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Subdivisions of the Spectra for Generalized Difference Operator over Certain Sequence Spaces

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Abstract : The fine spectrum of the generalized difference operator $B(r,s)$ represented by a double band matrix over the sequence spaces c_0 and c was studied by Altay and Başar [B. Altay, F. Başar, On the fine spectrum of the generalized difference operator $B(r,s)$ over the sequence spaces c_0 and c, Int. J. Math. Math. Sci. 2005 (18) (2005) 3005–3013], over the sequence spaces ℓ_p and bv_p was worked by Bilgiç and Furkan [H. Bilgiç, H. Furkan, On the fine spectrum of the generalized difference operator $B(r,s)$ over the sequence spaces ℓ_p and bv_p, $(1 < p < \infty)$, Nonlinear Anal. 68 (3) (2008) 499–506]; where bv_p denotes the space of all sequences (x_k) such that $(x_k - x_{k-1})$ in ℓ_p with $1 < p < \infty$ and was studied by Başar and Altay [F. Başar, B. Altay, On the space of sequences of p-bounded variation and related matrix mappings, Ukrainian Math. J. 55 (1) (2003) 136–147]. In this paper, the approximate point spectrum, defect spectrum and compression spectrum of the generalized difference operator $B(r,s)$ over the sequence spaces c_0, c, ℓ_p and bv_p have been determined.

Keywords : Fine spectrum; Approximate point spectrum; Defect spectrum;

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Compression spectrum; Generalized difference operator $B(r, s)$; Spectrum.
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1 Preliminaries, Background and Notation

Let X and Y be the Banach spaces and $T : X \to Y$ also be a bounded linear operator. By $R(T)$, we denote the range of T, i.e.,

$$R(T) = \{y \in Y : y = Tx,\ x \in X\}.$$

By $B(X)$, we also denote the set of all bounded linear operators on X into itself. If X is any Banach space and $T \in B(X)$ then the adjoint T^* of T is a bounded linear operator on the dual X^* of X defined by $(T^*f)(x) = f(Tx)$ for all $f \in X^*$ and $x \in X$.

Let $X \neq \{\theta\}$ be a non trivial complex normed space and $T : \mathcal{D}(T) \to X$ a linear operator defined on subspace $\mathcal{D}(T) \subseteq X$. We do not assume that $\mathcal{D}(T)$ is dense in X, or that T has closed graph $\{(x, Tx) : x \in \mathcal{D}(T)\} \subseteq X \times X$. We mean by the expression 'T is *invertible*' that there exists a bounded linear operator $S : R(T) \to X$ for which $ST = I$ on $D(T)$ and $\overline{R(T)} = X$; such that $S = T^{-1}$ is necessarily uniquely determined, and linear; the boundedness of S means that T must be *bounded below*, in the sense that there is $k > 0$ for which $\|Tx\| \geq k\|x\|$ for all $x \in D(T)$. Associated with each complex number λ is perturbed operator

$$T_\lambda = \lambda I - T,$$

defined on the same domain $\mathcal{D}(T)$ as T. The *spectrum* $\sigma(T, X)$ consist of those $\lambda \in \mathbb{C}$ for which T_λ is not invertible, and the *resolvent* is the mapping from the complement $\sigma(T, X)$ of the spectrum into the algebra of bounded linear operators on X defined by $\lambda \mapsto T_\lambda^{-1}$.

2 Subdivisions of the spectrum

In this section, we mention from the parts point spectrum, continuous spectrum, residual spectrum, approximate point spectrum, defect spectrum and compression spectrum of the spectrum. There are many different ways to subdivide the spectrum of a bounded linear operator. Some of them are motivated by applications to physics, in particular, quantum mechanics.

2.1 The point spectrum, continuous spectrum and residual spectrum

The name *resolvent* is appropriate, since T_λ^{-1} helps to solve the equation $T_\lambda x = y$. Thus, $x = T_\lambda^{-1}y$ provided T_λ^{-1} exists. More important, the investigation of properties of T_λ^{-1} will be basic for an understanding of the operator T itself.

