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Some Properties of Intuitionistic Fuzzy Ideals of Γ-Rings

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Abstract : In this paper, using the notion of intuitionistic fuzzy ideals of Γ-rings, we study the properties of normal intuitionistic fuzzy ideals, completely normal intuitionistic fuzzy ideals and maximal intuitionistic fuzzy ideals of Γ-rings.

Keywords : Γ-ring; Intuitionistic fuzzy set; Intuitionistic fuzzy ideal; Normal intuitionistic fuzzy ideal; Completely normal intuitionistic fuzzy ideal and maximal intuitionistic fuzzy ideal.
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1 Introduction

The notion of a fuzzy set was introduced by Zadeh [1], and since then this concept has been applied to various algebraic structures. The idea of 'Intuitionistic Fuzzy Set' was first introduced by Atanassov [2] as a generalization of the notion of fuzzy set. Nobusawa [3] introduced the notion of a Γ-ring, as more general than

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a ring. Barnes [4] weakened slightly the conditions in the definition of Γ-rings in the sense of Nobusawa. Barnes [4], Kyuno [5, 6] and Luh [7] studied the structures of Γ-rings and obtained various generalizations analogous to corresponding parts in ring theory.

Jun and Lee [8] applied the concept of fuzzy sets to the theory of Γ-rings. They introduced the notion of fuzzy ideals in Γ-rings and studied some of its properties. Later Kim, Jun and Oztruk [9] introduced the notion of intuitionistic fuzzy ideals in Γ-rings and studied various properties on them. In this paper, using the notion of intuitionistic fuzzy ideals, we investigate the properties of normal, completely normal and maximal intuitionistic fuzzy ideals of Γ-rings.

2 Preliminaries

In this section, the notion of Γ-rings in the sense of Nobusawa and Barnes are discussed with examples. The concept of a fuzzy sets, fuzzy ideals and level sets are studied.

Definition 2.1 ([4]). *Let $(M,+),(\Gamma,+)$ be additive Abelian groups. If there exists a mapping $M\times\Gamma\times M\longrightarrow M$ [the image of (a,α,b) is denoted by $a\alpha b$ for $a,b\in M,\alpha\in\Gamma$] satisfying the following conditions :*

(1) $x\alpha y\in M$,

(2) $(x+y)\alpha z=x\alpha z+y\alpha z, x(\alpha+\beta)y=x\alpha y+x\beta y, x\alpha(y+z)=x\alpha y+x\alpha z$,

(3) $(x\alpha y)\beta z=x(\alpha y\beta)z=x\alpha(y\beta z)$, *for all* $x,y,z\in M$ *and* $\alpha,\beta\in\Gamma$,

then M is called a Γ-ring. If these conditions are strengthened to

(1′) $x\alpha y\in M$, $\alpha x\beta\in\Gamma$,

(2′) $(x+y)\alpha z=x\alpha z+y\alpha z, x(\alpha+\beta)y=x\alpha y+x\beta y, x\alpha(y+z)=x\alpha y+x\alpha z$,

(3′) $(x\alpha y)\beta z=x(\alpha y\beta)z=x\alpha(y\beta z)$,

(4′) $x\alpha y=0$ *for all* $x,y\in M$ *implies* $\alpha=0$,

we then have a Γ-ring in the sense of Nobusawa [3].

As indicated in [3], an example of a Γ-ring is obtained by letting X and Y be Abelian groups, $M=Hom(X,Y)$, $\Gamma=Hom(Y,X)$ and $x\alpha y$ be the usual composite map. (While Nobusawa does not explicitly require that M and Γ be Abelian groups, it appears clear that this is intended). We may note that it follows from $(1)-(3)$ that $0\alpha y=x0y=x\alpha 0=0$ for all $x,y\in M$ and all $\alpha\in\Gamma$.

Example 2.2. *If G and G' are two additive Abelian groups, $M=Hom(G,G'),\Gamma=Hom(G',G)$ then M is a Γ-ring with respect to point-wise addition and composition of mappings.*

Example 2.3. *Let U, V be vector spaces over the same field F, $M = Hom(U, V)$, $\Gamma = Hom(V, U)$ then M is a Γ-ring with respect to point-wise addition and composition of mappings.*

Definition 2.4 ([4])**.** *A subset A of a Γ-ring M is a left (resp. right) ideal of M if A is an additive subgroup of M such that $M\Gamma A \subseteq A$ (resp. $A\Gamma M \subseteq A$), where $M\Gamma A = \{x\alpha y | x \in M, \alpha \in \Gamma, y \in A\}$ and $A\Gamma M = \{y\alpha x | y \in A, \alpha \in \Gamma, x \in M\}$. If A is both a left and a right ideal, than A is a two sided ideal or simply an ideal of M.*

Definition 2.5 ([10])**.** *A fuzzy set A in M is a function $A : M \to [0, 1]$.*

Definition 2.6 ([11])**.** *A fuzzy set μ in a Γ-ring M is called a fuzzy left (resp. right) ideal of M, if it satisfies:*

(i) $\mu(x - y) \geq \{\mu(x) \wedge \mu(y)\}$,

(ii) $\mu(x\alpha y) \geq \mu(y)$ $\big[$*resp.* $\mu(x\alpha y) \geq \mu(x)\big]$,

for all $x, y \in M$ and $\alpha \in \Gamma$. If μ is both a fuzzy left and right ideal of M, then μ is called a fuzzy ideal of M.

Definition 2.7 ([11])**.** *Let μ be a fuzzy set in a Γ-ring M. For any $t \in [0, 1]$, the set $U(\mu, t) = \{x \in M \mid \mu(x) \geq t\}$ is called a level set of μ.*

3 Intuitionistic fuzzy ideals

In this section, we study the notion of intuitionistic fuzzy ideals, normal intuitionistic fuzzy ideals, completely normal intuitionistic fuzzy ideals, intuitionistic fuzzy maximal ideals of Γ-rings with examples and investigate its various properties. Through out this paper, let M denotes a Γ-ring.

