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Strong Convergence Theorems for Generalized Equilibrium Problems, Variational Inequality and Fixed Point Problems of Asymptotically Strict Pseudocontractive Mappings in the Intermediate Sense

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Abstract : In this paper, we introduce two iterative schemes based on the extragradient method and hybrid projection method for finding a common element of the set of a generalized equilibrium problem, the set of solutions of the variational inequality problem for a γ-inverse strongly monotone mapping and the set of fixed points of an asymptotically κ-strict pseudocontractive mappings in the intermediate sense which is not necessarily Lipschitzian in a real Hilbert space. We prove that two sequences converge strongly to a common element of the above three sets under some parameters controlling conditions. Our results improve and extend the corresponding results announced by many others.

Keywords : Asymptotically κ-strict pseudocontractive mappings; Generalized equilibrium problem; Variational inequality; γ-inverse strongly monotone mapping; Fixed point; Strong convergence.
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1 Introduction and Preliminaries

Throughout this paper, we always assume that H is a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. Let $f : C\times C\to\mathbb{R}$ be a bifunction and $B : C\to H$ a monotone mapping. The generalized equilibrium problem (for short, GEP) for f and B is to find $u\in C$ such that

$$f(u,v)+\langle Bu, v-u\rangle\geq 0,\ \forall v\in C. \tag{1.1}$$

The set of solutions for the problem (1.1) is denoted by $GEP(f,B)$, i.e.,

$$GEP(f,B)=\{u\in C : f(u,v)+\langle Bu,v-u\rangle\geq 0,\forall v\in C\}.$$

If $B=0$ in (1.1), then GEP reduces to the classical equilibrium problem and $GEP(f,0)$ is denoted by $EP(f)$, i.e.,

$$EP(f)=\{u\in C : f(u,v)\geq 0,\forall v\in C\}.$$

If $f=0$ in (1.1), then GEP reduces to the classical variational inequality and $GEP(0,B)$ is denoted by $VI(C,B)$, i.e.,

$$VI(C,B)=\{u\in C : \langle Bu,v-u\rangle\geq 0,\forall v\in C\}.$$

It is easy to see that the following is true:

$$u\in VI(C,B)\Leftrightarrow u=P_C(u-\lambda Bu),\quad \lambda>0. \tag{1.2}$$

The problem (1.1) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, Min-max problems, the Nash equilibrium problems in noncooperative games and others; see, for example, Blum-Oettli [1] and Moudafi [2].

For solving the generalized equilibrium problem, let us assume that f satisfies the following conditions:

(A1) $f(x,x)=0$ for all $x\in C$;

(A2) f is monotone, that is, $f(x,y)+f(y,x)\leq 0$ for all $x,y\in C$;

(A3) for each $x,y,z\in C$, $\lim_{t\to 0} f(tz+(1-t)x,y)\leq f(x,y)$;

(A4) for each $x\in C$, the function $y\mapsto f(x,y)$ is convex and lower semicontinuous.

A mapping A of C into H is called monotone if

$$\langle Au-Av,u-v\rangle\geq 0$$

for all $u,v\in C$. A is called γ-inverse strongly monotone if there exists a positive real number γ such that

$$\langle Au-Av,u-v\rangle\geq\gamma\|Au-Av\|^2$$

for all $u, v \in C$. It is obvious that any γ-inverse strongly monotone mapping A is monotone and Lipschitz continuous.

It is well known that for every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|$$

for all $y \in C$. P_C is called the metric projection of H onto C. P_C is a nonexpansive mapping of H onto C and satisfies the following properties:

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2,\ \forall x, y \in H,$$

$$\|x - P_C x\|^2 \leq \|x - y\|^2 - \|y - P_C x\|^2,\ \forall x \in H,\ y \in C, \tag{1.3}$$

$$\|x - P_C y\|^2 \leq \|x - y\|^2 - \|y - P_C y\|^2,\ \forall x \in C,\ y \in H. \tag{1.4}$$

Moreover, given $x \in H$, $z \in C$, $z = P_C x$ if and only if

$$\langle x - z, y - z\rangle \leq 0,\ \forall y \in C. \tag{1.5}$$

Let $T : C \to C$ be a mapping. In this paper, we denote the fixed point set of T by $F(T)$. Recall that T is said to be uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x - y\|,\ \forall x, y \in C, \forall n \geq 1. \tag{1.6}$$

T is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|,\ \forall x, y \in C. \tag{1.7}$$

T is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\}$ in $[1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n\|x - y\|,\ \forall x, y \in C, \forall n \geq 1. \tag{1.8}$$

T is said to be asymptotically nonexpansive in the intermediate sense if it is continuous and the following inequality holds:

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \leq 0. \tag{1.9}$$

Observe that if we define

$$\tau_n = \max\left\{0, \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|)\right\},$$

then $\tau_n \to 0$ as $n \to \infty$. It follows that (1.9) is reduced to

$$\|T^n x - T^n y\| \leq \|x - y\| + \tau_n,\ \forall x, y \in C, \forall n \geq 1.$$

The class of mappings which are asymptotically nonexpansive in the intermediate sense was introduced by Bruck, Kuczumow and Reich [3]. It is worth mentioning that the class of mappings which are asymptotically nonexpansive in the intermediate sense contains properly the class of asymptotically nonexpansive mappings.