Naturally, many properties of T_λ and T_λ^{-1} depend on λ, and spectral theory is concerned with those properties. For instance, we shall be interested in the set of all λ's in the complex plane such that T_λ^{-1} exists. Boundedness of T_λ^{-1} is another property that will be essential. We shall also ask for what λ's the domain of T_λ^{-1} is dense in X, to name just a few aspects. A *regular value* λ of T is a complex number such that T_λ^{-1} exists and bounded and whose domain is dense in X. For our investigation of T, T_λ and T_λ^{-1}, we need some basic concepts in spectral theory which are given as follows (see [1, pp. 370–371]):

The *resolvent set* $\rho(T, X)$ of T is the set of all regular values λ of T. Furthermore, the spectrum $\sigma(T, X)$ is partitioned into three disjoint sets as follows:

The *point (discrete) spectrum* $\sigma_p(T, X)$ is the set such that T_λ^{-1} does not exist. An $\lambda \in \sigma_p(T, X)$ is called an *eigenvalue* of T.

The *continuous spectrum* $\sigma_c(T, X)$ is the set such that T_λ^{-1} exists and is unbounded and the domain of T_λ^{-1} is dense in X.

The *residual spectrum* $\sigma_r(T, X)$ is the set such that T_λ^{-1} exists (and may be bounded or not) but the domain of T_λ^{-1} is not dense in X.

Therefore, these three subspectras form a disjoint subdivisions

$$\sigma(T, X) = \sigma_p(T, X) \cup \sigma_c(T, X) \cup \sigma_r(T, X). \tag{2.1}$$

To avoid trivial misunderstandings, let us say that some of the sets defined above, may be empty. This is an existence problem which we shall have to discuss. Indeed, it is well-known that $\sigma_c(T, X) = \sigma_r(T, X) = \emptyset$ and the spectrum $\sigma(T, X)$ consists of only the set $\sigma_p(T, X)$ in the finite dimensional case.

2.2 The approximate point spectrum, defect spectrum and compression spectrum

In this subsection, following Appell et al. [2], we give the definitions of the three more subdivisions of the spectrum called as the *approximate point spectrum*, *defect spectrum* and *compression spectrum*.

Given a bounded linear operator T in a Banach space X, we call a sequence (x_k) in X as a *Weyl sequence* for T if $\|x_k\| = 1$ and $\|Tx_k\| \to 0$, as $k \to \infty$.

In what follows, we call the set

$$\sigma_{ap}(T, X) := \{\lambda \in \mathbb{C} : \text{there exists a Weyl sequence for } \lambda I - T\} \tag{2.2}$$

the *approximate point spectrum of* T. Moreover, the subspectrum

$$\sigma_\delta(T, X) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not surjective}\} \tag{2.3}$$

is called *defect spectrum* of T.

The two subspectra given by (2.2) and (2.3) form a (not necessarily disjoint) subdivisions

$$\sigma(T, X) = \sigma_{ap}(T, X) \cup \sigma_\delta(T, X)$$

of the spectrum. There is another subspectrum,

$$\sigma_{co}(T,X) = \{\lambda \in \mathbb{C} : \overline{R(\lambda I - T)} \neq X\}$$

which is often called *compression spectrum* in the literature. The compression spectrum gives rise to another (not necessarily disjoint) decomposition

$$\sigma(T,X) = \sigma_{ap}(T,X) \cup \sigma_{co}(T,X)$$

of the spectrum. Clearly, $\sigma_p(T,X) \subseteq \sigma_{ap}(T,X)$ and $\sigma_{co}(T,X) \subseteq \sigma_\delta(T,X)$. Moreover, comparing these subspectra with those in (2.1) we note that

$$\sigma_r(T,X) = \sigma_{co}(T,X) \backslash \sigma_p(T,X)$$

and

$$\sigma_c(T,X) = \sigma(T,X) \backslash [\sigma_p(T,X) \cup \sigma_{co}(T,X)].$$

Sometimes it is useful to relate the spectrum of a bounded linear operator to that of its adjoint. Building on classical existence and uniqueness results for linear operator equations in Banach spaces and their adjoints.

