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Stability of a Mixed Type Cubic and Quartic Functional Equation in non-Archimedean ℓ-Fuzzy Normed Spaces

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Abstract : In this paper, we prove the generalized Hyres–Ulam–Rassias stability of the mixed type cubic and quartic functional equation

$$f(x+2y)+f(x-2y)=4(f(x+y)+f(x-y))-24f(y)-6f(x)+3f(2y)$$

in non-Archimedean ℓ-fuzzy normed spaces.

Keywords : Generalized Hyers–Ulam–Rassias stability; Cubic functional equation; Quartic functional equation; ℓ-fuzzy metric and normed spaces; Non-Archimedean ℓ-fuzzy normed spaces.
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1 Introduction

The stability problem of functional equations originated from a question of Ulam [1] in 1940, concerning the stability of group homomorphisms. We are

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looking for situations when the homomorphisms are stable, i.e., if a mapping is almost a homomorphism, then there exists a true homomorphism near it. The case of approximately additive mappings was solved by Hyers [2] under the assumption that G_1 and G_2 are Banach spaces. In 1978, a generalized version of the theorem of Hyers for approximately linear mappings was given by Rassias [3]. This new concept is known as Hyers–Ulam–Rassias stability of functional equations (see [4–22]).

Jun and Kim [7] introduced the following functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x) \tag{1.2}$$

and they established the general solution and the generalized Hyers–Ulam–Rassias stability for the functional equation (1.2). The function $f(x)=x^3$ satisfies the functional equation (1.2), which is thus called a cubic functional equation. Every solution of the cubic functional equation is said to be a cubic function. Jun and Kim proved that a function f between real vector spaces X and Y is a solution of (1.2) if and only if there exists a unique function $C : X\times X\times X \to Y$ such that $f(x)=C(x,x,x)$ for all $x\in X$, and C is symmetric for each fixed one variable and is additive for fixed two variables. For more detailed definitions of such terminologies, we can refer to [17–58].

Rassias [8, 9] studied the stability of quartic functional equations. In the following Park [59] studied the quartic functional equation

$$f(x+2y)+f(x-2y)=4(f(x+y)+f(x-y))+24f(y)-6f(x). \tag{1.3}$$

In fact they proved that a function f between real vector spaces X and Y is a solution of (1.3) if and only if there exists a unique symmetric multi-additive function $Q : X\times X\times X\times X \to Y$ such that $f(x)=Q(x,x,x,x)$ for all $x\in X$. It is easy to show that the function $f(x)=x^4$ satisfies the functional equation (1.3), which is called a quartic functional equation and every solution of the quartic functional equation is said to be a quartic function.

The theory of fuzzy sets was introduced by Zadeh [60] in 1965. After the pioneering work of Zadeh, there has been a great effort to obtain fuzzy analogues of classical theories. Among other fields, a progressive development is made in the field of fuzzy topology [61–64]. Saadati and Park [65] introduced and studied the concept of intuitionistic fuzzy normed spaces (see also [20]). The pioneering work of Zadeh provided some influence to several mathematicians to study fuzzy analogues of classical theories connected with functional equations in the framework of mathematical analysis.

A triangular norm (shortly, t-norm) is a binary operation $T : [0,1]\times[0,1]\to[0,1]$ which is commutative, associative, monotone and has 1 as the unit element. A t-norm T can be extended (by associativity) in a unique way to an n-ary operation taking, for all $(x_1,...,x_n)\in[0,1]^n$, the value $T(x_1,...,x_n)$ defined by

$$T_{i=1}^{0}x_i=1,\quad T_{i=1}^{n}x_i=T\left(T_{i=1}^{n-1}x_i,x_n\right)=T(x_1,...,x_n).$$