Recall that T is said to be a κ-strict pseudocontraction if there exists a constant $\kappa \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \kappa\|(I - T)x - (I - T)y\|^2, \ \forall x, y \in C. \tag{1.10}$$

T is said to be an asymptotically κ-strict pseudocontraction with sequence $\{\mu_n\}$ if there exist a constant $\kappa \in [0, 1)$ and a sequence $\{\mu_n\} \subset [0, \infty)$ with $\mu_n \to 0$ as $n \to \infty$ such that

$$\|T^n x - T^n y\|^2 \le (1 + \mu_n)\|x - y\|^2 + \kappa\|(I - T^n)x - (I - T^n)y\|^2, \ \forall x, y \in C, n \ge 1. \tag{1.11}$$

It is important to note that every asymptotically κ-strict pseudocontractive mapping with sequence $\{\mu_n\}$ is a uniformly L-Lipschitzian mapping with

$$L = \sup\left\{\frac{\kappa + \sqrt{1 + (1 - \kappa)\mu_n}}{1 + \kappa} : n \in N\right\}.$$

Recently, Sahu, Xu and Yao [4] introduced a class of new mappings: asymptotically κ-strict pseudocontractive mappings in the intermediate sense. Recall that T is said to be an asymptotically κ-strict pseudocontraction in the intermediate sense with sequence $\{\mu_n\}$ if there exist a constant $\kappa \in [0, 1)$ and a sequence $\{\mu_n\} \subset [0, \infty)$ with $\mu_n \to 0$ as $n \to \infty$ such that

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\|^2 - (1 + \mu_n)\|x - y\|^2 - \kappa\|(I - T^n)x - (I - T^n)y\|^2) \le 0. \tag{1.12}$$

Throughout this paper we assume that

$$c_n = \max\{0, \sup_{x,y\in C} (\|T^n x - T^n y\|^2 - (1 + \mu_n)\|x - y\|^2 - \kappa\|(I - T^n)x - (I - T^n)y\|^2)\}.$$

It follows that $c_n \to 0$ as $n \to \infty$ and (1.12) is reduced to the relation

$$\|T^n x - T^n y\|^2 \le (1 + \mu_n)\|x - y\|^2 + \kappa\|(I - T^n)x - (I - T^n)y\|^2 + c_n, \ \forall x, y \in C. \tag{1.13}$$

They studied the demiclosedness principle and obtained weak and strong convergence theorems of modified Mann iterative processes for the class of mappings which is not necessarily Lipschitzian; see [4] for more details.

Recently, many authors studied the problem of finding a common element of the set of fixed points of nonexpansive mappings or strict pseudocontractive mappings, the set of solutions of an equilibrium problem and the set of solutions of the variational inequality problem in the frame work of Hilbert spaces and Banach spaces respectively; see, for instance, [5–8] and the references therein.

For solving the variational inequality problem in the finite-dimensional Euclidean space R^n, Korpelevich [9] (1976) introduced the following so-called extragradient method:

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = P_C(x_n - \lambda A y_n), \ \ \forall n \geq 0, \end{cases} \tag{1.14}$$

In 2006, Nadezhkina and Takahashi [10] and Zeng and Yao [11] proposed some iterative schemes for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of a variational inequality problem by so-called extragradient method. Further, Ceng et al. [12] introduced and studied a relaxed extragradient method for finding solutions of a general system of variational inequalities with inverse-strongly monotone mappings in a real Hilbert space.

Motivated and inspired by the above works, in this paper, we introduce two iterative processes based on extragradient method and hybrid projection method for finding a common element of the set of a generalized equilibrium problem, the set of solutions of the variational inequality problem for a γ-inverse strongly monotone mapping and the set of fixed points of an asymptotically κ-strict pseudocontractive mappings in the intermediate sense in a real Hilbert space. We establish some strong convergence theorems for our iterative schemes.

In order to prove our main results, we also need the following lemmas.

Lemma 1.1 ([13])**.** *Let $(E, \langle \cdot, \cdot \rangle)$ be an inner product space. Then, for all $x, y, z \in E$ and $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma = 1$, we have*

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x - y\|^2 - \alpha\gamma\|x - z\|^2 - \beta\gamma\|y - z\|^2.$$

Lemma 1.2 ([4])**.** *Let C be a nonempty subset of a Hilbert space H and $T : C \to C$ a uniformly continuous asymptotically κ-strict pseudocontractive mapping in the intermediate sense with sequence $\{\mu_n\}$. Let $\{x_n\}$ be a sequence in C such that $\|x_n - x_{n+1}\| \to 0$ and $\|x_n - T^n x_n\| \to 0$ as $n \to \infty$. Then $\|x_n - T x_n\| \to 0$ as $n \to \infty$.*

Lemma 1.3 ([4, Proposition 3.1])**.** *Let C be a nonempty closed convex subset of a Hilbert space H and $T : C \to C$ a continuous asymptotically κ-strict pseudocontractive mapping in the intermediate sense. Then $I - T$ is demiclosed at zero in the sense that if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x \in C$ and $\limsup_{m\to\infty} \limsup_{n\to\infty} \|x_n - T^m x_n\| = 0$, then $(I - T)x = 0$.*

Lemma 1.4 ([14])**.** *Let C be a closed convex subset of a real Hilbert space H, let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1)-(A4)$, and let B be a monotone mapping from C into H. Then, for $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$f(z, y) + \langle Bx, y - z\rangle + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \ \forall y \in C.$$

Lemma 1.5 ([14])**.** *Let C be a closed convex subset of a real Hilbert space H. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1)-(A4)$ and let B be a monotone mapping from C into H. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \{z \in C : f(z, y) + \langle Bx, y - z\rangle + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall y \in C\}$$

for all $x \in H$. Then, the following hold:

(1) T_r is single-valued;

(2) T_r is firmly nonexpansive, i.e., for any $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(3) $F(T_r) = GEP(f, B)$;

(4) $GEP(f, B)$ is closed and convex;

(5) T_r is quasi-ϕ-nonexpansive;

(6) $\|T_r x - q\|^2 + \|T_r x - x\|^2 \leq \|x - q\|^2,\ \forall q \in F(T_r)$.