Proposition 2.1 ([2, Proposition 1.3, p. 28]). *Spectra and subspectra of an operator $T \in B(X)$ and its adjoint $T^* \in B(X^*)$ are related by the following relations:*

(a) $\sigma(T^*, X^*) = \sigma(T,X)$.

(b) $\sigma_c(T^*, X^*) \subseteq \sigma_{ap}(T,X)$.

(c) $\sigma_{ap}(T^*, X^*) = \sigma_\delta(T,X)$.

(d) $\sigma_\delta(T^*, X^*) = \sigma_{ap}(T,X)$.

(e) $\sigma_p(T^*, X^*) = \sigma_{co}(T,X)$.

(f) $\sigma_{co}(T^*, X^*) \supseteq \sigma_p(T,X)$.

(g) $\sigma(T,X) = \sigma_{ap}(T,X) \cup \sigma_p(T^*, X^*) = \sigma_p(T,X) \cup \sigma_{ap}(T^*, X^*)$.

The relations (c)–(f) show that the approximate point spectrum is in a certain sense dual to defect spectrum, and the point spectrum dual to the compression spectrum.

The last equality (g) implies, in particular, that $\sigma(T,X) = \sigma_{ap}(T,X)$ if X is a Hilbert space and T is normal. Roughly speaking, this shows that normal (in particular, self-adjoint) operators on Hilbert spaces are most similar to matrices in finite dimensional spaces (see [2]).

2.3 Goldberg's classification of spectrum

If X is a Banach space and $T \in B(X)$, then there are three possibilities for $R(T)$:

(I) $R(T) = X$.

(II) $R(T) \neq \overline{R(T)} = X$.

(III) $\overline{R(T)} \neq X$.

and

(1) T^{-1} exists and is continuous.

(2) T^{-1} exists but is discontinuous.

(3) T^{-1} does not exist.

If these possibilities are combined in all possible ways, nine different states are created. These are labelled by: I_1, I_2, I_3, II_1, II_2, II_3, III_1, III_2, III_3. If an operator is in state III_2 for example, then $\overline{R(T)} \neq X$ and T^{-1} exist but is discontinuous (see [3]).

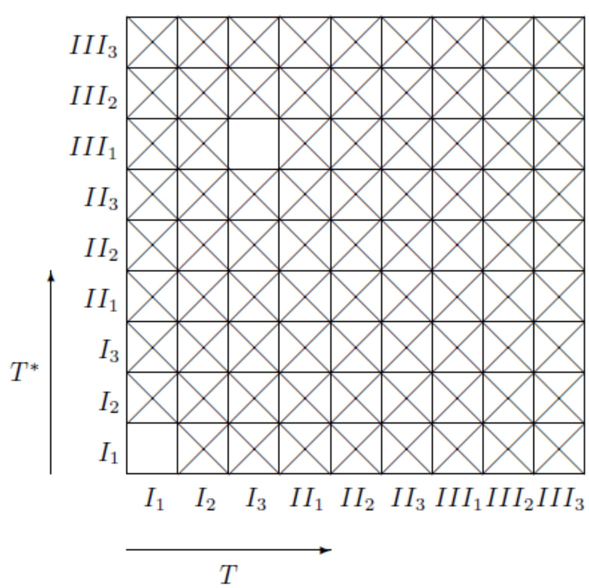

Table 1.1: State diagram for $B(X)$ and $B(X^*)$ for a non-reflective Banach space X

If λ is a complex number such that $T_\lambda = \lambda I - T \in I_1$ or $T_\lambda = \lambda I - T \in II_1$, then $\lambda \in \rho(T, X)$. All scalar values of λ not in $\rho(T, X)$ comprise the spectrum of

T. The further classification of $\sigma(T, X)$ gives rise to the fine spectrum of T. That is, $\sigma(T, X)$ can be divided into the subsets $I_2\sigma(T, X) = \emptyset$, $I_3\sigma(T, X)$, $II_2\sigma(T, X)$, $II_3\sigma(T, X)$, $III_1\sigma(T, X)$, $III_2\sigma(T, X)$, $III_3\sigma(T, X)$. For example, if $T_\lambda = \lambda I - T$ is in a given state, III_2 (say), then we write $\lambda \in III_2\sigma(T, X)$.