A t-norm T can also be extended to a countable operation taking, for any sequence $\{x_n\}_{n\in N}$ in $[0,1]$, the value

$$T_{i=1}^{\infty}x_i = \lim_{n\to\infty} T_{i=1}^{n}x_i.$$

Definition 1.1 ([66])**.** *Let $\ell = (L, \leq_L)$ be a complete lattice and let U be a nonempty set called the universe. An ℓ-fuzzy set in U is defined as a mapping $A : U \to L$. For each u in U, $A(u)$ represents the degree (in L) to which u is an element of U.*

Consider the set L^* and operation $\leq_{L^*}$ defined by

$$L^* = \{(x_1, x_2) :\ (x_1, x_2) \in [0,1]^2 \text{ and } x_1 + x_2 \leq 1\},$$

$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \Longleftrightarrow x_1 \leq y_1, x_2 \geq y_2$$

for all $(x_1, x_2), (y_1, y_2) \in L^*$. Then $(L^*, \leq_{L^*})$ is a complete lattice (see [67]).

Definition 1.2. *A triangular norm (t-norm) on L is a mapping $T : L^2 \to L$ satisfying the following conditions:*

(1) $T(x, 1_L) = x$ for all $x \in L$; (boundary condition).

(2) $T(x, y) = T(y, x)$ for all $(x, y) \in L^2$; (commutativity).

(3) $T(x, T(y, z)) = T(T(x, y), z)$ for all $(x, y, z) \in L^3$; (associativity).

(4) $x \leq_L x', y \leq_L y' \Longrightarrow T(x, y) \leq_L T(x', y')$ for all $(x, x', y, y') \in L^4$; (monotonicity).

A t-norm T on ℓ is said to be continuous if, for any $x, y \in \ell$ and any sequences $\{x_n\}$ and $\{y_n\}$ which converge to x and y, respectively,

$$\lim_{n\to\infty} T(x_n, y_n) = T(x, y).$$

A t-norm T can also be defined recursively as an $(n+1)$-ary operation ($n \in N$) by $T^1 = T$ and

$$T^n(x_1, ..., x_{n+1}) = T\left(T^{n-1}(x_1, ..., x_n), x_{n+1}\right)$$

for all $n \geq 2$ and $x_i \in L$.

Definition 1.3.

(1) A negator on ℓ is any decreasing mapping $N : L \to L$ satisfying $N(0_L) = 1_L$ and $N(1_L) = 0_L$.

(2) If $N(N(x)) = x$ for all $x \in L$, then N is called an involutive negator.

(3) The negator N_s on $([0,1], \leq)$ defined as $N_s(x) = 1 - x$ for all $x \in [0,1]$ is called the standard negator on $([0,1], \leq)$.

Definition 1.4. *The triple (X, M, T) is said to be an ℓ-fuzzy metric space if X is an arbitrary (non-empty) set, T is a continuous t-norm on L and M is an ℓ-fuzzy set on $X^2\times]0, +\infty[$ satisfying the following conditions: for all $x, y, z \in X$ and $t, s \in]0, +\infty[$,*

(1) $M(x, y, t) >_L 0_L$;

(2) $M(x, y, t) = 1_L$ *for all* $t > 0$ *if and only if* $x = y$;

(3) $M(x, y, t) = M(y, x, t)$;

(4) $T(M(x, y, t), M(y, z, s)) \leq_L M(x, z, t + s)$;

(5) $M(x, y, .) :]0, +\infty[\to L$ *is continuous.*

In this case, M is called an ℓ-fuzzy metric.

Definition 1.5. *The triple (V, P, T) is said to be an ℓ-fuzzy normed space if V is a vector space, T is a continuous t-norm on L and P is an ℓ-fuzzy set on $V\times]0, +\infty[$ satisfying the following conditions: for all $x, y \in V$ and $t, s \in]0, +\infty[$,*

(1) $P(x, t) >_L 0_L$;

(2) $P(x, t) = 1_L$ *if and only if* $x = 0$;

(3) $P(\alpha x, t) = P(x, \frac{t}{|\alpha|})$ *for each* $\alpha \neq 0$;

(4) $T(P(x, t), P(y, s)) \leq_L P(x + y, t + s)$;

(5) $P(x, .) :]0, +\infty[\to L$ *is continuous.*

(6) $\lim_{t\to 0} P(x, t) = 0_L$ *and* $\lim_{t\to\infty} P(x, t) = 1_L$.