Definition 3.1 ([2])**.** *Let X be a non-empty fixed set. An intuitionistic fuzzy set A in X is an object having the form $A = \left\{ \langle x, \mu_A(x), \nu_A(x)\rangle / x \in X \right\}$, where the functions $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ denote the degree of membership and the degree of non-membership of each element $x \in X$ to the set A, respectively, and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for every $x \in X$.*

Notation. For the sake of simplicity, we shall denote the intuitionistic fuzzy set (IFS in short) $A = \left\{ \langle x, \mu_A(x), \nu_A(x)\rangle / x \in X\right\}$ by $A = \langle \mu_A, \nu_A\rangle$.

Definition 3.2 ([2])**.** *Let X be a non-empty set and let $A = \langle \mu_A, \nu_A\rangle$ and $B = \langle \mu_B, \nu_B\rangle$ be $IFSs$ in X. Then*

(1) $A \subset B$ *iff* $\mu_A \leq \mu_B$ *and* $\nu_A \geq \nu_B$.

(2) $A = B$ *iff* $A \subset B$ *and* $B \subset A$.

(3) $A^c = \langle \nu_A, \mu_A\rangle$.

(4) $A \cap B = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$.

(5) $A \cup B = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B)$.

(6) $\Box A = (\mu_A, 1 - \mu_A)$, $\Diamond A = (1 - \nu_A, \nu_A)$.

Definition 3.3 ([9])**.** *An IFS $A = \langle \mu_A, \nu_A \rangle$ in M is called an intuitionistic fuzzy left (resp. right) ideal of a Γ-ring M if*

(i) $\mu_A(x - y) \geq \{\mu_A(x) \wedge \mu_A(y)\}$ *and* $\mu_A(x\alpha y) \geq \mu_A(y)$ *(resp.* $\mu_A(x\alpha y) \geq \mu_A(x)$*),*

(ii) $\nu_A(x-y) \leq \{\nu_A(x) \vee \nu_A(y)\}$ *and* $\nu_A(x\alpha y) \leq \nu_A(y)$ *(resp.* $\nu_A(x\alpha y) \leq \nu_A(x)$*),*

for all $x, y \in M$ and $\alpha \in \Gamma$.

Example 3.4. [Intuitionistic fuzzy ideal] *Let R be the set of all integers. Then R is a ring. Take $M = \Gamma = R$. Let $a, b \in M$, $\alpha \in \Gamma$. Suppose $a\alpha b$ is the product of $a, \alpha, b \in M$. Then M is a Γ-ring. Define an IFS $A = \langle \mu_A, \nu_A \rangle$ in M as follows.*
$\mu_A(0) = 1$ *and* $\mu_A(\pm 1) = \mu_A(\pm 2) = \cdots = t$ *and*
$\nu_A(0) = 0$ *and* $\nu_A(\pm 1) = \nu_A(\pm 2) = \cdots = s$,
where $t \in [0, 1]$, $s \in [0, 1]$ and $t + s \leq 1$. By routine calculations, clearly A is an intuitionistic fuzzy ideal of a Γ-ring M.

Like in fuzzy ideal, if $A = \langle \mu_A, \nu_A \rangle$ is an intuitionistic fuzzy left (resp. right) ideal of M then $\mu_A(0) \geq \mu_A(x)$ and $\nu_A(0) \leq \nu_A(x)$ for all $x \in M$, where 0 is the zero element of M. Also we note that $\mu_A(0)$ and $\nu_A(0)$ are the largest value and the smallest value of μ_A and ν_A respectively. It is often convenient to have $\mu_A(0) = 1$ and $\nu_A(0) = 0$.

Theorem 3.5. *Let $A = \langle \mu_A, \nu_A \rangle$ be an intuitionistic fuzzy left (resp. right) ideal of M. Then the set $M_A = \{x \in M | \mu_A(x) = \mu_A(0)$ and $\nu_A(x) = \nu_A(0)\}$ is an intuitionistic fuzzy left (resp. right) ideal of M.*

Proof. Let A be an intuitionistic fuzzy left (resp. right) ideal of M and let $x, y \in M_A$ and $\alpha \in \Gamma$. Then $\mu_A(x - y) \geq \{\mu_A(x) \wedge \mu_A(y)\} = \mu_A(0)$ and $\nu_A(x - y) \leq \{\nu_A(x) \vee \nu_A(y)\} = \nu_A(0)$, and so $\mu_A(x-y) = \mu_A(0)$ and $\nu_A(x-y) = \nu_A(0)$ or $x-y \in M_A$. Also $\mu_A(x\alpha y) \geq \mu_A(y) = \mu_A(0)$ [resp. $\mu_A(x\alpha y) \geq \mu_A(x) = \mu_A(0)$] and $\nu_A(x\alpha y) \leq \nu_A(y) = \nu_A(0)$ [resp. $\nu_A(x\alpha y) \leq \nu_A(x) = \nu_A(0)$]. Hence $\mu_A(x\alpha y) = \mu_A(0)$ and $\nu_A(x\alpha y) = \nu_A(0)$ or xαy$\in M_A$. Therefore M_A is an intuitionistic fuzzy left (resp. right) ideal of M. □

Definition 3.6. *An intuitionistic fuzzy ideal A of M is said to be normal if $\mu_A(0) = 1$ and $\nu_A(0) = 0$. Denote by $NIFLI(M)$ the set of all normal intuitionistic fuzzy left ideals of M. Note that $NIFLI(M)$ is a poset under set inclusion.*

