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Inequalities Concerning Maximum Modulus of Polynomials

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Abstract : If $p(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j,\ 1 \le \mu \le n$ is a polynomial of degree n having no zeros in $|z| < k,\ k > 0$, then for $0 \le r \le \rho \le k$, [A. Aziz, W.M. Shah, Inequalities for a polynomial and its derivative, Math. Inequal. and Appl. 7 (3) (2004) 397–391],

$$M(p^{'}, \rho) \le n\rho^{\mu-1} \frac{(\rho^{\mu} + k^{\mu})^{\frac{n}{\mu}-1}}{(k^{\mu} + r^{\mu})^{\frac{n}{\mu}}} \{M(p, r) - m(p, k)\}.$$

In this paper, we have generalized as well as improved upon the above inequality by involving the coefficients of the polynomial $p(z)$. Besides, our result gives interesting refinements of some well-known results.

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1 Introduction and Statement of results

Let $p(z)$ be an entire function, $M(p,r) = \max_{|z|=r} |p(z)|$ and $m(p,r) = \min_{|z|=r} |p(z)|$. If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree atmost n and $p^{'}(z)$ its derivative, then

$$M(p^{'},1) \leq nM(p,1). \tag{1.1}$$

Inequality (1.1) is a well known result of S. Bernstein (for reference see [1, 2]. If we restrict ourselves to a class of polynomials having no zeros in $|z| < 1$, then

$$M(p^{'},1) \leq \frac{n}{2}M(p,1). \tag{1.2}$$

Inequality (1.2) was conjectured by Erdös and latter verified by Lax [3]. This inequality was further improved by Aziz and Dawood [4]. In fact they proved with the same hypothesis, that

$$M(p^{'},1) \leq \frac{n}{2}\{M(p,1) - m(p,1)\}. \tag{1.3}$$

As an extension of (1.2), Malik [5] proved that if $p(z)$ is a polynomial of degree n which does not vanish in $|z| < k,\ k \geq 1$, then

$$M(p^{'},1) \leq \frac{n}{1+k}M(p,1), \tag{1.4}$$

whereas Govil [6], under the same hypothesis proved that

$$M(p^{'},1) \leq \frac{n}{1+k}\{M(p,1) - m(p,k)\}. \tag{1.5}$$

As a generalization of inequality (1.4), Dewan and Bidkham [7] proved that if $p(z)$ is a polynomial of degree n having no zeros in $|z| < k,\ k \geq 1$, then for $0 \leq r \leq \rho \leq k$,

$$M(p^{'},\rho) \leq n\frac{(\rho+k)^{n-1}}{(k+r)^n}M(p,r). \tag{1.6}$$

Recently, Dewan and Mir [8] generalized inequality (1.6) and proved the following result:

Theorem A. If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n having no zeros in $|z| < k,\ k \geq 1$, then for $0 \leq r \leq \rho \leq k$,

$$M(p^{'},\rho) \leq n\frac{(\rho+k)^{n-1}}{(k+r)^n}\left\{1 - \frac{k(k-\rho)(n|a_0| - k|a_1|)n}{n|a_0|(k^2+\rho^2) + 2k^2\rho|a_1|}\left(\frac{\rho-r}{k+\rho}\right)\left(\frac{k+r}{k+\rho}\right)^{n-1}\right\}M(p,r). \tag{1.7}$$

As a generalization of inequality (1.5), Aziz and Shah [9] proved the following:

Theorem B. If $p(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j,\ 1 \leq \mu \leq n$ is a polynomial of degree n having no zeros in $|z| < k,\ k > 0$, then for $0 \leq r \leq \rho \leq k$,

$$M(p^{'},\rho) \leq n\rho^{\mu-1}\frac{(\rho^\mu + k^\mu)^{\frac{n}{\mu}-1}}{(k^\mu + r^\mu)^{\frac{n}{\mu}}}\{M(p,r) - m(p,k)\}. \tag{1.8}$$

In this paper, we prove the following more general result which improves as well as generalizes Theorems A and B. More precisely, we prove:

Theorem 1.1. *If $p(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j,\ 1 \leq \mu \leq n$ is a polynomial of degree n having no zeros in $|z| < k,\ k > 0$, then for $0 \leq r \leq \rho \leq k$,*

$$M(p',\rho) \leq \frac{n\rho^{\mu-1}(\rho^\mu + k^\mu)^{\frac{n}{\mu}-1}}{(k^\mu + r^\mu)^{\frac{n}{\mu}}}\Bigg[\Bigg\{1 - \frac{k^\mu(k-\rho)(n|a_0| - k^\mu\mu|a_\mu|)n}{\mu(n|a_0|(\rho^{\mu+1} + k^{\mu+1}) + \mu|a_\mu|(k^{2\mu}\rho + k^{\mu+1}\rho^\mu))}$$
$$\times\left(\frac{\rho^\mu - r^\mu}{k^\mu + \rho^\mu}\right)\left(\frac{k^\mu + r^\mu}{k^\mu + \rho^\mu}\right)^{\frac{n}{\mu}-1}\Bigg\}M(p,r) - \frac{(r^\mu + k^\mu)^{\frac{n}{\mu}+1}}{(\rho^\mu + k^\mu)^{\frac{n}{\mu}}}$$
$$\times\Bigg\{\frac{(n|a_0|\rho + \mu|a_\mu|k^{\mu+1})}{n|a_0|(k^{\mu+1} + \rho^{\mu+1}) + \mu|a_\mu|(k^{2\mu}\rho + k^{\mu+1}\rho^\mu)}\Bigg(\left(\frac{\rho^\mu + k^\mu}{r^\mu + k^\mu}\right)^{\frac{n}{\mu}} - 1$$
$$- \frac{n}{\mu}\frac{(\rho^\mu - r^\mu)}{(r^\mu + k^\mu)}\Bigg)\Bigg\}m(p,k)\Bigg]. \tag{1.9}$$

Equality in (1.9) holds for the polynomials of the form $p(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$, where n is a mutiple of μ.

Remark 1.2. *It is well known (for example see [10], proof of Lemma 2.4) that if $p(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j,\ 1 \leq \mu \leq n$ having no zeros in $|z| < k,\ k > 0$, then*

$$\frac{\mu}{n}\frac{|a_\mu|}{|a_0| - m(p,k)}k^\mu \leq 1,$$

which also implies

$$\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^\mu \leq 1.$$

The above theorem with $0 < r < \rho \leq k$ and $\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^\mu < 1$ gives a bound that is much better than abtainable from Theorems A and B.

Again, If we put $n|a_0| = \mu|a_\mu|k^\mu$ in above theorem, we get the following result.

Corollary 1.3. *If $p(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j,\ 1 \leq \mu \leq n$ is a polynomial of degree n having no zeros in $|z| < k,\ k > 0$, then for $0 \leq r \leq \rho \leq k$,*

$$M(p',\rho) \leq n\rho^{\mu-1}\frac{(\rho^\mu + k^\mu)^{\frac{n}{\mu}-1}}{(k^\mu + r^\mu)^{\frac{n}{\mu}}}\Bigg[M(p,r) - \left(\frac{r^\mu + k^\mu}{\rho^\mu + k^\mu}\right)\Bigg\{1 - \left(\frac{r^\mu + k^\mu}{\rho^\mu + k^\mu}\right)^{\frac{n}{\mu}}$$
$$- \frac{n}{\mu}\frac{(\rho^\mu - r^\mu)(r^\mu + k^\mu)^{\frac{n}{\mu}-1}}{(k^\mu + \rho^\mu)^{\frac{n}{\mu}}}\Bigg\}m(p,k)\Bigg].$$

The result is best possible and equality holds for $p(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$, where n is a mutiple of μ.

2 Lemmas

For the proof of Theorem 1.1, we require the following lemmas:

Lemma 2.1. *If* $p(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j,\ 1 \le \mu \le n$ *be a polynomial of degree* n *having no zeros in* $|z| < k,\ k \ge 1$, *then*

$$M(p^{'}, 1) \le \frac{n}{1 + k^{\mu}} M(p, 1). \tag{2.1}$$

The above result is due to Chan and Malik [11].