Lemma 1.6 ([4])**.** *Let C be a nonempty closed convex subset of a Hilbert space H and $T : C \to C$ a continuous asymptotically κ-strict pseudocontractive mapping in the intermediate sense. Then $F(T)$ is closed and convex.*

We denote by $N_C(v)$ the normal cone for $C \subset H$ at a point $v \in C$, that is $N_C(v) = \{x \in H : \langle v - y, x\rangle \geq 0,\ \forall y \in C\}$. We shall use the following lemma.

Lemma 1.7 ([15])**.** *Let C be a nonempty closed convex subset of a Banach space E and let A be a monotone and hemicontinuous operator of C into E^* with $C = D(A)$. Let $S \subset E \times E^*$ be an operator defined as follows:*

$$Sv = \begin{cases} Av + N_C(v), & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then S is maximal monotone and $S^{-1}(0) = VI(C, A)$.

2 Main Results

Theorem 2.1. *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ a uniformly continuous asymptotically κ-strict pseudocontractive mapping in the intermediate sense with sequence $\{\mu_n\}$. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1)$–$(A4)$ and B a continuous monotone mapping of C into H. Let A be a γ-inverse strongly monotone mapping of C into H such that*

$F = F(T)\bigcap GEP(f,B)\bigcap VI(C,A) \neq \emptyset$ and F is bounded. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in C generated by the following iterative process:

$$\begin{cases} x_1 \in C = C_1, \\ z_n = P_C(x_n - \lambda_n A x_n), \\ u_n \in C,\ f(u_n, y) + \langle Bz_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - z_n\rangle \geq 0,\ \ \forall y \in C, \\ y_n = \alpha_n u_n + \beta_n T^n u_n + \gamma_n P_C(z_n - \lambda_n A z_n), \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \beta_n \theta_n\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \geq 1, \end{cases} \tag{2.1}$$

where $\theta_n = c_n + \mu_n \cdot \Delta_n$, $\Delta_n = \sup\{\|x_n - p\|^2 : p \in F\} < \infty$. Assume that $\{r_n\} \subset [a, \infty)$ for some $a > 0$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $[0,1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ such that $\alpha_n \geq \eta > \kappa$, $\beta_n \geq \zeta > 0$ and $\{\lambda_n\}$ is a sequence in $(0, 2\gamma)$ such that $0 < s \leq \lambda_n < 2\gamma$. Then the sequence $\{x_n\}$ given by (2.1) converges strongly to $x^* \in F$, where $x^* = P_F x_1$.

Proof. Since $\mu_n \to 0$ and $c_n \to 0$, we get $\theta_n \to 0$ as $n \to \infty$. For any x, $y \in C$ and $\lambda_n \in (0, 2\gamma)$, we note that

$$\begin{aligned} &\|(I - \lambda_n A)x - (I - \lambda_n A)y\|^2 \\ =&\|x - y - \lambda_n(Ax - Ay)\|^2 \\ =&\|x - y\|^2 - 2\lambda_n\langle x - y, Ax - Ay\rangle + \lambda_n^2\|Ax - Ay\|^2 \\ \leq&\|x - y\|^2 + \lambda_n(\lambda_n - 2\gamma)\|Ax - Ay\|^2 \\ \leq&\|x - y\|^2, \end{aligned}$$

which implies that $I - \lambda_n A$ is nonexpansive. For any $p \in F$, from the definition of T_r we have $u_n = T_{r_n} z_n$. It follows that

$$\begin{aligned} \|u_n - p\| = \|T_{r_n} z_n - p\| \leq \|z_n - p\| &= \|P_C(x_n - \lambda_n A x_n) - p\| \\ &\leq \|(I - \lambda_n A)x_n - (I - \lambda_n A)p\| \\ &\leq \|x_n - p\|. \end{aligned} \tag{2.2}$$

Next, we divide the proof of Theorem 2.1 into eight steps.

Step 1. C_n is closed and convex for each $n \geq 1$.

By the assumption, we see that $C_1 = C$ is closed and convex. Suppose that C_n is closed and convex for some integer $n > 1$. Next, we show that C_{n+1} is closed and convex. For any $z \in C_n$ such that

$$\|y_n - z\|^2 \leq \|x_n - z\|^2 + \beta_n \theta_n.$$

This inequality is equivalent to the inequality:

$$2\langle x_n - y_n, z\rangle \leq \|x_n\|^2 - \|y_n\|^2 + \beta_n \theta_n.$$

It is easy to see that C_{n+1} is closed and convex. Then, for all $n \in N$, C_n is closed and convex.

Step 2. $F \subset C_n$ for each $n \geq 1$.

This can be proved by induction on $n \in N$. Indeed, for $n = 1$, we have $F \subset C = C_1$. Suppose that $F \subset C_n$ for some $n \geq 0$. Let $p \in F$. From $p = P_C(I - \lambda_n A)p$, Lemma 1.1 and (2.2), we have