In this case, P is called an ℓ-fuzzy norm.

Definition 1.6.

(1) A sequence $\{x_n\}_{n\in N}$ in an ℓ-fuzzy normed space (V, P, T) is called a Cauchy sequence if, for each $\epsilon \in L\backslash\{0_L\}$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that, for all $n, m \geq n_0$,

$$P(x_n - x_m, t) >_L N(\epsilon),$$

where N is a negator on ℓ.

(2) A sequence $\{x_n\}_{n\in N}$ is said to be convergent to $x \in V$ in the ℓ-fuzzy normed space (V, P, T), which is denoted by $x_n \to x$ if $P(x_n - x, t) \to 1_\ell$, whenever $n \to +\infty$ for all $t > 0$.

(3) An ℓ- fuzzy normed space (V, P, T) is said to be complete if and only if every Cauchy sequence in V is convergent.

Note that, if P is an ℓ-fuzzy norm on V, then the following are satisfied:

(1) $P(x, t)$ is nondecreasing with respect to t for all $x \in V$.

(2) $P(x-y,t) = P(y-x,t)$ for all $x, y \in V$ and $t \in]0,+\infty[$.

Let (V, P, T) be an ℓ-fuzzy normed space. If we define

$$M(x,y,t) = P(x-y,t)$$

for all $x, y \in V$ and $t \in]0,+\infty[$, then M is an ℓ-fuzzy metric on V, which is called the ℓ-fuzzy metric induced by the ℓ-fuzzy norm P.

In 1897, Hensel [68] introduced a field with a valuation in which does not have the Archimedean property.

Definition 1.7. *Let K be a field. A non-Archimedean absolute value on K is a function $|.| : K \to [0,+\infty[$ such that, for any $a, b \in K$,*

(1) $|a| \geq 0$ *and equality holds if and only if* $a = 0$,

(2) $|ab| = |a||b|$,

(3) $|a+b| \leq max\{|a|, |b|\}$ *(the strict triangle inequality).*

Note that $|n| \leq 1$ for each integer n. We always assume, in addition, that $|.|$ is non-trivial, i.e., there exists an $a_0 \in K$ such that $|a_0| \neq 0, 1$.

Definition 1.8. *A non-Archimedean ℓ-fuzzy normed space is a triple (V, P, T), where V is a vector space, T is a continuous t-norm on L and P is an ℓ-fuzzy set on $V \times]0,+\infty[$ satisfying the following conditions: for all $x, y \in V$ and $t, s \in]0,+\infty[$,*

(1) $0_L <_L P(x,t)$;

(2) $P(x,t) = 1_L$ *if and only if* $x = 0$;

(3) $P(\alpha x, t) = P(x, \frac{t}{|\alpha|})$ *for all* $\alpha \neq 0$;

(4) $T(P(x,t), P(y,s)) \leq_L P(x+y, max\{t,s\})$;

(5) $P(x,.) :]0,\infty[\to L$ *is continuous;*

(6) $\lim_{t\to 0} P(x,t) = 0_L$ *and* $\lim_{t\to\infty} P(x,t) = 1_L$.

Recently, Gordji and Savadkouhi [29] proved the stability of cubic and quartic functional equations in non-Archimedean spaces. For more detailed definitions of such terminologies, we can refer to [30–32].

In 2010, Shakeri, Saadati and Park [69] investigated the classical quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

and proved the generalized Hyers–Ulam stability in the context of non-Archimedean $l-$fuzzy normed spaces. In the same year Xu, Rassias and Xu [70] investigated as well the stability of a mixed type additive cubic functional equation in non–Archimedean fuzzy normed spaces.