Example 3.7. [Normal intuitionistic fuzzy ideal] *Let R be the set of all integers. Then R is a ring. Take $M = \Gamma = R$. Let $a, b \in M$, $\alpha \in \Gamma$. Suppose $a\alpha b$*

is the product of $a, \alpha, b \in M$. *Then* M *is a* Γ*-ring. Define* $\mu_A : M \to [0,1]$ *and* $\nu_A : M \to [0,1]$ *by*

$$\mu_A(x) = \begin{cases} 1, & x = 0 \\ \alpha, & Otherwise \end{cases} \tag{3.1}$$

$$\nu_A(x) = \begin{cases} 0, & x = 0 \\ \beta, & Otherwise \end{cases} \tag{3.2}$$

where $\alpha \in [0,1)$, $\beta \in (0,1]$ *and* $\alpha + \beta \leq 1$. *By routine calculations, clearly* A *is an IF ideal of a* Γ*-ring* M. *Also* $\mu_A(0) = 1$ *and* $\nu_A(0) = 0$ *then* A *is a normal IF ideal of a* Γ*-ring* M.

Lemma 3.8. *For an intuitionistic fuzzy ideal* A *of* M, *if we define an intuitionistic fuzzy set by*

$$\mu_A(x) := \begin{cases} 1, & if x \in A \\ 0, & Otherwise \end{cases} \tag{3.3}$$

$$\nu_A(x) := \begin{cases} 0, & if x \in A \\ 1, & Otherwise \end{cases} \tag{3.4}$$

for all $x \in M$, *then* M_A *is a normal intuitionistic fuzzy ideal of* M *and* $M_{M_A} = A$.

Proof. Straight forward. □

Theorem 3.9. *Let* $A = \langle \mu_A, \nu_A \rangle$ *be an intuitionistic fuzzy left (resp. right) ideal of* M *and let* ${\mu_A}^+(x) = \mu_A(x) + 1 - \mu_A(0), {\nu_A}^+(x) = \nu_A(x) - \nu_A(0)$. *If* ${\mu_A}^+(x) + {\nu_A}^+(x) \leq 1$ *for all* $x \in M$, *then* $A^+ = \langle {\mu_A}^+, {\nu_A}^+ \rangle$ *is a normal intuitionistic fuzzy left (resp. right) ideal of* M *containing* A.

Proof. We first observe that ${\mu_A}^+(0) = 1, {\nu_A}^+(0) = 0$, where ${\mu_A}^+(x), {\nu_A}^+(x) \in [0,1]$ for every $x \in M$. So, A^+ is a normal intuitionistic fuzzy set. To prove that it is an intuitionistic fuzzy left (resp. right) ideal, let $x, y \in M$ and $\alpha \in \Gamma$. Then

$$\begin{aligned} {\mu_A}^+(x-y) &= \mu_A(x-y) + 1 - \mu_A(0) \geq \{\mu_A(x) \wedge \mu_A(y)\} + 1 - \mu_A(0) \\ &= \{\mu_A(x) + 1 - \mu_A(0)\} \wedge \{\mu_A(y) + 1 - \mu_A(0)\} = {\mu_A}^+(x) \wedge {\mu_A}^+(y). \end{aligned}$$

$$\begin{aligned} {\nu_A}^+(x-y) &= \nu_A(x-y) - \nu_A(0) \leq \{\nu_A(x) \vee \nu_A(y)\} - \nu_A(0) \\ &= \{\nu_A(x) - \nu_A(0)\} \vee \{\nu_A(y) - \nu_A(0)\} = {\nu_A}^+(x) \vee {\nu_A}^+(y). \end{aligned}$$

$$\begin{aligned} {\mu_A}^+(x\alpha y) &= \mu_A(x\alpha y) + 1 - \mu_A(0) \geq \mu_A(y) + 1 - \mu_A(0) = {\mu_A}^+(y) \\ &[resp. {\mu_A}^+(x\alpha y) \geq {\mu_A}^+(x)] \text{ and} \end{aligned}$$

$$\nu_A{}^+(x\alpha y) = \nu_A(x\alpha y) - \nu_A(0) \leq \nu_A(y) - \nu_A(0) = \nu_A{}^+(y).$$
$$[resp.\, \nu_A{}^+(x\alpha y) \leq \nu_A{}^+(x)]$$

This shows that A^+ is an intuitionistic fuzzy left (resp. right) ideal of M. So, A^+ is a normal intuitionistic fuzzy left (resp. right) ideal of M. □

Corollary 3.10. *If A is normal, then $A^+ = A$ where $A^+ = \langle \mu_A{}^+, \nu_A{}^+ \rangle$, $\mu_A{}^+(x) = \mu_A(x) + 1 - \mu_A(0)$ and $\nu_A{}^+(x) = \nu_A(x) - \nu_A(0)$.*

Lemma 3.11. *If A is an intuitionistic fuzzy ideal of M satisfying $A^+(x) = (0, 1)$ for some $x \in M$, then $\mu_A(x) = 0$ and $\nu_A(x) = 1$.*

Proof. Let A be an intuitionistic fuzzy ideal of M such that $A^+(x)$=(0,1). Then A^+ is normal intuitionistic fuzzy ideal of M containing A. Hence A is normal and $A = A^+$. Therefore $A(x) = (0, 1) \Rightarrow \mu_A(x) = 0$ and $\nu_A(x) = 1$. This completes the proof. □

Using an intuitionistic fuzzy ideal $A = \langle\, \mu_A, \nu_A \,\rangle$ of M, we will construct a new intuitionistic fuzzy ideal. Let $t > 0$ be a real number, and define a mapping $\mu_A{}^t : M \to [0, 1]$ by $\mu_A{}^t(x) = (\mu_A(x))^t$ and $\nu_A{}^t : M \to [0, 1]$ by $\nu_A{}^t(x) = (\nu_A(x))^t$ for all $x \in M$, where $(\mu_A(x))^t = \sqrt[2t]{\mu_A(x)}$ and $(\nu_A(x))^t = \sqrt[2t]{\nu_A(x)}$ when $0 < t < 1$.