$$
\begin{aligned}
&\|y_n - p\|^2 \\
=&\|\alpha_n(u_n - p) + \beta_n(T^n u_n - p) + \gamma_n(P_C(I - \lambda_n A)z_n - p)\|^2 \\
=&\alpha_n\|u_n - p\|^2 + \beta_n\|T^n u_n - p\|^2 + \gamma_n\|P_C(I - \lambda_n A)z_n - p\|^2 \\
&- \alpha_n\beta_n\|u_n - T^n u_n\|^2 - \alpha_n\gamma_n\|u_n - P_C(I - \lambda_n A)z_n\|^2 \\
&- \beta_n\gamma_n\|T^n u_n - P_C(I - \lambda_n A)z_n\|^2 \\
=&\alpha_n\|u_n - p\|^2 + \beta_n\|T^n u_n - p\|^2 + \gamma_n\|P_C(I - \lambda_n A)z_n - P_C(I - \lambda_n A)p\|^2 \\
&- \alpha_n\beta_n\|u_n - T^n u_n\|^2 - \alpha_n\gamma_n\|u_n - P_C(I - \lambda_n A)z_n\|^2 \\
&- \beta_n\gamma_n\|T^n u_n - P_C(I - \lambda_n A)z_n\|^2 \\
\leq&\alpha_n\|u_n - p\|^2 + \beta_n((1 + \mu_n)\|u_n - p\|^2 + \kappa\|u_n - T^n u_n\|^2 + c_n) \\
&+ \gamma_n\|z_n - p\|^2 - \alpha_n\beta_n\|u_n - T^n u_n\|^2 - \alpha_n\gamma_n\|u_n - P_C(I - \lambda_n A)z_n\|^2 \\
\leq&(\alpha_n + \beta_n + \beta_n\mu_n)\|u_n - p\|^2 + \beta_n c_n + \gamma_n\|x_n - p\|^2 \\
&- \beta_n(\alpha_n - \kappa)\|u_n - T^n u_n\|^2 - \alpha_n\gamma_n\|u_n - P_C(I - \lambda_n A)z_n\|^2 \\
\leq&\|x_n - p\|^2 + \beta_n\mu_n\|x_n - p\|^2 + \beta_n c_n - \beta_n(\alpha_n - \kappa)\|u_n - T^n u_n\|^2 \\
&- \alpha_n\gamma_n\|u_n - P_C(I - \lambda_n A)z_n\|^2 \\
\leq&\|x_n - p\|^2 + \beta_n\mu_n\|x_n - p\|^2 + \beta_n c_n \\
\leq&\|x_n - p\|^2 + \beta_n\theta_n,
\end{aligned}
\tag{2.3}
$$

where $\theta_n = c_n + \mu_n \cdot \Delta_n$ and $\Delta_n = \sup\{\|x_n - p\|^2 : p \in F\} < \infty$ for each $n \geq 1$. This shows that $p \in C_{n+1}$ and $F \subset C_{n+1}$. Hence $F \subset C_n$ for each $n \geq 1$. This means that the iterative algorithm (2.1) is well defined.

Step 3. $\lim_{n\to\infty} \|x_n - x_1\|$ exists and $\{x_n\}$ is bounded.

Noticing that $x_n = P_{C_n}x_1$ and (1.3), we have

$$\|x_n - x_1\|^2 \leq \|x_1 - p\|^2 - \|x_n - p\|^2 \leq \|x_1 - p\|^2,$$

which implies that $\|x_n - x_1\| \leq \|x_1 - p\|$ for all $p \in F$ and $n \geq 1$. This shows that the sequence $\{\|x_n - x_1\|\}$ is bounded. From $x_n = P_{C_n}x_1$ and $x_{n+1} = P_{C_{n+1}}x_1 \in C_{n+1} \subset C_n$, we obtain that

$$\|x_n - x_1\| \leq \|x_{n+1} - x_1\|, \ \forall n \geq 1.$$

It follows that $\{\|x_n - x_1\|\}$ is nondecreasing. Therefore, $\lim_{n\to\infty} \|x_n - x_1\|$ exists and $\{x_n\}$ is bounded.

Step 4. $x_{n+1} - x_n \to 0$ and $x_n \to x^* \in C$.

For any positive integer $m \geq n$, we know $x_m = P_{C_m} x_1 \in C_m \subset C_n$. It follows from (1.4) that

$$\|x_m - x_n\|^2 = \|x_m - P_{C_n} x_1\|^2 \leq \|x_m - x_1\|^2 - \|x_1 - P_{C_n} x_1\|^2 = \|x_m - x_1\|^2 - \|x_n - x_1\|^2.$$

In view of step 3 we deduce that $\|x_m - x_n\| \to 0$ as $m, n \to \infty$. Hence $\{x_n\}$ is a Cauchy sequence of C and

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{2.4}$$

Since H is a real Hilbert space and C is a closed subset of H, there exists a point $x^* \in C$ such that $x_n \to x^*$ as $n \to \infty$.

Step 5. $x^* \in F(T)$.

Noticing that $x_{n+1} \in C_{n+1}$, we obtain

$$\|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + \beta_n \theta_n.$$

From (2.4) and $\theta_n \to 0$ we have

$$\lim_{n\to\infty} \|y_n - x_{n+1}\| = 0.$$

Furthermore, it follows from $\|x_n - y_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\|$ that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{2.5}$$

For any $p \in F$, we have

$$\begin{aligned}\|u_n - p\|^2 = \|T_{r_n} z_n - T_{r_n} p\|^2 &\leq \langle T_{r_n} z_n - T_{r_n} p, z_n - p\rangle \\ &= \langle u_n - p, z_n - p\rangle \\ &= \frac{1}{2}(\|u_n - p\|^2 + \|z_n - p\|^2 - \|u_n - z_n\|^2)\end{aligned}$$

and hence

$$\|u_n - p\|^2 \leq \|z_n - p\|^2 - \|u_n - z_n\|^2 \leq \|x_n - p\|^2 - \|u_n - z_n\|^2.$$