In the present paper we introduce the following functional equation

$$f(x+2y) + f(x-2y) = 4(f(x+y) + f(x-y)) - 24f(y) - 6f(x) + 3f(2y)$$

and prove the generalized Hyers-Ulam-Rassias stability in non-Archimedean ℓ-fuzzy normed spaces.

2 Main Results

In this section, we investigate the generalized Hyers–Ulam–Rassias stability of the mixed type cubic and quartic functional equation

$$f(x+2y)+f(x-2y)=4(f(x+y)+f(x-y))-24f(y)-6f(x)+3f(2y).$$

Let Ψ be an ℓ-fuzzy set on $X\times X\times[0,\infty)$ such that $\Psi(x,y,.)$ is nondecreasing,

$$\Psi(cx,cx,t)\geq_L \Psi\left(x,x,\frac{t}{|c|}\right),\ \ \forall x\in X,\ \ c\neq 0$$

and

$$\lim_{t\to\infty}\Psi(x,y,t)=1_\ell,\ \ \forall x,y\in X,\ \ t>0.$$

Theorem 2.1. *Let K be a non-Archimedean field, X a vector space over K and (Y,P,T) a non-Archimedean ℓ-fuzzy Banach space over K. Suppose that $f:X\to Y$ is an odd mapping satisfying*

$$P(f(x+2y)+f(x-2y)-4f(x+y)-4f(x-y)+24f(y)+6f(x)-3f(2y),t)\geq_L \Psi(x,y,t) \tag{2.1}$$

for all $x,y\in X$ and $t>0$. If there exists an $\alpha\in\mathbb{R}$ and an integer k, $k\geq 2$ with $|2^k|<\alpha$ and $|2|\neq 0$ such that

$$\Psi(2^{-k}x,2^{-k}y,t)\geq_L \Psi(x,y,\alpha t),\ \ \forall x\in X,\ \ t>0, \tag{2.2}$$

$$\lim_{n\to\infty}T_{j=n}^{\infty}M\left(x,\frac{\alpha^j t}{|2|^{kj}}\right)=1_\ell,\ \ \forall x\in X,\ \ t>0,$$

then there exists a unique cubic mapping $C:X\to Y$ such that

$$P(f(x)-C(x),t)\geq T_{i=0}^{\infty}M(x,\frac{\alpha^{i+1}t}{|2|^{ki}}),\ \ \forall x\in X,\ \ t>0, \tag{2.3}$$

where

$$M(x,t):=T(\Psi(0,x,3t),\Psi(0,2x,3t),...,\Psi(0,2^{k-1}x,3t))$$

for all $x\in X$, $t>0$.

Proof. First, we show, by induction on j, that, for all $x\in X$, $t>0$ and $j\geq 1$,

$$P(f(2^jx)-8^jf(x),t)\geq_L M_j(x,t):=T(\Psi(0,x,3t),...,\Psi(0,2^{j-1}x,3t)). \tag{2.4}$$

Putting $x=0$ in (2.1), we obtain

$$P(3f(2y)-24f(y),t)\geq_L \Psi(0,y,t), \tag{2.5}$$

for all $y\in X$ and $t>0$. If we replace y in (2.5) by x, we get

$$P(f(2x)-8f(x),t)\geq_L \Psi(0,x,3t),$$

for all $x \in X$ and $t > 0$. This proves (2.4) for $j = 1$. Let (2.4) holds for some $j > 1$. Putting $x = 0$ and $y = 2^j x$ in (2.1), we get