Theorem 3.12. *Let $t > 0$ be a real number. If A is a normal intuitionistic fuzzy ideal of M then A^t is also a normal intuitionistic fuzzy ideal of M and $M_A{}^t = M_A$.*

Proof. For any $x, y \in M$ and $\alpha \in \Gamma$, we have

$$\begin{aligned}\mu_A{}^t(x - y) &= (\mu_A(x - y))^t \geq (\mu_A(x) \wedge \mu_A(y))^t \\ &= (\mu_A(x))^t \wedge (\mu_A(y))^t = \mu_A{}^t(x) \wedge \mu_A{}^t(y).\end{aligned}$$

$$\begin{aligned}\nu_A{}^t(x - y) &= (\nu_A(x - y))^t \leq (\nu_A(x) \vee \nu_A(y))^t \\ &= (\nu_A(x))^t \vee (\nu_A(y))^t = \nu_A{}^t(x) \vee \nu_A{}^t(y).\end{aligned}$$

$$\mu_A{}^t(x\alpha y) = (\mu_A(x\alpha y))^t \geq (\mu_A(y))^t = \mu_A{}^t(y)[resp.\, (\mu_A(x\alpha y))^t \geq \mu_A{}^t(x)] \text{ and}$$

$$\nu_A{}^t(x\alpha y) = (\nu_A(x\alpha y))^t \leq (\nu_A(y))^t = \nu_A{}^t(y)[resp.\, (\nu_A(x\alpha y))^t \leq \nu_A{}^t(x).$$

Note that $\mu_A{}^t(0) = (\mu_A(0))^t = 1^t = 1$ and $\nu_A{}^t(0) = (\nu_A(0))^t = 0^t = 0$. Hence A^t is a normal intuitionistic fuzzy ideal of M. Now

$$\begin{aligned}M_A{}^t &= \{x \in M |\ \mu_A{}^t(x) = \mu_A{}^t(0) \text{ and } \nu_A{}^t(x) = \nu_A{}^t(0)\} \\ &= \{x \in M |\ (\mu_A(x))^t = 1 \text{ and } (\nu_A(x))^t = 0\} \\ &= \{x \in M |\ \mu_A(x) = \mu_A(0) \text{ and } \nu_A(x) = \nu_A(0)\} \\ &= M_A.\end{aligned}$$

This completes the proof. □

Definition 3.13. *A normal intuitionistic fuzzy left ideal A of M is called completely normal if there exists $x_0 \in M$ such that $A(x_0) = (0,1)$. Denote by $CNIFLI(M)$ the set of all completely normal intuitionistic fuzzy left ideals of M. Clearly $CNIFLI(M) \subseteq NIFLI(M)$.*

Example 3.14. [Completely normal IF ideal] *Let R be the set of all integers. Then R is a ring. Take $M = \Gamma = R$. Let $a, b \in M$, $\alpha \in \Gamma$. Suppose $a\alpha b$ is the product of $a, \alpha, b \in M$. Then M is a Γ-ring. Define $\mu_A : M \to [0,1]$ and $\nu_A : M \to [0,1]$ by*

$$\mu_A(x) = \begin{cases} 1, & x = 0 \\ 0, & Otherwise \end{cases} \tag{3.5}$$

$$\nu_A(x) = \begin{cases} 0, & x = 0 \\ 1, & Otherwise \end{cases} \tag{3.6}$$

Clearly $\mu_A(x) \in [0,1]$ and $\nu_A(x) \in [0,1]$, for all $x \in M$ and $\mu_A(x) + \nu_A(x) \leq 1$. By routine calculations, clearly A is an IF ideal of M. Also $\mu_A(x = 0) = 1$ and $\nu_A(x = 0) = 0$ then A is a normal IF ideal of M. Again for all $x(x \neq 0)$, $\mu_A(x) = 0$ and $\nu_A(x) = 1$ then A is a completely normal IF ideal of M.

Definition 3.15. *An intuitionistic fuzzy ideal A of M is said to be intuitionistic fuzzy maximal if it satisfies:*

(i) *A is non-constant.*

(ii) *A^+ is a maximal element of $NIFI(M)$, where $NIFI(M)$ denote the set of all normal intuitionistic fuzzy ideal of M.*

Example 3.16. [Maximal intuitionistic fuzzy ideal] *Let $R = \{0, 1, 2, 3, 4, 5\}$ be a non-empty set with two binary operations addition modulo 6 and multiplication modulo 6 defined below.*

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

.	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	3	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

Clearly $(R, +, \cdot)$ is a Ring. Take $M = \Gamma = R$. Let $a, b \in M$ and $\alpha \in \Gamma$.
Suppose $a\alpha b$ is the product of a, α and $b \in M$ [Take α = 2] defined as below.

$\alpha = 2$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	2	4	0	2	4
2	0	4	2	0	4	2
3	0	0	0	0	0	0
4	0	2	4	0	2	4
5	0	4	2	0	4	2

Since $\alpha = 2$ is arbitrary, by routine calculations M is a Γ-ring. Let $A = \{0, 2, 4\}$ be a subset of M. Then $\mu_A(x)$ and $\nu_A(x)$ of A is defined as follows.

$$\mu_A(x) = \begin{cases} 1, & x = \{0, 2, 4\} \\ 0, & Otherwise \end{cases} \tag{3.7}$$

$$\nu_A(x) = \begin{cases} 0, & x = \{0, 2, 4\} \\ 1, & Otherwise \end{cases} \tag{3.8}$$

Clearly $\mu_A(x) \in [0, 1], \nu_A(x) \in [0, 1]$ and $\mu_A(x) + \nu_A(x) \leq 1$. By routine calculations, clearly A is a normal intuitionistic fuzzy ideal of M. Hence $A = \{0, 2, 4\}$ is a maximal intuitionistic fuzzy ideal of M, since there is no ideal bigger than A in M.