In view of (2.3), we obtain

$$\begin{aligned}&\|y_n - p\|^2 \\ \leq&(\alpha_n + \beta_n + \beta_n \mu_n)\|u_n - p\|^2 + \beta_n c_n + \gamma_n \|x_n - p\|^2 \\ \leq&(\alpha_n + \beta_n)\|u_n - p\|^2 + \beta_n \theta_n + \gamma_n \|x_n - p\|^2 \\ \leq&(\alpha_n + \beta_n)(\|x_n - p\|^2 - \|u_n - z_n\|^2) + \beta_n \theta_n + \gamma_n \|x_n - p\|^2 \\ =&\|x_n - p\|^2 - (\alpha_n + \beta_n)\|u_n - z_n\|^2 + \beta_n \theta_n.\end{aligned} \tag{2.6}$$

It follows from the assumption conditions $\alpha_n \geq \eta > \kappa$ and $\beta_n \geq \zeta > 0$ that

$$\begin{aligned}(\eta + \zeta)\|u_n - z_n\|^2 &\leq (\alpha_n + \beta_n)\|u_n - z_n\|^2 \\ &\leq \|x_n - p\|^2 - \|y_n - p\|^2 + \beta_n \theta_n \\ &= (\|x_n - p\| - \|y_n - p\|)(\|x_n - p\| + \|y_n - p\|) + \beta_n \theta_n \\ &\leq \|x_n - y_n\|(\|x_n - p\| + \|y_n - p\|) + \beta_n \theta_n.\end{aligned}$$

From (2.5) we arrive at

$$\lim_{n\to\infty}\|u_n - z_n\| = 0. \tag{2.7}$$

Using (2.3), we have

$$\|y_n - p\|^2 \le \|x_n - p\|^2 + \beta_n\mu_n\|x_n - p\|^2 + \beta_n c_n - \beta_n(\alpha_n - \kappa)\|u_n - T^n u_n\|^2.$$

It follows that

$$\begin{aligned}\zeta(\eta-\kappa)\|u_n - T^n u_n\|^2 &\le \beta_n(\alpha_n - \kappa)\|u_n - T^n u_n\|^2\\ &\le \|x_n - p\|^2 - \|y_n - p\|^2 + \beta_n\theta_n\\ &\le \|x_n - y_n\|(\|x_n - p\| + \|y_n - p\|) + \beta_n\theta_n.\end{aligned}$$

and hence

$$\lim_{n\to\infty}\|u_n - T^n u_n\| = 0. \tag{2.8}$$

Using (2.3) again, we obtain

$$\begin{aligned}&\|y_n - p\|^2\\ \le&\|x_n - p\|^2 + \beta_n\theta_n - \beta_n(\alpha_n - \kappa)\|u_n - T^n u_n\|^2 - \alpha_n\gamma_n\|u_n - P_C(I - \lambda_n A)z_n\|^2\\ \le&\|x_n - p\|^2 + \beta_n\theta_n - \alpha_n\gamma_n\|u_n - P_C(I - \lambda_n A)z_n\|^2\end{aligned}$$

It follows that

$$\begin{aligned}\eta\gamma_n\|u_n - P_C(I - \lambda_n A)z_n\|^2 &\le \alpha_n\gamma_n\|u_n - P_C(I - \lambda_n A)z_n\|^2\\ &\le \|x_n - p\|^2 - \|y_n - p\|^2 + \beta_n\theta_n\\ &\le \|x_n - y_n\|(\|x_n - p\| + \|y_n - p\|) + \beta_n\theta_n\end{aligned}$$

which implies that

$$\lim_{n\to\infty}\gamma_n\|u_n - P_C(I - \lambda_n A)z_n\|^2 = 0.$$

So,

$$\lim_{n\to\infty}\sqrt{\gamma_n}\|u_n - P_C(I - \lambda_n A)z_n\| = 0. \tag{2.9}$$

Since $y_n = \alpha_n u_n + \beta_n T^n u_n + \gamma_n P_C(z_n - \lambda_n A z_n)$, we have

$$\begin{aligned}\|y_n - u_n\| &= \|\beta_n(T^n u_n - u_n) + \gamma_n(P_C(z_n - \lambda_n A z_n) - u_n)\|\\ &\le \beta_n\|T^n u_n - u_n\| + \sqrt{\gamma_n}\sqrt{\gamma_n}\|P_C(z_n - \lambda_n A z_n) - u_n)\|.\end{aligned}$$

It follows from (2.8) and (2.9) that

$$\lim_{n\to\infty}\|y_n - u_n\| = 0. \tag{2.10}$$

Noticing that $\|x_n - z_n\| \le \|x_n - y_n\| + \|y_n - u_n\| + \|u_n - z_n\|$, we obtain

$$\lim_{n\to\infty}\|x_n - z_n\| = 0, \tag{2.11}$$

and hence $z_n \to x^*$ as $n \to \infty$. Combining (2.7) and (2.11) we get

$$\lim_{n\to\infty} \|x_n - u_n\| = 0, \tag{2.12}$$

which implies $u_n \to x^*$ as $n \to \infty$. Since

$$\|u_{n+1} - u_n\| \le \|u_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| + \|x_n - u_n\|,$$

it follows from (2.4) and (2.12) that

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0. \tag{2.13}$$

Note that T is uniformly continuous, from (2.8), (2.13) and Lemma 1.2 we obtain that $\|u_n - Tu_n\| \to 0$ as $n \to \infty$. Moreover, we see that $\|u_n - T^m u_n\| \to 0$ for any $m \in N$. By Lemma 1.3 we obtain $x^* \in F(T)$.

Step 6. $x^* \in VI(C, A)$.