By the definitions given above, we can illustrate the subdivisions (2.1) in the following table:

		1	2	3
		T_λ^{-1} exists and is bounded	T_λ^{-1} exists and is unbounded	T_λ^{-1} does not exist
I	$R(\lambda I - T) = X$	$\lambda \in \rho(T, X)$	–	$\lambda \in \sigma_p(T, X)$ $\lambda \in \sigma_{ap}(T, X)$
II	$\overline{R(\lambda I - T)} = X$	$\lambda \in \rho(T, X)$	$\lambda \in \sigma_c(T, X)$ $\lambda \in \sigma_{ap}(T, X)$ $\lambda \in \sigma_\delta(T, X)$	$\lambda \in \sigma_p(T, X)$ $\lambda \in \sigma_{ap}(T, X)$ $\lambda \in \sigma_\delta(T, X)$
III	$\overline{R(\lambda I - T)} \neq X$	$\lambda \in \sigma_r(T, X)$ $\lambda \in \sigma_\delta(T, X)$ $\lambda \in \sigma_{co}(T, X)$	$\lambda \in \sigma_r(T, X)$ $\lambda \in \sigma_{ap}(T, X)$ $\lambda \in \sigma_\delta(T, X)$ $\lambda \in \sigma_{co}(T, X)$	$\lambda \in \sigma_p(T, X)$ $\lambda \in \sigma_{ap}(T, X)$ $\lambda \in \sigma_\delta(T, X)$ $\lambda \in \sigma_{co}(T, X)$

Table 1.2: Subdivisions of spectrum of a linear operator

Observe that the case in the first row and second column cannot occur in a Banach space X, by the closed graph theorem. If we are not in the third column, i.e., if λ is not an eigenvalue of T, we may always consider the resolvent operator T_λ^{-1} (on a possibly "thin" domain of definition) as "algebraic" inverse of $\lambda I - T$.

By a *sequence space*, we understand a linear subspace of the space $\omega = \mathbb{C}^{\mathbb{N}_1}$ of all complex sequences which contains ϕ, the set of all finitely non–zero sequences, where $\mathbb{N}_1$ denotes the set of positive integers. We write ℓ_∞, c, c_0 and bv for the spaces of all bounded, convergent, null and bounded variation sequences which are the Banach spaces with the sup-norm $\|x\|_\infty = \sup_{k\in\mathbb{N}} |x_k|$ and $\|x\|_{bv} = \sum_{k=0}^{\infty} |x_k - x_{k+1}|$ while ϕ is not a Banach space with respect to any norm, respectively. Also by ℓ_p, we denote the space of all p–absolutely summable sequences which is a Banach space with the norm $\|x\|_p = \left(\sum_{k=0}^{\infty} |x_k|^p\right)^{1/p}$, where $1 \leq p < \infty$.

Başar, Durna and Yıldırım [4] have recently determined the divisions approximation point spectrum, defect spectrum and compression spectrum of difference operator Δ over the sequence spaces c, c_0, ℓ_p and bv_p, where $1 < p < \infty$. In this paper, our main focus is the generalized difference operator $B(r, s)$ represented by the following double band matrix

$$B(r, s) = \begin{pmatrix} r & 0 & 0 & 0 & \cdots \\ s & r & 0 & 0 & \cdots \\ 0 & s & r & 0 & \cdots \\ 0 & 0 & s & r & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (s \neq 0).$$

Following Başar, Durna and Yıldırım [4], we give the subdivisions of the spectrum of the generalized difference matrix $B(r,s)$ on the spaces c, c_0, ℓ_p and bv_p, where $1 < p < \infty$. Since the generalized difference matrix $B(r,s)$ is reduced in the case $r = 1$, $s = -1$ to the difference matrix Δ, the results given in the present paper are more general than the corresponding results of Başar, Durna and Yıldırım [4].