Theorem 3.17. *Let A be a non-constant normal intuitionistic fuzzy left ideal of M, which is maximal in the poset of normal intuitionistic fuzzy left ideals under set inclusion. Then both μ_A and ν_A takes only the values $(0, 1)$ and $(1, 0)$ respectively.*

Proof. Let $A = \langle \mu_A, \nu_A \rangle \in NIFLI(M)$ be a non-constant maximal element of $(NIFLI(M), \subseteq)$. Then $\mu_A(0) = 1$ and $\nu_A(0) = 0$. Let $x \in M$ be such that $\mu_A(x) \neq 1$. We claim that $\mu_A(x) = 0$. If not, then there exists $c \in M$ such that $0 < \mu_A(c) < 1$. Let $A_c = \langle \lambda_A, \rho_A \rangle$ be an IFS in M defined by $\lambda_A(x) = \frac{1}{2}\{\mu_A(x) + \mu_A(c)\}$ and $\rho_A(x) = \frac{1}{2}\{\nu_A(x) + \nu_A(c)\}$, for all $x \in M$. Then clearly the IFS A_c is well defined and $\lambda_A(0) = \frac{1}{2}\{\mu_A(0) + \mu_A(c)\} \geq \frac{1}{2}\{\mu_A(x) + \mu_A(c)\} = \lambda_A(x)$, $\rho_A(0) = \frac{1}{2}\{\nu_A(0) + \nu_A(c)\} \leq \frac{1}{2}\{\nu_A(x) + \nu_A(c)\} = \rho_A(x)$, for all $x \in M$.

Also let $x, y \in M$, we have

$$\begin{aligned} \lambda_A(x - y) = \tfrac{1}{2}\{\mu_A(x - y) + \mu_A(c)\} &\geq \tfrac{1}{2}[\{\mu_A(x) \wedge \mu_A(y)\} + \mu_A(c)] \\ &= \tfrac{1}{2}\{\mu_A(x) + \mu_A(c)\} \wedge \tfrac{1}{2}\{\mu_A(y) + \mu_A(c)\} \\ &= \lambda_A(x) \wedge \lambda_A(y), \end{aligned}$$

$$\begin{aligned} \rho_A(x - y) = \tfrac{1}{2}\{\nu_A(x - y) + \nu_A(c)\} &\leq \tfrac{1}{2}[\{\nu_A(x) \vee \nu_A(y)\} + \nu_A(c)] \\ &= \tfrac{1}{2}\{\nu_A(x) + \nu_A(c)\} \vee \tfrac{1}{2}\{\nu_A(y) + \nu_A(c)\} \\ &= \rho_A(x) \vee \rho_A(y). \end{aligned}$$

Also

$$\lambda_A(x\alpha y) = \tfrac{1}{2}\{\mu_A(x\alpha y) + \mu_A(c)\} \geq \tfrac{1}{2}\{\mu_A(y) + \mu_A(c)\} = \lambda_A(y),$$
$$\rho_A(x\alpha y) = \tfrac{1}{2}\{\nu_A(x\alpha y) + \nu_A(c)\} \leq \tfrac{1}{2}\{\nu_A(y) + \nu_A(c)\} = \rho_A(y).$$

Therefore A_c is an intuitionistic fuzzy left ideal of M. By Theorem 3.9, ${A_c}^+ = \langle {\lambda_A}^+, {\rho_A}^+ \rangle$ where ${\lambda_A}^+ = \lambda_A(x) + 1 - \lambda_A(0) = \frac{1}{2}\{1 + \mu_A(x)\}$ and ${\rho_A}^+(x) = \rho_A(x) - \rho_A(0) = \frac{1}{2}\nu_A(x)$ belongs to $NIFLI(M)$. Clearly $A \subseteq {A_c}^+$.

Since ${\lambda_A}^+(x) = \frac{1}{2}\{1 + \mu_A(x)\} > \mu_A(x)$, A is a proper subset of ${A_c}^+$. Obviously ${\lambda_A}^+(x) < 1 = {\lambda_A}^+(0)$. Hence ${A_c}^+$ is non-constant and A is not a maximal element of $NIFLI(M)$. This is a contradiction. Therefore μ_A takes only two values: 0 and 1.

Analogously we can prove that ν_A also takes the values 0 and 1. This means that for A, the possible values are $(0,0)$, $(0,1)$ and $(1,0)$. If A takes these three values, then

$$M_{\langle 0,0\rangle} = \{x \in M \mid \mu_A(x) \geq 0, \nu_A(x) \leq 0\} = \{x \in M \mid \nu_A(x) = 0\}$$
$$M_{\langle 1,0\rangle} = \{x \in M \mid \mu_A(x) \geq 1, \nu_A(x) \leq 0\} = \{x \in M \mid \mu_A(x) = 1, \nu_A(x) = 0\}$$
$$M_{\langle 0,1\rangle} = \{x \in M \mid \mu_A(x) \geq 0, \nu_A(x) \leq 1\} = M$$

are nonempty left ideals of M such that $M_{\langle 1,0\rangle} \subset M_{\langle 0,0\rangle} \subset M_{\langle 0,1\rangle} = M$. An IFS $B = \langle \mu_B, \nu_B \rangle$ defined by