Since A is Lipschitz continuous, from $x_n - z_n \to 0$ we have

$$\lim_{n\to\infty} \|Ax_n - Az_n\| = 0. \tag{2.14}$$

Let

$$Sv = \begin{cases} Av + N_C(v), & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

By Lemma 1.7, S is maximal monotone and $S^{-1}(0) = VI(C, A)$. Let $(v, w) \in G(S)$. Since $w \in Sv = Av + N_C(v)$, we have $w - Av \in N_C(v)$. It follows from $z_n \in C$ that

$$\langle v - z_n, w - Av\rangle \ge 0. \tag{2.15}$$

On the other hand, from $z_n = P_C(x_n - \lambda_n Ax_n)$ we obtain that

$$\langle v - z_n, z_n - (x_n - \lambda_n Ax_n)\rangle \ge 0,$$

and hence

$$\langle v - z_n, \frac{x_n - z_n}{\lambda_n} - Ax_n\rangle \le 0. \tag{2.16}$$

Then, from (2.15) and (2.16), we have

$$\begin{aligned}
\langle v - z_n, w\rangle &\ge \langle v - z_n, Av\rangle \\
&\ge \langle v - z_n, Av\rangle + \langle v - z_n, \frac{x_n - z_n}{\lambda_n} - Ax_n\rangle \\
&= \langle v - z_n, Av - Ax_n + \frac{x_n - z_n}{\lambda_n}\rangle \\
&= \langle v - z_n, Av - Az_n\rangle + \langle v - z_n, Az_n - Ax_n\rangle + \langle v - z_n, \frac{x_n - z_n}{\lambda_n}\rangle \\
&\ge -\|v - z_n\| \cdot \|Az_n - Ax_n\| - \|v - z_n\| \cdot \frac{\|x_n - z_n\|}{s}.
\end{aligned}$$

Hence we have $\langle v - x^*, w\rangle \geq 0$ as $n \to \infty$. Since S is maximal monotone, we have $x^* \in S^{-1}(0)$ and hence $x^* \in VI(C, A)$.

Step 7. $x^* \in GEP(f, B) = F(T_r)$.

Since $u_n = T_{r_n} z_n$, we obtain that

$$f(u_n, y) + \langle Bz_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - z_n\rangle \geq 0, \quad \forall y \in C.$$

From $(A2)$, we have

$$\langle Bz_n, y - u_n\rangle + \langle y - u_n, \frac{u_n - z_n}{r_n}\rangle \geq -f(u_n, y) \geq f(y, u_n), \ \forall y \in C. \tag{2.17}$$

For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)x^*$. Since $y \in C$ and $x^* \in C$, we have $y_t \in C$. From (2.17) we obtain that

$$\begin{aligned}
&\langle By_t, y_t - u_n\rangle \\
\geq &\langle By_t, y_t - u_n\rangle - \langle Bz_n, y_t - u_n\rangle - \langle y_t - u_n, \frac{u_n - z_n}{r_n}\rangle + f(y_t, u_n) \\
= &\langle By_t - Bu_n, y_t - u_n\rangle + \langle Bu_n - Bz_n, y_t - u_n\rangle - \langle y_t - u_n, \frac{u_n - z_n}{r_n}\rangle + f(y_t, u_n).
\end{aligned}$$

By the continuity of B and the fact that z_n, $u_n \to x^*$ as $n \to \infty$, we know that $Bu_n - Bz_n \to 0$ as $n \to \infty$. Since B is monotone, we obtain that $\langle By_t - Bu_n, y_t - u_n\rangle \geq 0$. Thus, it follows from (2.7), $(A4)$ and the assumption $r_n \geq a$ that

$$f(y_t, x^*) \leq \liminf_{n\to\infty} f(y_t, u_n) \leq \lim_{n\to\infty}\langle By_t, y_t - u_n\rangle = \langle By_t, y_t - x^*\rangle.$$

Now, from $(A1)$ and $(A4)$ we have

$$\begin{aligned}
0 = f(y_t, y_t) &\leq tf(y_t, y) + (1-t)f(y_t, x^*) \\
&\leq tf(y_t, y) + (1-t)\langle By_t, y_t - x^*\rangle \\
&\leq tf(y_t, y) + (1-t)t\langle By_t, y - x^*\rangle,
\end{aligned}$$

and hence $f(y_t, y) + (1-t)\langle By_t, y - x^*\rangle \geq 0$. Letting $t \to 0$, from $(A3)$, we have $f(x^*, y) + \langle Bx^*, y - x^*\rangle \geq 0$ for all $y \in C$. This implies that $x^* \in GEP(f, B)$. Therefore, in view of steps 5, 6 we have $x^* \in F$.

Step 8. $x^* = P_F x_1$.

From $x_n = P_{C_n} x_1$, we get

$$\langle x_n - z, x_1 - x_n\rangle \geq 0, \ \forall z \in C_n.$$

Since $F \subset C_n$ for all $n \geq 1$, we arrive at

$$\langle x_n - p, x_1 - x_n\rangle \geq 0, \ \forall p \in F.$$

Letting $n \to \infty$, we have

$$\langle x^* - p, x_1 - x^*\rangle \geq 0, \ \forall p \in F$$

and hence $x^* = P_F x_1$. This completes the proof. $\square$

Corollary 2.2. *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ a uniformly continuous asymptotically κ-strict pseudocontractive mapping in the intermediate sense with sequence $\{\mu_n\}$. Let A be a γ-inverse strongly monotone mapping of C into H such that $F = F(T)\bigcap VI(C, A) \neq \emptyset$ and F is bounded. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in C generated by the following iterative process:*

$$\begin{cases} x_1 \in C = C_1, \\ z_n = P_C(x_n - \lambda_n A x_n), \\ y_n = \alpha_n z_n + \beta_n T^n z_n + \gamma_n P_C(z_n - \lambda_n A z_n), \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \beta_n \theta_n\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \geq 1, \end{cases} \tag{2.18}$$

where $\theta_n = c_n + \mu_n \cdot \Delta_n$, $\Delta_n = \sup\{\|x_n - p\|^2 : p \in F\} < \infty$. Assume that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $[0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ such that $\alpha_n \geq \eta > \kappa$, $\beta_n \geq \zeta > 0$ and $\{\lambda_n\}$ is a sequence in $(0, 2\gamma)$ such that $0 < s \leq \lambda_n < 2\gamma$. Then the sequence $\{x_n\}$ given by (2.18) converges strongly to $x^* \in F$, where $x^* = P_F x_1$.