3 The subdivisions of the spectrum of the matrix $B(r,s)$ on the spaces c_0, c, ℓ_p and bv_p

In 2005, Altay and Başar [5] determined the spectra and the fine spectra of generalized difference operator $B(r,s)$ on the sequence spaces c_0 and c. In 2008, Bilgiç and Furkan [6] worked the spectra and the fine spectra of generalized difference operator $B(r,s)$ on the sequence spaces ℓ_p and bv_p, where bv_p denotes the space of sequences of p-bounded variation introduced by Başar and Altay [7] consisting of all sequences $x = (x_k)$ such that $(x_k - x_{k-1}) \in \ell_p$ and $1 \le p < \infty$. In this paper, we have derived the approximate point spectrum, defect spectrum and compression spectrum of the matrix operator $B(r,s)$ over the sequence spaces c_0, c, ℓ_p and bv_p, where $1 < p < \infty$.

3.1 Subdivisions of the spectrum of $B(r,s)$ on c_0

In this subsection, we deal with the subdivisions of the spectrum of the generalized difference operator $B(r,s)$ over the sequence space c_0.

Theorem 3.1. *The following results hold:*

(a) $\sigma_{ap}(B(r,s), c_0) = \{\lambda \in \mathbb{C} : |\lambda - r| \le |s|\} \setminus \{r\}$.

(b) $\sigma_{\delta}(B(r,s), c_0) = \{\lambda \in \mathbb{C} : |\lambda - r| \le |s|\}$.

(c) $\sigma_{co}(B(r,s), c_0) = \{\lambda \in \mathbb{C} : |\lambda - r| < |s|\}$.

Proof. (a) Since, $\sigma_{ap}(B(r,s), c_0) = \sigma(B(r,s), c_0) \setminus III_1\sigma(B(r,s), c_0)$,

$$\sigma_{ap}(B(r,s), c_0) = \{\lambda \in \mathbb{C} : |\lambda - r| \le |s|\} \setminus \{r\}$$

is obtained by Theorem 2.1 and Theorem 2.6 of Altay and Başar [5].

(b) Since

$$\sigma_{\delta}(B(r,s), c_0) = \sigma(B(r,s), c_0) \setminus I_3\sigma(B(r,s), c_0)$$

from Table 1.2 and

$$I_3\sigma(B(r,s), c_0) = II_3\sigma(B(r,s), c_0) = III_3\sigma(B(r,s), c_0) = \emptyset$$

is obtained by Theorem 2.2 of Altay and Başar [5], we therefore derive that $\sigma_{\delta}(B(r,s), c_0) = \sigma(B(r,s), c_0)$.

(c) Since the equality

$$\sigma_{co}\left(B(r,s),c_0\right) = III_1\sigma(B(r,s),c_0) \cup III_2\sigma(B(r,s),c_0) \cup III_3\sigma(B(r,s),c_0)$$

holds from Table 1.2, it can be easily seen by Theorems 2.2 and 2.3 of Altay and Başar [5] that $\sigma_\delta\left(B(r,s),c_0\right)$ is the set of $\lambda \in \mathbb{C}$ such that $|\lambda - r| < |s|$. □

The next corollary is an immediate consequence of Proposition 2.1:

Corollary 3.2. *The following results hold:*

(a) $\sigma_{ap}\left(B(r,s)^*, \ell_1\right) = \{\lambda \in \mathbb{C} : |\lambda - r| \leq |s|\}$.

(b) $\sigma_\delta\left(B(r,s)^*, \ell_1\right) = \{\lambda \in \mathbb{C} : |\lambda - r| \leq |s|\} \setminus \{r\}$.

(c) [5, Theorem 2.3] $\sigma_p\left(B(r,s)^*, \ell_1\right) = \{\lambda \in \mathbb{C} : |\lambda - r| < |s|\}$.

3.2 Subdivisions of the spectrum of $B(r,s)$ on c

In the present subsection, we give the subdivisions of the spectrum of the generalized difference operator $B(r,s)$ over the sequence space c.

Theorem 3.3. *The following results hold:*

(a) $\sigma_{ap}(B(r,s),c) = \{\lambda \in \mathbb{C} : |\lambda - r| \leq |s|\} \setminus \{r\}$.

(b) $\sigma_\delta(B(r,s),c) = \{\lambda \in \mathbb{C} : |\lambda - r| \leq |s|\}$.