$$P(f(2^{j+1}x) - 8f(2^j x), t) \geq_L \Psi(0, 2^j x, 3t),$$

for all $x \in X$ and $t > 0$. Since $|2| < 1$, it follows that

$$\begin{aligned} &P(f(2^{j+1}x) - 8^{j+1}f(x), t) \\ &\geq_L T(P(f(2^{j+1}x) - 8f(2^j x), t), P(8f(2^j x) - 8^{j+1}f(x), t)) \\ &= T\left(P(f(2^{j+1}x) - 8f(2^j x), t), P\left(f(2^j x) - 8^j f(x), \frac{t}{|8|}\right)\right) \\ &\geq_L T(P(f(2^{j+1}x) - 8f(2^j x), t), P(f(2^j x) - 8^j f(x), t)) \\ &= T(\Psi(0, 2^j x, 3t), M_j(x, t)) \\ &= M_{j+1}(x, t), \end{aligned}$$

for all $x \in X$ and $t > 0$. Thus (2.4) holds for all $j \geq 1$. In particular, we have

$$P(f(2^k x) - 8^k f(x), t) \geq_L M(x, t), \tag{2.6}$$

for all $x \in X$ and $t > 0$. Replacing x by $2^{-(kn+k)}x$ in (2.6) and using the inequality (2.2), we obtain

$$P\left(f\left(\frac{x}{2^{kn}}\right) - 8^k f\left(\frac{x}{2^{kn+k}}\right), t\right) \geq_L M\left(\frac{x}{2^{kn+k}}, t\right) \geq_L M\left(x, \alpha^{n+1}t\right)$$

for all $x \in X$, $t > 0$ and $n \geq 0$. Thus we have

$$\begin{aligned} P\left((2^{3k})^n f\left(\frac{x}{(2^k)^n}\right) - (2^{3k})^{n+1} f\left(\frac{x}{(2^k)^{n+1}}\right), t\right) &\geq_L M\left(x, \frac{\alpha^{n+1}}{|(2^{3k})^n|}t\right) \\ &\geq_L M\left(x, \frac{\alpha^{n+1}}{|(2^k)^n|}t\right) \end{aligned}$$

for all $x \in X$, $t > 0$ and $n \geq 0$. Hence it follows that

$$\begin{aligned} &P\left((2^{3k})^n f\left(\frac{x}{(2^k)^n}\right) - (2^{3k})^{n+p} f\left(\frac{x}{(2^k)^{n+p}}\right), t\right) \\ &\geq_L T_{j=n}^{n+p-1} P\left((2^{3k})^j f\left(\frac{x}{(2^k)^j}\right) - (2^{3k})^{j+1} f\left(\frac{x}{(2^k)^{j+1}}\right), t\right) \\ &\geq_L T_{j=n}^{n+p-1} M\left(x, \frac{\alpha^{j+1}}{|(2^k)^j|}t\right) \end{aligned}$$

for all $x \in X$, $t > 0$ and $n \geq 0$. Since $\lim_{n\to\infty} T_{j=n}^{\infty} M(x, \frac{\alpha^{j+1}}{|(2^k)^j|}t) = 1_\ell$ for all $x \in X$ and $t > 0$, $\{(2^{3k})^n f(\frac{x}{(2^k)^n})\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the non-Archimedean ℓ-fuzzy Banach space (Y, P, T). Hence we can define a mapping $C : X \to Y$ such that

$$\lim_{n\to\infty} P\left((2^{3k})^n f\left(\frac{x}{(2^k)^n}\right) - C(x), t\right) = 1_\ell \tag{2.7}$$

for all $x \in X$ and $t > 0$. Next, for all $n \geq 1$, $x \in X$ and $t > 0$, we have

$$\begin{aligned}
&P\left(f(x) - (2^{3k})^n f\left(\frac{x}{(2^k)^n}\right), t\right) \\
&= P\left(\sum_{i=0}^{n-1}\left[(2^{3k})^i f\left(\frac{x}{(2^k)^i}\right) - (2^{3k})^{i+1} f\left(\frac{x}{(2^k)^{i+1}}\right)\right], t\right) \\
&\geq_L T_{i=0}^{n-1}\left(P\left((2^{3k})^i f\left(\frac{x}{(2^k)^i}\right) - (2^{3k})^{i+1} f\left(\frac{x}{(2^k)^{i+1}}\right), t\right)\right) \\
&\geq_L T_{i=0}^{n-1} M\left(x, \frac{\alpha^{i+1}}{|2^k|^i} t\right)
\end{aligned}$$