Corollary 2.3. *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ a uniformly continuous asymptotically κ-strict pseudocontractive mapping in the intermediate sense with sequence $\{\mu_n\}$. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4) such that $F = F(T)\bigcap EP(f) \neq \emptyset$ and F is bounded. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in C generated by the following iterative process:*

$$\begin{cases} x_1 \in C = C_1, \\ u_n \in C, \ f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \ \forall y \in C, \\ y_n = \alpha_n u_n + (1 - \alpha_n) T^n u_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \beta_n \theta_n\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \geq 1, \end{cases} \tag{2.19}$$

where $\theta_n = c_n + \mu_n \cdot \Delta_n$, $\Delta_n = \sup\{\|x_n - p\|^2 : p \in F\} < \infty$. Assume that $\{r_n\} \subset [a, \infty)$ for some $a > 0$ and $\{\alpha_n\}$ is sequence in $[0, 1]$ such that $1 > \zeta \geq \alpha_n \geq \eta > \kappa$. Then the sequence $\{x_n\}$ given by (2.19) converges strongly to $x^* \in F$, where $x^* = P_F x_1$.

As the proof of Theorem 2.1, we can prove the following strong convergence theorem for generalized equilibrium problem, the variational inequality problem for a γ-inverse strongly monotone mapping and an asymptotically κ-strict pseudo-contractive mappings in the intermediate sense by using of routine method.

Theorem 2.4. *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ a uniformly continuous asymptotically κ-strict pseudocontractive mapping in the intermediate sense with sequence $\{\mu_n\}$. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4) and B a continuous monotone mapping of C into H. Let A be a γ-inverse strongly monotone mapping of C into H such that*

$F = F(T)\bigcap GEP(f,B)\bigcap VI(C,A) \neq \emptyset$ and F is bounded. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in C generated by the following iterative process:

$$\begin{cases} x_1 \in C, \\ z_n = P_C(x_n - \lambda_n A x_n), \\ u_n \in C,\ f(u_n, y) + \langle Bz_n, y-u_n\rangle + \frac{1}{r_n}\langle y-u_n, u_n - z_n\rangle \geq 0,\ \ \forall y \in C, \\ y_n = \alpha_n u_n + \beta_n T^n u_n + \gamma_n P_C(z_n - \lambda_n A z_n), \\ H_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \beta_n\theta_n\}, \\ W_n = \{z \in C : \langle x_n - z, x_1 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{H_n\cap W_n} x_1, \quad \forall n \geq 1, \end{cases} \tag{2.20}$$

where $\theta_n = c_n + \mu_n \cdot \Delta_n$, $\Delta_n = \sup\{\|x_n - p\|^2 : p \in F\} < \infty$. Assume that $\{r_n\} \subset [a,\infty)$ for some $a > 0$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $[0,1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ such that $\alpha_n \geq \eta > \kappa$, $\beta_n \geq \zeta > 0$ and $\{\lambda_n\}$ is a sequence in $(0, 2\gamma)$ such that $0 < s \leq \lambda_n < 2\gamma$. Then the sequence $\{x_n\}$ given by (2.20) converges strongly to $x^* \in F$, where $x^* = P_F x_1$.

Proof. It is obvious that $H_n \cap W_n$ is closed and convex for each $n \geq 1$. Now we show that $F \subset H_n \cap W_n$ for all $n \geq 1$. Note that H_n is actually C_{n+1} in Theorem 2.1, so we have $F \subset H_n$ for all $n \geq 1$. Next we show by induction that $F \subset H_n \cap W_n$. From $W_1 = C$, we have $F \subset H_1 \cap W_1$. Assume that $F \subset H_k \cap W_k$ for some $k \geq 1$. Then there exist a $x_{k+1} \in H_k \cap W_k$ such that

$$x_{k+1} = P_{H_k\cap W_k} x_1.$$

Since $F \subset H_k \cap W_k$, from the definition of x_{k+1} and (1.5), for all $p \in F$ we have

$$\langle x_{k+1} - p, x_1 - x_{k+1}\rangle \geq 0,$$

and hence $p \in W_{k+1}$. So we have $F \subset W_{k+1}$. Therefore we get $F \subset H_{K+1} \cap W_{k+1}$. Thus we prove that $F \subset H_n \cap W_n$ for all $n \geq 1$. This means that the iterative algorithm (2.20) is well defined.