(c) $\sigma_{co}(B(r,s),c) = \{\lambda \in \mathbb{C} : |\lambda - r| < |s|\} \cup \{r+s\}$.

Proof. (a) $\sigma_{ap}(B(r,s),c) = \sigma(B(r,s),c) \backslash III_1\sigma(B(r,s),c)$ is obtained from Table 1.2. Now, the validity of the present part of the theorem follows from Theorem 2.10 of Altay and Başar [5].

(b) $\sigma_\delta(B(r,s),c) = \sigma(B(r,s),c) \backslash I_3\sigma(B(r,s),c)$ is obtained from Table 1.2. Moreover, since

$$\sigma_p(B(r,s),c) = I_3\sigma(B(r,s),c) \cup II_3\sigma(B(r,s),c) \cup III_3\sigma(B(r,s),c) = \emptyset$$

then $I_3\sigma(B(r,s),c) = \emptyset$ by Theorem 2.11 of Altay and Başar [5]. Hence, $\sigma_\delta(B(r,s),c) = \sigma(B(r,s),c)$.

(c) From Table 1.2

$$\sigma_{co}(B(r,s),c) = III_1\sigma(B(r,s),c) \cup III_2\sigma(B(r,s),c) \cup III_3\sigma(B(r,s),c),$$

$$III_1\sigma(B(r,s),c) \cup III_2\sigma(B(r,s),c) = \sigma_r(B(r,s),c)$$

and $III_3\sigma(B(r,s),c) = \emptyset$ from Theorems 2.11 and 2.13 of Altay and Başar [5], we have

$$\sigma_{co}(B(r,s),c) = \{\lambda \in \mathbb{C} : |\lambda - r| < |s|\} \cup \{r+s\}.$$

□

As a consequence of Proposition 2.1, we have:

Corollary 3.4. *The following results hold:*

(a) $\sigma_{ap}\left(B(r,s)^*, \ell_1\right) = \{\lambda \in \mathbb{C} : |\lambda - r| \leq |s|\}$.

(b) $\sigma_{\delta}\left(B(r,s)^*, \ell_1\right) = \{\lambda \in \mathbb{C} : |\lambda - r| \leq |s|\} \setminus \{r\}$.

(c) *[5, Theorem 2.12]* $\sigma_{p}\left(B(r,s)^*, \ell_1\right) = \{\lambda \in \mathbb{C} : |\lambda - r| < |s|\} \cup \{r+s\}$.

3.3 Subdivisions of the spectrum of $B(r,s)$ on ℓ_p, $(1 < p < \infty)$

In this subsection, we give the subdivisions of the spectrum of the generalized difference operator $B(r,s)$ over the sequence space ℓ_p, where $1 < p < \infty$.

Theorem 3.5. *The following results hold:*

(a) $\sigma_{ap}\left(B(r,s), \ell_p\right) = \{\lambda \in \mathbb{C} : |\lambda - r| \leq |s|\} \setminus \{r\}$.

(b) $\sigma_{\delta}\left(B(r,s), \ell_p\right) = \{\lambda \in \mathbb{C} : |\lambda - r| \leq |s|\}$.

(c) $\sigma_{co}\left(B(r,s), \ell_p\right) = \{\lambda \in \mathbb{C} : |\lambda - r| < |s|\}$.

Proof. (a) Since, $\sigma_{ap}\left(B(r,s), \ell_p\right) = \sigma\left(B(r,s), \ell_p\right) \setminus III_1\sigma\left(B(r,s), \ell_p\right)$, one can derive from Theorems 2.3 and 2.9 of Bilgiç and Furkan [6] that

$$\sigma_{ap}\left(B(r,s), \ell_p\right) = \{\lambda \in \mathbb{C} : |\lambda - r| \leq |s|\} \setminus \{r\}.$$

(b) Since $\sigma_{\delta}\left(B(r,s), \ell_p\right) = \sigma\left(B(r,s), \ell_p\right) \setminus I_3\sigma\left(B(r,s), \ell_p\right)$ from Table 1.2 and

$$I_3\sigma\left(B(r,s), \ell_p\right) = II_3\sigma\left(B(r,s), \ell_p\right) = III_3\sigma\left(B(r,s), \ell_p\right) = \emptyset$$

is obtained by Theorem 2.4 of Bilgiç and Furkan [6], one can therefore see that $\sigma_{\delta}\left(B(r,s), \ell_p\right) = \sigma\left(B(r,s), \ell_p\right)$.