Lemma 2.2. *If* $p(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j,\ 1 \le \mu \le n$ *be a polynomial of degree* n *having no zeros in* $|z| < k$, $k \ge 1$, *then*

$$M(p^{'}, 1) \le n \frac{1 + \frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right| k^{\mu+1}}{1 + k^{\mu+1} + \frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|(k^{\mu+1} + k^{2\mu})} M(p, 1), \tag{2.2}$$

where $q(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$.

The above lemma is due to Qazi [12].

Lemma 2.3. *If* $p(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j,\ 1 \le \mu \le n$ *be a polynomial of degree* n *having no zeros in* $|z| < k,\ k > 0$, *then for* $0 \le r \le \rho \le k$,

$$M(p, r) \ge \left(\frac{r^{\mu} + k^{\mu}}{\rho^{\mu} + k^{\mu}}\right)^{\frac{n}{\mu}} M(p, \rho) + \left[1 - \left(\frac{r^{\mu} + k^{\mu}}{\rho^{\mu} + k^{\mu}}\right)^{\frac{n}{\mu}}\right] m(p, k). \tag{2.3}$$

This lemma is due to Dewan, Yadav and Puktha [13].

Lemma 2.4. *If* $p(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j,\ 1 \le \mu \le n$ *is a polynomial of degree* n *having no zeros in* $|z| < k,\ k > 0$, *then for* $0 \le r \le \rho \le k$,

$$\begin{aligned} M(p, \rho) \quad \le \quad & \left(\frac{k^{\mu} + \rho^{\mu}}{k^{\mu} + r^{\mu}}\right)^{\frac{n}{\mu}} \Bigg[1 - \frac{k^{\mu}(k - \rho)(n|a_0| - k^{\mu}\mu|a_\mu|)n}{\mu(n|a_0|(\rho^{\mu+1} + k^{\mu+1}) + \mu|a_\mu|(k^{2\mu}\rho + k^{\mu+1}\rho^{\mu}))} \\ & \times \left(\frac{\rho^{\mu} - r^{\mu}}{k^{\mu} + \rho^{\mu}}\right)\left(\frac{k^{\mu} + r^{\mu}}{k^{\mu} + \rho^{\mu}}\right)^{\frac{n}{\mu} - 1}\Bigg] M(p, r) \\ & - \Bigg[\frac{(n|a_0|\rho + \mu|a_\mu|k^{\mu+1})(r^{\mu} + k^{\mu})}{n|a_0|(k^{\mu+1} + \rho^{\mu+1}) + \mu|a_\mu|(k^{2\mu}\rho + k^{\mu+1}\rho^{\mu})} \\ & \times \left\{\left(\left(\frac{\rho^{\mu} + k^{\mu}}{r^{\mu} + k^{\mu}}\right)^{\frac{n}{\mu}} - 1\right) - \frac{n}{\mu}\frac{(\rho^{\mu} - r^{\mu})}{(r^{\mu} + k^{\mu})}\right\}\Bigg] m(p, k). \end{aligned} \tag{2.4}$$

Proof. Since $p(z)$ has no zeros in $|z|<k,\ k>0$, therefore the polynomial $T(z)=p(tz)$ has no zero in $|z|<\frac{k}{t}$, where $0\le t\le k$. Applying Lemma 2.2 to the polynomial $T(z)$, we get

$$M(T^{'},1)\le n\frac{1+\frac{\mu}{n}\left|\frac{t^{\mu}a_{\mu}}{a_0}\right|(k/t)^{\mu+1}}{1+(k/t)^{\mu+1}+\frac{\mu}{n}\left|\frac{t^{\mu}a_{\mu}}{a_0}\right|((k/t)^{\mu+1}+(k/t)^{2\mu})}M(T,1),$$

which implies

$$M(p^{'},t)\le nt^{\mu-1}\left\{\frac{n|a_0|t+\mu|a_{\mu}|k^{\mu+1}}{n|a_0|(k^{\mu+1}+t^{\mu+1})+\mu|a_{\mu}|(k^{2\mu}t+k^{\mu+1}t^{\mu})}\right\}M(p,t). \tag{2.5}$$

