^*\|)\end{aligned}$$

where

$$\delta_n = 1 - 2\alpha_n\beta_n L^2 - \alpha_n L(\alpha_n(L+1)+\beta_n) - 2\alpha_n k_n - 2\beta_n k_n.$$

Since $\sum_{n=0}^{\infty}(\alpha_n+\beta_n)^2 < \infty$ implies $(\alpha_n+\beta_n), \alpha_n^2, \alpha_n\beta_n \to 0$ and $k_n \to 1$ as $n \to \infty$, then there exists a positive integer n_0 such that

$$\frac{1}{2} < 1 - \left[\alpha_n\beta_n\left(2L^2+L\right) + \alpha_n^2\left(L^2+L\right) + 2k_n(\alpha_n+\beta_n)\right] \leq 1$$

for all $n \geq n_0$. From (2.3), we have

$$\begin{aligned}\|x_n - x^*\|^2 &\leq (1 + 2[2\alpha_n\beta_n L^2 + 2\alpha_n L(\alpha_n(L+1)+\beta_n) + 2\alpha_n k_n + 2\beta_n k_n \\ &\quad -2\alpha_n + \alpha_n^2 - 2\beta_n + 2\alpha_n\beta_n + \beta_n^2])\|x_{n-1}-x^*\|^2 \\ &\quad -(2\alpha_n+2\beta_n)\phi(\|x_n-x^*\|) \\ &\leq (1 + 2[2\alpha_n\beta_n L^2 + 2\alpha_n L(\alpha_n(L+1)+\beta_n) + 2\alpha_n k_n + 2\beta_n k_n \\ &\quad -2\alpha_n + \alpha_n^2 - 2\beta_n + 2\alpha_n\beta_n + \beta_n^2])\|x_{n-1}-x^*\|^2 \\ &\quad -(\alpha_n+\beta_n)\phi(\|x_n-x^*\|) \qquad (2.4)\end{aligned}$$

for all $n \geq n_0$. Since $\phi(x) \geq 0$ for all $x \geq 0$, then for all $n \geq n_0$, we obtain:

$$\begin{aligned}\|x_n - x^*\|^2 &\leq (1 + 2[2\alpha_n\beta_n(L^2+L) + 2\alpha_n^2(L^2+L) \\ &\quad +2(\alpha_n+\beta_n)(k_n-1) + (\alpha_n+\beta_n)^2])\|x_{n-1}-x^*\|^2.\end{aligned}$$

Since $\sum_{n=0}^{\infty}(\alpha_n+\beta_n)^2 < \infty$ implies $\sum_{n=0}^{\infty}\alpha_n^2 < \infty$ and $\sum_{n=0}^{\infty}\alpha_n\beta_n < \infty$ and by condition (iv), $\sum_{n=0}^{\infty}(\alpha_n+\beta_n)(k_n-1) < \infty$, we have

$$2\sum_{n=0}^{\infty} 2[2\alpha_n\beta_n\left(L^2+L\right) + 2\alpha_n^2\left(L^2+L\right) + 2(\alpha_n+\beta_n)(k_n-1) + (\alpha_n+\beta_n)^2] < \infty.$$

It follows from Lemma 1.10 that $\lim_{n\to\infty} \|x_n - x^*\|$ exists. Therefore, the sequence $\{\|x_n - x^*\|\}$ is bounded. Without loss of generality, we can assume that $\|x_n - x^*\|^2 \le M$, where M is a positive constant.