$$\mu_B(x) = \begin{cases} 1 & if \quad x \in M_{\langle 0,0\rangle} \\ 0 & if \quad x \notin M_{\langle 0,0\rangle} \end{cases} \quad \text{and} \quad \nu_B(x) = \begin{cases} 0 & if \quad x \in M_{\langle 0,0\rangle} \\ 1 & if \quad x \notin M_{\langle 0,0\rangle} \end{cases}$$

is an intuitionistic fuzzy left ideal of M. It is normal. More over, $\nu_A(x) \neq 0$ for $x \in M \backslash M_{\langle 0,0\rangle}$. Thus $\nu_A(x) = 1$, consequently $\mu_A(x) = 0$. This implies $A(x) = B(x)$ for $x \in M \backslash M_{\langle 0,0\rangle}$. For $x \in M_{\langle 0,0\rangle}$, we have $\nu_A(x) = 0 = \nu_B(x)$ and $\mu_A(x) = 1 = \mu_B(x)$. Hence $A \subset B$. Since $\mu_A(x) = 0 < \mu_B(x)$ for $x \in M_{\langle 0,0\rangle} \backslash M_{\langle 1,0\rangle}$, an intuitionistic fuzzy left ideal A is a proper subset of B. This is a contradiction. So, a non-constant maximal element of $(NIFLI(M), \subseteq)$ takes only two values $(0,1)$ and $(1,0)$. This completes the proof. □

Let $I(M)$ (resp. $NIFI(M)$) denote the set of all ideals (resp. normal intuitionistic fuzzy ideals) of M. We define functions $\phi : I(M) \to NIFI(M)$ and $\chi : NIFI(M) \to I(M)$ by $\phi(A) = A^+$ (i.e. $\phi(\mu_A) = {\mu_A}^+$ and $\phi(\nu_A) = {\nu_A}^+$) and $\chi(A^+) = M_A$ (i.e. $\chi({\mu_A}^+) = M_{\mu_A}$ and $\chi({\nu_A}^+) = M_{\nu_A}$), respectively, for all $A \in I(M)$ and $A^+ \in NIFI(M)$. Then $\chi\phi = (0,1)_{I(M)}$ and $\phi\chi(A^+) = \phi(M_A) = {M_A}^+$.

Lemma 3.18. *Let A and B be intuitionistic fuzzy ideals of M. Then $A \subseteq B$ if and only if $\mu_A \subseteq \mu_B$ and $\nu_A \supseteq \nu_B$.*

Proof. Straight forward. □

Theorem 3.19. *If A is an intuitionistic fuzzy maximal ideal of M, then*

(i) A is normal.

(ii) A^+ takes only the values $(0,1)$ and $(1,0)$.

(iii) $M_A = A$.

(iv) M_A is a maximal ideal of M.

Proof. Let A be an intuitionistic fuzzy maximal ideal of M. Then A is non-constant and A^+ is a maximal element of the poset $(NIFI(M), \subseteq)$. It follows from Theorem 3.17, both ${\mu_A}^+$ and ${\nu_A}^+$ takes only the values $(0,1)$ and $(1,0)$ respectively. Note that ${\mu_A}^+(x) = 0$ if and only if $\mu_A(x) = \mu_A(0) - 1$ and ${\nu_A}^+(x) =$

1 if and only if $\nu_A(x) = \nu_A(0) + 1$. By Lemma 3.11, $\mu_A(x) = 0$,that is, $\mu_A(0) = 0$ and $\nu_A(x) = 1$, that is, $\nu_A(0) = 1$. Hence A is normal. This proves (i) and (ii).

(iii) Clearly $M_A \subseteq A$ and M_A takes only the values 0 and 1. Let $x \in M$. If $\mu_A(x) = 0$ and $\nu_A(x) = 1$ then obviously $A \subseteq M_A$. If $\mu_A(x) = 1$ and $\nu_A(x) = 0$ then $x \in A$ and so $M_A(x) = (1, 0)$. This shows that $A \subseteq M_A$.

(iv) M_A is a proper ideal of M because A is non-constant. Let A be an intuitionistic fuzzy ideal of M such that $M_A \subseteq A$. Using (iii) and Lemma 3.18, we have $A = A_{M_A} \subseteq M_A$. Since $A, M_A \in NIFI(M)$ and $A = A^+$ is a maximal element of $NIFI(M)$, it follows that either $A = M_A$ or $M_A = (1, 0)$, where $(1, 0) : M \to [0, 1]$ is an intuitionistic fuzzy set defined by $\mu_{1<x>} = 1$ and $\nu_{1<x>} = 0$ for all $x \in M$. The later case implies that $A = M_A$. If $A = M_A$ then $M_A = M_{M_A} = A$ by Lemma 3.8. This proves that M_A is a maximal ideal of M. □

Theorem 3.20. *A maximal $A \in IFLI(M)$ is normal and takes only two values $(0, 1)$ and $(1, 0)$.*

Proof. Let $A \in IFLI(M)$ be maximal. Then A^+ is a non-constant maximal element of $(NIFLI, \subseteq)$ and by Theorem 3.17, the possible values of A^+ are $(0, 1)$ and $(1, 0)$. Then ${\mu_A}^+$ and ${\nu_A}^+$ takes only two values 0 and 1. Clearly ${\mu_A}^+(x) = 1$ if and only if $\mu_A(x) = \mu_A(0)$ and ${\mu_A}^+(x) = 0$ if and only if $\mu_A(x) = \mu_A(0) - 1$. Similarly, ${\nu_A}^+(x) = 1$ if and only if $\nu_A(x) = 1$ and ${\nu_A}^+(x) = 0$ if and only if $\nu_A(x) = 0$. But $A \subseteq A^+$ (Theorem 3.9), so $\mu_A(x) \leq {\mu_A}^+(x)$ and $\nu_A(x) \geq {\nu_A}^+(x)$ for all $x \in M$. Thus ${\mu_A}^+(x) = 0$ implies $\mu_A(x) = 0$ and ${\nu_A}^+(x) = 1$ implies $\nu_A(x) = 1$, hence $\mu_A(0) = 1$ and $\nu_A(0) = 0$. This proves that A is normal. □