Now for $0\le r\le\rho\le k$ and $0\le\theta\le 2\pi$, we have by using (2.5)

$$\begin{aligned}|p(\rho e^{i\theta})-p(re^{i\theta})| &\le \int_r^{\rho}|p^{'}(te^{i\theta})|dt\\ &\le \int_r^{\rho} nt^{\mu-1}\left\{\frac{n|a_0|t+\mu|a_{\mu}|k^{\mu+1}}{n|a_0|(k^{\mu+1}+t^{\mu+1})+\mu|a_{\mu}|(k^{2\mu}t+k^{\mu+1}t^{\mu})}\right\}M(p,t).\end{aligned}$$

Using Lemma 2.3 with $\rho=t$ and noting that $0\le r\le t\le\rho\le k$, it follows that

$$\begin{aligned}|p(\rho e^{i\theta})-p(re^{i\theta})| \le \int_r^{\rho} nt^{\mu-1}&\left\{\frac{n|a_0|t+\mu|a_{\mu}|k^{\mu+1}}{n|a_0|(k^{\mu+1}+t^{\mu+1})+\mu|a_{\mu}|(k^{2\mu}t+k^{\mu+1}t^{\mu})}\right\}\\ &\times\left(\frac{t^{\mu}+k^{\mu}}{r^{\mu}+k^{\mu}}\right)^{\frac{n}{\mu}}\left\{M(p,r)-\left(1-\left(\frac{r^{\mu}+k^{\mu}}{t^{\mu}+k^{\mu}}\right)^{\frac{n}{\mu}}\right)m(p,k)\right\}dt.\end{aligned}$$

Therefore,

$$\begin{aligned}M(p,\rho) \le\ & \Bigg[1+\frac{(k^{\mu}+\rho^{\mu})}{(k^{\mu}+r^{\mu})^{\frac{n}{\mu}}}\left\{\frac{n|a_0|\rho+\mu|a_{\mu}|k^{\mu+1}}{n|a_0|(k^{\mu+1}+\rho^{\mu+1})+\mu|a_{\mu}|(k^{2\mu}\rho+k^{\mu+1}\rho^{\mu})}\right\}\\ &\times\int_r^{\rho} nt^{\mu-1}(k^{\mu}+t^{\mu})^{\frac{n}{\mu}-1}dt\Bigg]M(p,r)\\ &-\Bigg[\left\{\frac{n|a_0|\rho+\mu|a_{\mu}|k^{\mu+1}}{n|a_0|(k^{\mu+1}+\rho^{\mu+1})+\mu|a_{\mu}|(k^{2\mu}\rho+k^{\mu+1}\rho^{\mu})}\right\}\\ &\times\int_r^{\rho}\left(\left(\frac{t^{\mu}+k^{\mu}}{r^{\mu}+k^{\mu}}\right)^{\frac{n}{\mu}}-1\right)nt^{\mu-1}dt\Bigg]m(p,k)\end{aligned}$$