(II) Now, we consider (2.4) and prove that $x_n \to x^*$. Taking $\theta_n = \|x_{n-1} - x^*\|$, $\lambda_n = \alpha_n + \beta_n$ and

$$\sigma_n = 2\left[2\alpha_n (\alpha_n + \beta_n) \left(L^2 + L\right) + 2 (\alpha_n + \beta_n) (k_n - 1) + (\alpha_n + \beta_n)^2\right] M,$$

we can write (2.3) as

$$\theta_{n+1}^2 \le \theta_n^2 - \lambda_n \phi(\theta_{n+1}) + \sigma_n, \ \forall n \ge n_0.$$

Then $0 \le \lambda_n \le 1$ by condition (i), $\sum_{n=0}^{\infty} \alpha_n = \infty$ implies $\sum_{n=0}^{\infty} \lambda_n = \infty$, so $\lim_{n\to\infty} \frac{\sigma_n}{\lambda_n} = 0$ and all the conditions of Lemma 1.9 are satisfied. Hence

$$\lim_{n\to\infty} \|x_n - x^*\| = 0.$$

□

Remark 2.2. *Theorem* 2.1 *extends and improves the corresponding results of Chang et al. [1–3], Cho et al. [4], Ofoedu [5], Schu [6], Zeng [7, 8], Qin et al. [26] and Gu [27].*

The following theorem deals with one family of mappings and can be obtained from Theorem 2.1 immediately:

Theorem 2.3. *Let E be a real Banach space, K be a nonempty closed convex subset of E, $T_i : K \to K$, $i = 1, 2, ..., N$ be a finite family of uniformly L_i-Lipschitzian mappings where $L = \max\{L_1, L_2, ..., L_N\}$. Let $F = \cap_{i=1}^{N} F(T_i)$, the set of the common fixed points of T_i, be nonempty. Let x^* be a point in F. Let $\{k_n\} \subset [1, \infty)$ be a sequence with $k_n \to 1$ and $\{\alpha_n\}$ be a sequence in $[0, 1]$ satisfying the following conditions:*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$;

(iii) $\sum_{n=0}^{\infty} \alpha_n (k_n - 1) < \infty$;

(iv) $L\alpha_n < 1, \forall n \ge 1$.

For any $x_0 \in K$, let $\{x_n\}$ be the iterative sequence defined by

$$x_n = (1 - \alpha_n) x_{n-1} + \alpha_n T_{i(n)}^{k(n)} x_n, \ \forall n \ge 1.$$

where $T_n = T_{n(modN)}$,$n = (k-1)N + i, i \in I = \{1, 2, ..., N\}$. If there exists a strict increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$\left\langle T_{i(n)}^{k(n)} x - x^*, \ j(x - x^*)\right\rangle \le k_n \|x - x^*\|^2 - \phi(\|x - x^*\|),$$

for all $j(x - x^*) \in J(x - x^*)$ and $x \in K$, then $\{x_n\}$ converges strongly to x^*.

Proof. Taking $\beta_n = 0$ in Theorem 2.1, the conclusion can be obtained immediately. □

Our next result is for two mappings.

Theorem 2.4. *Let E be a real Banach space, K be a nonempty closed convex subset of E, $T, S : K \to K$ be two uniformly L-Lipschitzian mappings. Let $F = F(T) \cap F(S)$, the set of the common fixed points of T and S, be nonempty. Let x^* be a point in F. Let $\{k_n\} \subset [1, \infty)$ be a sequence with $k_n \to 1$ and $\{\alpha_n\}$, $\{\beta_n\}$ be two sequences in $[0, 1]$ satisfying the following conditions:*

(i) $\alpha_n + \beta_n \leq 1, \forall n \geq 1$;

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) $\sum_{n=0}^{\infty} (\alpha_n + \beta_n)^2 < \infty$;

(iv) $\sum_{n=0}^{\infty} (\alpha_n + \beta_n)(k_n - 1) < \infty$;

(v) $L\beta_n < 1, \forall n \geq 1$.