Theorem 3.21. *A $(1, 0)$–level subset of a maximal intuitionistic fuzzy left ideal of M is a maximal left ideal of M.*

Proof. Let S be a $(1, 0)$–level subset of a maximal intuitionistic fuzzy left ideal A of M. That is, $S = M_{\langle 1,0\rangle} = \{x \in M \mid \mu_A(x) = 1\}$. It is not difficult to verify that S is a left ideal of M. $S \neq M$ because μ_A takes two values. Let I be a left ideal of M containing S. Then $\mu_S \subseteq \mu_I$. Since $\mu_A = \mu_S$ and μ_A takes only two values, μ_I also takes these two values. But, by assumption, A is a maximal intuitionistic fuzzy left ideal of M, so $\mu_S = \mu_A = \mu_I$ or $\mu_I(x) = 1$ for all x ∈ M. In the last case $S = M$, which is not possible. So, $\mu_S = \mu_A = \mu_I$, which implies $S = I$. This means that S is a maximal left ideal of M. □

Theorem 3.22. *A non-constant maximal element of $(NIFLI(M), \subseteq)$ is also a maximal element of $(CNIFLI(M), \subseteq)$.*

Proof. Let A be a non-constant maximal element of $(NIFLI(M), \subseteq)$. By Theorem 3.17, A takes only the values $(0, 1)$ and $(1, 0)$, so $A(0) = (1, 0)$ and $A(x_0) = (0, 1)$ for some $x_0 \in M$. Hence $A \in CNIFLI(M)$. Assume that there exists $B \in CNIFLI(M)$ such that $A \subseteq B$. It follows that $A \subseteq B$ in $NIFLI(M)$. Since A is maximal in $(NIFLI(M), \subseteq)$ and since B is non-constant, therefore $A = B$. Thus A is maximal element of $(CNIFILI(M), \subseteq)$. This completes the proof. □

Theorem 3.23. *Every maximal intuitionistic fuzzy left ideal of M is completely normal.*

Proof. Let an intuitionistic fuzzy left ideal A of M be maximal. Then by Theorem 3.14, it is normal and $A = A^+$ takes only two values $(0,1)$ and $(1,0)$. Since A is non-constant, it follows that $A(0) = (1,0)$ and $A(x_0) = (0,1)$ for some $x_0 \in M$. Hence A is completely normal. This completes the proof. $\square$

Theorem 3.24. [Converse of Theorem 3.23] *A completely normal intuitionistic fuzzy left ideal of M is a maximal intuitionistic fuzzy left ideal of M.*

Proof. Let A be a completely normal intuitionistic fuzzy left ideal of M. Then A is normal and takes only the values $(0,1)$ and $(1,0)$ so that $A(0) = (1,0)$ and $A(x_0) = (0,1)$, for some $x_0 \in M$. So $A \in (NIFLI(M), \subseteq)$. Assume that $B \in NIFLI(M)$ such that $A \subseteq B$. It follows that $A \subseteq B$ in $(CNIFLI(M), \subseteq)$. Since A is maximal in (CNIFLIM, $\subseteq$) and B is non-constant, therefore $A = B$. Thus A is a maximal element of $(NIFLI(M), \subseteq)$. This completes the proof. $\square$

Theorem 3.25. *Let $f : [0,1] \to [0,1]$ be an increasing function and let $A = \langle \mu_A, \nu_A \rangle$ be an IFS of a Γ-ring M. Then $f^{-1}(A) = \big\langle \mu_{f^{-1}}(A), \nu_{f^{-1}}(A) \big\rangle$, where $\mu_{f^{-1}}(A)(x) = f(\mu_A(x))$ and $\nu_{f^{-1}}(A)(x) = f\big(\nu_A(x)\big)$ is an intuitionistic fuzzy left ideal if and only if $A = \langle \mu_A, \nu_A \rangle$ is an intuitionistic fuzzy left ideal. Moreover if $f(\mu_A(0)) = 1$ and $f(\nu_A(0)) = 0$, then $f^{-1}(A)$ is normal.*

Proof. $(\Rightarrow)$ Let $f^{-1}(A) = \big\langle \mu_{f^{-1}}(A), \nu_{f^{-1}}(A) \big\rangle$ be an intuitionistic fuzzy left ideal of M. For all $x, y \in M$,

$$\begin{aligned} f\big(\mu_A(x-y)\big) = \mu_{f^{-1}}(A)(x-y) &\geq \mu_{f^{-1}}(A)(x) \wedge \mu_{f^{-1}}(A)(y) \\ &= f\big(\mu_A(x)\big) \wedge f\big(\mu_A(y)\big) \\ &= f\big(\mu_A(x) \wedge \mu_A(y)\big). \end{aligned}$$

Since f is increasing, $\mu_A(x-y) \geq \mu_A(x) \wedge \mu_A(y)$. Similarly

$$\begin{aligned} f\big(\nu_A(x-y)\big) = \nu_{f^{-1}}(A)(x-y) &\leq \nu_{f^{-1}}(A)(x) \vee \nu_{f^{-1}}(A)(y) \\ &= f\big(\nu_A(x)\big) \vee f\big(\nu_A(y)\big) \\ &= f\big(\nu_A(x) \vee \nu_A(y)\big). \end{aligned}$$

Since f is increasing, $\nu_A(x-y) \leq \nu_A(x) \vee \nu_A(y)$.

$$f\big(\mu_A(x\alpha y)\big) = \mu_{f^{-1}}(A)(x\alpha y) \geq \mu_{f^{-1}}(A)(y) = f\big(\mu_A(y)\big)$$

and

$$f\big(\nu_A(x\alpha y)\big) = \nu_{f^{-1}}(A)(x\alpha y) \leq \nu_{f^{-1}}(A)(y) = f\big(\nu_A(y)\big).$$

Since f is increasing, $\mu_A(x\alpha y) \geq \mu_A(y)$ and $\nu_A(x\alpha y) \leq \nu_A(y)$.