$$
\begin{aligned}
\leq\quad & \Bigg[1-\frac{(k^\mu+\rho^\mu)(n|a_0|\rho+\mu|a_\mu|k^{\mu+1})}{n|a_0|(k^{\mu+1}+\rho^{\mu+1})+\mu|a_\mu|(k^{2\mu}\rho+k^{\mu+1}\rho^\mu)} \\
& +\frac{(k^\mu+\rho^\mu)(n|a_0|\rho+\mu|a_\mu|k^{\mu+1})}{n|a_0|(k^{\mu+1}+\rho^{\mu+1})+\mu|a_\mu|(k^{2\mu}\rho+k^{\mu+1}\rho^\mu)}\left(\frac{k^\mu+\rho^\mu}{k^\mu+r^\mu}\right)^{\frac{n}{\mu}}\Bigg]M(p,r) \\
& -\Bigg[\left\{\frac{n|a_0|\rho+\mu|a_\mu|k^{\mu+1}}{n|a_0|(k^{\mu+1}+\rho^{\mu+1})+\mu|a_\mu|(k^{2\mu}\rho+k^{\mu+1}\rho^\mu)}\right\} \\
& \times\int_r^\rho\left(\left(\frac{t^\mu+k^\mu}{r^\mu+k^\mu}\right)^{\frac{n}{\mu}-1}-1\right)nt^{\mu-1}dt\Bigg]m(p,k) \\
=\quad & \Bigg[\frac{(k-\rho)(n|a_0|-k^\mu\mu|a_\mu|)k^\mu}{n|a_0|(k^{\mu+1}+\rho^{\mu+1})+\mu|a_\mu|(k^{2\mu}\rho+k^{\mu+1}\rho^\mu)} \\
& +\left\{1-\frac{(k-\rho)(n|a_0|-k^\mu\mu|a_\mu|)k^\mu}{n|a_0|(k^{\mu+1}+\rho^{\mu+1})+\mu|a_\mu|(k^{2\mu}\rho+k^{\mu+1}\rho^\mu)}\right\}\left(\frac{k^\mu+\rho^\mu}{k^\mu+r^\mu}\right)^{\frac{n}{\mu}}\Bigg]M(p,r) \\
& -\Bigg[\left\{\frac{n|a_0|\rho+\mu|a_\mu|k^{\mu+1}}{n|a_0|(k^{\mu+1}+\rho^{\mu+1})+\mu|a_\mu|(k^{2\mu}\rho+k^{\mu+1}\rho^\mu)}\right\}\frac{1}{(r^\mu+k^\mu)^{\frac{n}{\mu}-1}} \\
& \times\int_r^\rho\left((k^\mu+t^\mu)^{\frac{n}{\mu}-1}-(r^\mu+k^\mu)^{\frac{n}{\mu}-1}\right)nt^{\mu-1}dt\Bigg]m(p,k) \\
=\quad & \left(\frac{k^\mu+\rho^\mu}{k^\mu+r^\mu}\right)^{\frac{n}{\mu}}\Bigg[1-\frac{(k-\rho)(n|a_0|-k^\mu\mu|a_\mu|)k^\mu}{n|a_0|(k^{\mu+1}+\rho^{\mu+1})+\mu|a_\mu|(k^{2\mu}\rho+k^{\mu+1}\rho^\mu)} \\
& \times\left\{1-\left(\frac{k^\mu+r^\mu}{k^\mu+\rho^\mu}\right)^{\frac{n}{\mu}}\right\}\Bigg]M(p,r) \\
& -\Bigg[\left\{\frac{(n|a_0|\rho+\mu|a_\mu|k^{\mu+1})(r^\mu+k^\mu)}{n|a_0|(k^{\mu+1}+\rho^{\mu+1})+\mu|a_\mu|(k^{2\mu}\rho+k^{\mu+1}\rho^\mu)}\right\} \\
& \times\left\{\left(\left(\frac{k^\mu+\rho^\mu}{k^\mu+r^\mu}\right)^{\frac{n}{\mu}}-1\right)-\frac{n}{\mu}\left(\frac{\rho^\mu-r^\mu}{k^\mu+r^\mu}\right)\right\}\Bigg]m(p,k) \\
=\quad & \left(\frac{k^\mu+\rho^\mu}{k^\mu+r^\mu}\right)^{\frac{n}{\mu}}\Bigg[1-\frac{(k-\rho)(n|a_0|-k^\mu\mu|a_\mu|)k^\mu}{n|a_0|(k^{\mu+1}+\rho^{\mu+1})+\mu|a_\mu|(k^{2\mu}\rho+k^{\mu+1}\rho^\mu)} \\
& \times\frac{\rho^\mu-r^\mu}{(k^\mu+\rho^\mu)(1-\frac{k^\mu+r^\mu}{k^\mu+\rho^\mu})}\left\{1-\left(\frac{k^\mu+r^\mu}{k^\mu+\rho^\mu}\right)^{\frac{n}{\mu}}\right\}\Bigg]M(p,r) \\
& -\Bigg[\left\{\frac{(n|a_0|\rho+\mu|a_\mu|k^{\mu+1})(r^\mu+k^\mu)}{n|a_0|(k^{\mu+1}+\rho^{\mu+1})+\mu|a_\mu|(k^{2\mu}\rho+k^{\mu+1}\rho^\mu)}\right\}\left\{\left(\left(\frac{k^\mu+\rho^\mu}{k^\mu+r^\mu}\right)^{\frac{n}{\mu}}-1\right)\right. \\
& \left.-\frac{n}{\mu}\left(\frac{\rho^\mu-r^\mu}{k^\mu+r^\mu}\right)\right\}\Bigg]m(p,k)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \left(\frac{k^\mu+\rho^\mu}{k^\mu+r^\mu}\right)^{\frac{n}{\mu}}\left[1-\frac{k^\mu(k-\rho)(n|a_0|-k^\mu\mu|a_\mu|)n}{\mu(n|a_0|(\rho^{\mu+1}+k^{\mu+1})+\mu|a_\mu|(k^{2\mu}\rho+k^{\mu+1}\rho^\mu))}\right.\\
&\quad\left.\times\left(\frac{\rho^\mu-r^\mu}{k^\mu+\rho^\mu}\right)\left(\frac{k^\mu+r^\mu}{k^\mu+\rho^\mu}\right)^{\frac{n}{\mu}-1}\right]M(p,r)\\
&\quad-\left[\frac{(n|a_0|\rho+\mu|a_\mu|k^{\mu+1})(r^\mu+k^\mu)}{n|a_0|(k^{\mu+1}+\rho^{\mu+1})+\mu|a_\mu|(k^{2\mu}\rho+k^{\mu+1}\rho^\mu)}\left\{\left(\left(\frac{\rho^\mu+k^\mu}{r^\mu+k^\mu}\right)^{\frac{n}{\mu}}-1\right)\right.\right.\\
&\quad\left.\left.-\frac{n}{\mu}\frac{(\rho^\mu-r^\mu)}{(r^\mu+k^\mu)}\right\}\right]m(p,k).
\end{aligned}
$$