From the definition of W_n, we know that

$$\langle x_n - z, x_1 - x_n\rangle \geq 0,\ \forall z \in W_n.$$

So by (1.5) we have $x_n = P_{W_n} x_1$. If we instead C_n by W_n and C_{n+1} by H_n in the proof of Theorem 2.1 and notice that $x_{n+1} = P_{H_n\cap W_n} x_1 \in W_n$, we have

$$\begin{aligned} \lim_{n\to\infty} \|x_{n+1} - x_n\| &= \lim_{n\to\infty} \|x_n - y_n\| = \lim_{n\to\infty} \|y_n - z_n\| \\ &= \lim_{n\to\infty} \|z_n - u_n\| = \lim_{n\to\infty} \|u_{n+1} - u_n\| = \lim_{n\to\infty} \|u_n - T^n u_n\| = 0. \end{aligned}$$

Thus the proof that $\{x_n\}$ converges strongly to $P_F x_1$ follows on the lines of Theorem 2.1. $\square$

Corollary 2.5. *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ a uniformly continuous asymptotically κ-strict pseudocontractive mapping in the intermediate sense with sequence $\{\mu_n\}$. Let A be a γ-inverse strongly monotone mapping of C into H such that $F = F(T) \bigcap VI(C, A) \neq \emptyset$ and F is bounded. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in C generated by the following iterative process:*

$$\begin{cases} x_1 \in C \\ z_n = P_C(x_n - \lambda_n A x_n), \\ y_n = \alpha_n z_n + \beta_n T^n z_n + \gamma_n P_C(z_n - \lambda_n A z_n), \\ H_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \beta_n \theta_n\}, \\ W_n = \{z \in C : \langle x_n - z, x_1 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{H_n \cap W_n} x_1, \quad \forall n \geq 1, \end{cases} \tag{2.21}$$

where $\theta_n = c_n + \mu_n \cdot \Delta_n$, $\Delta_n = \sup\{\|x_n - p\|^2 : p \in F\} < \infty$. Assume that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $[0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ such that $\alpha_n \geq \eta > \kappa$, $\beta_n \geq \zeta > 0$ and $\{\lambda_n\}$ is a sequence in $(0, 2\gamma)$ such that $0 < s \leq \lambda_n < 2\gamma$. Then the sequence $\{x_n\}$ given by (2.21) converges strongly to $x^* \in F$, where $x^* = P_F x_1$.

Corollary 2.6. *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ a uniformly continuous asymptotically κ-strict pseudocontractive mapping in the intermediate sense with sequence $\{\mu_n\}$. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4) such that $F = F(T) \bigcap EP(f) \neq \emptyset$ and F is bounded. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in C generated by the following iterative process:*

$$\begin{cases} x_1 \in C \\ u_n \in C, \ f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \ \forall y \in C, \\ y_n = \alpha_n u_n + (1 - \alpha_n) T^n u_n, \\ H_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \beta_n \theta_n\}, \\ W_n = \{z \in C : \langle x_n - z, x_1 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{H_n \cap W_n} x_1, \quad \forall n \geq 1, \end{cases} \tag{2.22}$$

where $\theta_n = c_n + \mu_n \cdot \Delta_n$, $\Delta_n = \sup\{\|x_n - p\|^2 : p \in F\} < \infty$. Assume that $\{r_n\} \subset [a, \infty)$ for some $a > 0$ and $\{\alpha_n\}$ is sequence in $[0, 1]$ such that $1 > \zeta \geq \alpha_n \geq \eta > \kappa$. Then the sequence $\{x_n\}$ given by (2.22) converges strongly to $x^* \in F$, where $x^* = P_F x_1$.

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References

[1] E. Blum, W. Oettli, From optimization and variational inequalities to equilibrium problems, Math. Student 63 (1994) 123–145.

[2] A. Moudafi, Second-order differential proximal methods for equilibrium problems, J. Inequal. Pure Appl. Math. 4 (2003) 1–7.

[3] R.E. Bruck, T. Kuczumow, S. Reich, Convergence of iterates of asymptotically nonexpansive mappings in Banach spaces with the uniform Opial property, Colloq. Math. 65 (1993) 169–179.

[4] D.R. Sahu, H.K. Xu, J.C. Yao, Asymptotically strict pseudocontractive mappings in the intermediate sense, Nonlinear Anal. 70 (2009) 3502–3511.

[5] A. Tada, W. Takahashi, Weak and strong convergence theorems for a nonexpansive mapping and an equilibrium problem, J. Optim. Theory Appl. 133 (2007) 359–370.

[6] S. Takahashi, W. Takahashi, Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces, J. Math. Anal. Appl. 331 (2007) 506–515.

[7] C. Jaiboon, P. Kumam, Strong convergence theorems for solving equilibrium problems and fixed point problems of ξ-strict pseudo-contraction mappings by two hybrid projection methods, J. Comput. Appl. Math. 234 (2010) 722–732.

[8] Y. Yao, Y.C. Liou, S.M. Kang, Approach to common elements of variational inequality problems and fixed point problems via a relaxed extragradient method, Comput. Math. Appl. 59 (2010) 3472–3480.

[9] G.M. Korpelevich, An extragradient method for finding saddle points and for other problems, Ekon. Mat. Metody 12 (1976) 747–756.

[10] N. Nadezhkina, W. Takahashi, Weak convergence theorem by an extragradient method for nonexpansive mappings and monotone mappings, J. Optim. Theory Appl. 128 (2006) 191–201.

[11] L.C. Zeng, J.C. Yao, Strong convergence theorem by an extragradient method for fixed point problems and variational inequality problems, Taiwanese J. Math. 10 (2006) 1293–1303.

[12] L.C. Ceng, C. Wang, J.C. Yao, Strong convergence theorems by a relaxed extragradient method for a general system of variational inequalities, Math. Meth. Oper. Res. 67 (2008) 375–390.

[13] M.O. Osilike, D.I. Igbokwe, Weak and strong convergence theorems for fixed points of pseudocontractions and solutions of monotone type operator equations, Comput. Math. Appl. 40 (2000) 559–567.

[14] H. Zegeye, A hybrid iteration scheme for equilibrium problems, variational inequality problems and common fixed point problems in Banach spaces, Nonlinear Anal. 72 (2010) 2136–2146.

[15] R.T. Rockfellar, On the maximality of sums of nonlinear monotone operators, Trans. Amer. Math. Soc. 149 (1970) 75–88.

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