(⇐) Let $A = \langle \mu_A, \nu_A \rangle$ be an intuitionistic fuzzy left ideal of M. Then for all $x, y \in M$, we have

$$\begin{aligned}\mu_{f^{-1}}(A)(x-y) = f\big(\mu_A(x-y)\big) &\geq f\big(\mu_A(x) \wedge \mu_A(y)\big)\\ &= f\big(\mu_A(x)\big) \wedge f\big(\mu_A(y)\big)\\ &= \mu_{f^{-1}}(A)(x) \wedge \mu_{f^{-1}}(A)(y),\end{aligned}$$

$$\begin{aligned}\nu_{f^{-1}}(A)(x-y) = f\big(\nu_A(x-y)\big) &\leq f\big(\nu_A(x) \vee \nu_A(y)\big)\\ &= f\big(\nu_A(x)\big) \vee f\big(\nu_A(y)\big)\\ &= \nu_{f^{-1}}(A)(x) \vee \nu_{f^{-1}}(A)(y).\end{aligned}$$

Also

$$\mu_{f^{-1}}(A)(x\alpha y) = f\big(\mu_A(x\alpha y)\big) \geq f\big(\mu_A(y)\big) = \mu_{f^{-1}}(A)(y)$$

and

$$\nu_{f^{-1}}(A)(x\alpha y) = f\big(\nu_A(x\alpha y)\big) \leq f\big(\nu_A(y)\big) = \nu_{f^{-1}}(A)(y).$$

This proves the converse part. □

Definition 3.26. *Let $A = \langle \mu_A, \nu_A \rangle$ and $B = \langle \mu_B, \nu_B \rangle$ be two intuitionistic fuzzy subsets of a Γ-ring M. Then the product $A\Gamma B$ is defined by*

$$\mu_{A\Gamma B}(x) = \begin{cases} \bigvee\limits_{x=y\alpha z} \mu_A(y) \wedge \mu_A(z), & x = y\alpha z,\\ 0, & Otherwise\end{cases}$$

$$\nu_{A\Gamma B}(x) = \begin{cases} \bigwedge\limits_{x=y\alpha z} \nu_A(y) \vee \nu_A(z), & x = y\alpha z,\\ 1, & Otherwise.\end{cases}$$

Theorem 3.27. *If A is a left (resp. right) ideal of a Γ-ring M, then the following are equivalent:*

(1) A is an intuitionistic fuzzy left (resp. right) ideal of M.

(2) $M\Gamma A \subseteq A$ and $\mu_A(x-y) \geq \{\mu_A(x) \wedge \mu_A(y)\}, \nu_A(x-y) \leq \{\nu_A(x) \vee \nu_A(y)\}$.

Proof. (⇒) Suppose that (1) holds. Let a be any element of M. In the case when there exists $x, y \in M$ such that $a = x\alpha y$. Since A is the intuitionistic fuzzy ideal of M, we have

$$\begin{aligned}\mu_{M\Gamma A}(a) = \bigvee_{a=x\alpha y} \{\mu_M(x) \wedge \mu_A(y)\} &\leq \bigvee_{a=x\alpha y} \{1 \wedge \mu_A(x\alpha y)\}\\ &= \bigvee_{a=x\alpha y} \{1 \wedge \mu_A(a)\} = \mu_A(a),\end{aligned}$$

$$\begin{aligned}\nu_{M\Gamma A}(a) = \bigwedge_{a=x\alpha y}\{\nu_M(x)\vee\nu_A(y)\} &\geq \bigwedge_{a=x\alpha y}\{0\vee\nu_A(x\alpha y)\}\\ &= \bigwedge_{a=x\alpha y}\{0\vee\nu_A(a)\} = \nu_A(a).\end{aligned}$$

Otherwise, $\mu_{M\Gamma A}(a) = 0 \leq \mu_A(a), \nu_{M\Gamma A}(a) = 1 \geq \nu_A(a)$. So we have $M\Gamma A \subseteq A$. By the definition of intuitionistic fuzzy left (resp. right) ideal of A, we have

$$\mu_A(x-y) \geq \{\mu_A(x)\wedge\mu_A(y)\} \text{ and } \nu_A(x-y) \leq \{\nu_A(x)\vee\nu_A(y)\}.$$

($\Leftarrow$) Conversely assume (2) holds. Let $x, y \in M$, put $a = x\alpha y$. Since $M\Gamma A \subseteq A$, we have

$$\begin{aligned}\mu_A(x\alpha y) \geq \mu_{M\Gamma A}(a) = \bigvee_{a=x\alpha y}\{\mu_M(x)\wedge\mu_A(y)\} &\geq \{\mu_M(x)\wedge\mu_A(x\alpha y)\}\\ &= \{1\wedge\mu_A(y)\} = \mu_A(y),\end{aligned}$$

$$\begin{aligned}\nu_A(x\alpha y) \leq \nu_{M\Gamma A}(a) = \bigwedge_{a=x\alpha y}\{\nu_M(x)\vee\nu_A(y)\} &\leq \{\nu_M(x)\vee\nu_A(x\alpha y)\}\\ &= \Big\{0\vee\nu_A(y)\Big\} = \nu_A(y).\end{aligned}$$

This means that A is an intuitionistic fuzzy left (resp. right) ideal of M, thus (2) implies (1). □

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