For any $x_0 \in K$, let $\{x_n\}$ be the iterative sequence defined by

$$x_n = (1 - \alpha_n - \beta_n) x_{n-1} + \alpha_n S^n x_{n-1} + \beta_n T^n x_n, \ \forall n \geq 1.$$

If there exists a strict increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$\langle T^n x - x^*, j(x - x^*) \rangle \leq k_n \|x - x^*\|^2 - \phi(\|x - x^*\|),$$

and

$$\langle S^n x - x^*, j(x - x^*) \rangle \leq k_n \|x - x^*\|^2 - \phi(\|x - x^*\|),$$

for all $j(x - x^*) \in J(x - x^*)$ and $x \in K$, then $\{x_n\}$ converges strongly to x^*.

Proof. Take $N = 1$ in Theorem 2.1. □

Finally, we have a result for one mapping case.

Theorem 2.5. *Let E be a real Banach space, K be a nonempty closed convex subset of E, $T : K \to K$ be a uniformly L-Lipschitzian mapping. Let $F = F(T)$, the set of the fixed points of T, be nonempty. Let x^* be a point in F. Let $\{k_n\} \subset [1, \infty)$ be a sequence with $k_n \to 1$ and $\{\alpha_n\}$ be a sequence in $[0, 1]$ satisfying the following conditions:*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\sum_{n=0}^{\infty} \alpha_n (k_n - 1) < \infty$;

(iii) $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$;

(iv) $L\alpha_n < 1, \forall n \geq 1$.

For any $x_0 \in K$, let $\{x_n\}$ be the iterative sequence defined by

$$x_n = (1-\alpha_n)\,x_{n-1} + \alpha_n T^n x_n,\ \forall n \geq 1.$$

If there exists a strict increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that

$$\langle T^n x - x^*, j(x-x^*)\rangle \leq k_n \|x - x^*\|^2 - \phi(\|x-x^*\|),$$

for all $j(x-x^*) \in J(x-x^*)$ and $x \in K$, then $\{x_n\}$ converges strongly to x^*.

Proof. Take $S = T$ in Theorem 2.4. □

Remark 2.6.

(1) *Theorem 2.5 is also a generalization and improvement the Theorem 3.2 of Ofoedu [5].*

(2) *Under suitable conditions, the sequence $\{x_n\}$ defined by (1.4) can also be generalized to the iterative sequences with errors. Thus all the results proved in this paper can also be proved for the iterative process with errors. In this case our main iterative process (1.4) looks like*

$$x_n = (1-\alpha_n-\beta_n-\gamma_n)\,x_{n-1} + \alpha_n S_{i(n)}^{k(n)} x_{n-1} + \beta_n T_{i(n)}^{k(n)} x_n + \gamma_n u_n,\ \forall n \geq 1. \tag{2.5}$$

where $T_n = T_{n(modN)}$, $S_n = S_{n(modN)}$, $n = (k-1)N + i$, $i \in I$, $\{\alpha_n\}$, $\{\beta_n\}$ are two real sequences in $[0,1]$ satisfying $\alpha_n + \beta_n + \gamma_n \leq 1$ for all $n \geq 1$ and $\{u_n\}$ is a bounded sequence.

Remark 2.7. *If we take $\alpha_n = 0$, the iterative process (2.5) reduces to the following process:*

$$x_n = (1-\beta_n-\gamma_n)\,x_{n-1} + \beta_n T_{i(n)}^{k(n)} x_n + \gamma_n u_n,\ \forall n \geq 1,$$

therefore, our main results using (2.5) improve and extend the results for one family of mappings.

Remark 2.8. *Since the iterative process (2.5) is computationally simpler than the iterative process defined by Chang [3] as follows:*

$$\begin{cases} x_{n+1} = (1-\alpha_n-\beta_n)\,x_n + \beta_n T_n^n y_n + \gamma_n u_n, \\ y_n = (1-\gamma_n-\delta_n)\,x_n + \gamma_n T_n^n x_n + \delta_n v_n,\ \forall n \geq 1. \end{cases}$$

See also Chang [1] and Chang et al. [2]. Moreover, our result deals with two finite families of mappings, therefore, our result using (2.5) would be better.

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