This completes the proof. □

3 Proof of the Theorem

Proof of Theorem 1.1. Since $p(z)$ has no zeros in $|z|<k,\ k>0$, then for $0<\rho\leq k$, it follows that $T(z)=p(\rho z)$ has no zero in $|z|<\frac{k}{\rho}$, where $k/\rho\geq 1$. Applying Lemma 2.1 to the polynomial $T(z)$, we get

$$
M(T^{'},1)\leq\frac{n}{1+\left(\frac{k}{\rho}\right)^{\mu}}M(T,1),
$$

which gives

$$
M(p^{'},\rho)\leq\frac{n\rho^{\mu-1}}{k^\mu+\rho^\mu}M(p,\rho). \tag{3.1}
$$

Now if $0\leq r\leq\rho\leq k$, then from (3.1) it follows with the help of Lemma 2.4 that

$$
\begin{aligned}
M(p^{'},\rho) &\leq \frac{n\rho^{\mu-1}(\rho^\mu+k^\mu)^{\frac{n}{\mu}-1}}{(k^\mu+r^\mu)^{\frac{n}{\mu}}}\left[1-\frac{k^\mu(k-\rho)(n|a_0|-k^\mu\mu|a_\mu|)n}{\mu(n|a_0|(\rho^{\mu+1}+k^{\mu+1})+\mu|a_\mu|(k^{2\mu}\rho+k^{\mu+1}\rho^\mu))}\right.\\
&\quad\left.\times\left(\frac{\rho^\mu-r^\mu}{k^\mu+\rho^\mu}\right)\left(\frac{k^\mu+r^\mu}{k^\mu+\rho^\mu}\right)^{\frac{n}{\mu}-1}\right]M(p,r)\\
&\quad-n\left(\frac{r^\mu+k^\mu}{\rho^\mu+k^\mu}\right)\rho^{\mu-1}\left[\frac{(n|a_0|\rho+\mu|a_\mu|k^{\mu+1})}{n|a_0|(k^{\mu+1}+\rho^{\mu+1})+\mu|a_\mu|(k^{2\mu}\rho+k^{\mu+1}\rho^\mu)}\right.\\
&\quad\left.\times\left\{\left(\left(\frac{\rho^\mu+k^\mu}{r^\mu+k^\mu}\right)^{\frac{n}{\mu}}-1\right)-\frac{n}{\mu}\frac{(\rho^\mu-r^\mu)}{(r^\mu+k^\mu)}\right\}\right]m(p,k).
\end{aligned}
$$

This completes the proof of Theorem 1.1.

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