(c) Since the equality

$$\sigma_{co}\left(B(r,s), \ell_p\right) = III_1\sigma\left(B(r,s), \ell_p\right) \cup III_2\sigma\left(B(r,s), \ell_p\right) \cup III_3\sigma\left(B(r,s), \ell_p\right)$$

holds from Table 1.2, it can be easily seen by Theorems 2.3 and 2.4 of Bilgiç and Furkan [6] that $\sigma_{co}(B(r,s), \ell_p)$ consists of $\lambda \in \mathbb{C}$ such that $|\lambda - r| < |s|$. ∎

As a consequence of Proposition 2.1, we also have the following:

Corollary 3.6. *Let* $p^{-1} + q^{-1} = 1$*. Then, the following results hold:*

(a) $\sigma_{ap}\left(B(r,s)^*, \ell_q\right) = \{\lambda \in \mathbb{C} : |\lambda - r| \leq |s|\}$.

(b) $\sigma_{\delta}\left(B(r,s)^*, \ell_q\right) = \{\lambda \in \mathbb{C} : |\lambda - r| \leq |s|\} \setminus \{r\}$.

(c) *[6, Theorem 2.5]* $\sigma_{p}\left(B(r,s)^*, \ell_q\right) = \{\lambda \in \mathbb{C} : |\lambda - r| < |s|\}$.

3.4 Subdivisions of the spectrum of $B(r,s)$ on bv_p, $(1<p<\infty)$

In this subsection, we give the subdivisions of the spectrum of the generalized difference operator $B(r,s)$ over the sequence space bv_p, where $1<p<\infty$. Since the subdivisions of the spectrum of the operator $B(r,s)$ on the sequence space bv_p can be derived by analogy to that space ℓ_p, we omit the detail and give the related results without proof.

Theorem 3.7. *The following results hold:*

(a) $\sigma_{ap}\left(B(r,s),bv_p\right)=\{\lambda\in\mathbb{C}:|\lambda-r|\leq|s|\}\setminus\{r\}$.

(b) $\sigma_{\delta}\left(B(r,s),bv_p\right)=\{\lambda\in\mathbb{C}:|\lambda-r|\leq|s|\}$.

(c) $\sigma_{co}\left(B(r,s),bv_p\right)=\{\lambda\in\mathbb{C}:|\lambda-r|<|s|\}$.

The following corollary is also a consequence of Proposition 2.1:

Corollary 3.8. *The following results hold:*

(a) $\sigma_{ap}\left(B(r,s)^*,bv_p^*\right)=\{\lambda\in\mathbb{C}:|\lambda-r|\leq|s|\}$.

(b) $\sigma_{\delta}\left(B(r,s)^*,bv_p^*\right)=\{\lambda\in\mathbb{C}:|\lambda-r|\leq|s|\}\setminus\{r\}$.

(c) [6, Theorem 3.4.(ii)] $\sigma_{p}\left(B(r,s)^*,bv_p^*\right)=\{\lambda\in\mathbb{C}:|\lambda-r|<|s|\}$.

Conclusion

There is a wide literature related with the spectrum and fine spectrum of certain linear operators represented by particular limitation matrices over some sequence spaces. As the new subdivisions of spectrum in the present paper, the concepts of the approximate point spectrum, defect spectrum and compression spectrum have been introduced and given the subdivisions of the spectrum of the generalized difference matrix $B(r,s)$ over the sequence spaces c_0, c, ℓ_p and bv_p, where $1<p<\infty$. This is a development of the spectrum of an infinite matrix over a sequence space. Following the same way, it is natural that one can derive some new results, on the subdivisions of the spectrum of $B(r,s)$ or other particular limitation matrices, for example the triple band matrix $B(r,s,t)$, over the spaces which do not consider here, from the known results via Table 1.2, in the usual sense.

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