and so

$$\begin{aligned}
&P(f(x) - C(x), t) \\
&\geq_L T\left(P\left(f(x) - (2^{3k})^n f\left(\frac{x}{(2^k)^n}\right), t\right), P\left((2^{3k})^n f\left(\frac{x}{(2^k)^n}\right) - C(x), t\right)\right) \\
&\geq_L T\left(T_{i=0}^{n-1} M\left(x, \frac{\alpha^{i+1}}{|2^k|^i} t\right), P\left((2^{3k})^n f\left(\frac{x}{(2^k)^n}\right) - C(x), t\right)\right). \qquad (2.8)
\end{aligned}$$

Taking the limit as $n \to \infty$ in (2.8), we obtain

$$P\left(f(x) - C(x), t\right) \geq_L T_{i=0}^{\infty} M\left(x, \frac{\alpha^{i+1} t}{|2^k|^i}\right),$$

which proves (2.3). Replacing x, y by $2^{-kn}x, 2^{-kn}y$ in (2.1) and (2.2), we get

$$\begin{aligned}
P\Bigg((8^k)^n f\left(\frac{x+2y}{2^{kn}}\right) &+ (8^k)^n f\left(\frac{x-2y}{2^{kn}}\right) - 4(8^k)^n f\left(\frac{x+y}{2^{kn}}\right) - 4(8^k)^n f\left(\frac{x-y}{2^{kn}}\right) \\
&+ 24(8^k)^n f\left(\frac{y}{2^{kn}}\right) + 6(8^k)^n f\left(\frac{x}{2^{kn}}\right) - 3(8^k)^n f\left(\frac{2y}{2^{kn}}\right), t\Bigg) \\
&\geq_L \Psi\left(2^{-kn}x, 2^{-kn}y, \frac{t}{|2^{3k}|^n}\right) \\
&\geq_L \Psi\left(x, y, \frac{\alpha^n t}{|2^k|^n}\right)
\end{aligned}$$

for all $x, y \in X$ and $t > 0$. Since $\lim_{n\to\infty} \Psi(x, y, \frac{\alpha^n t}{|2^k|^n}) = 1_\ell$, we infer that C is a cubic mapping. For the uniqueness of C, let $C' : X \to Y$ be another cubic mapping such that

$$P\left(C'(x) - f(x), t\right) \geq_L T_{i=0}^{\infty} M\left(x, \frac{\alpha^{i+1} t}{|2^k|^i}\right)$$

for all $x \in X$ and $t > 0$. Then we have, for all $x, y \in X$ and $t > 0$,

$$\begin{aligned}
&P(C(x) - C'(x), t) \\
&\geq_L T\left(P\left(C(x) - (2^{3k})^n f\left(\frac{x}{(2^k)^n}\right), t\right), P\left((2^{3k})^n f\left(\frac{x}{(2^k)^n}\right) - C'(x), t\right)\right).
\end{aligned}$$

Therefore, from (2.7), we conclude that $C = C'$. This completes the proof. □

Theorem 2.2. *Let K be a non-Archimedean field, X a vector space over K and (Y, P, T) a non-Archimedean ℓ-fuzzy Banach space over K. Suppose that $f : X \to Y$ is an even mapping satisfying*

$$P(f(x+2y)+f(x-2y)-4f(x+y)-4f(x-y)+24f(y)+6f(x)-3f(2y), t) \geq_L \Psi(x, y, t) \tag{2.9}$$

for all $x, y \in X$ and $t > 0$. If there exist an $\alpha \in \mathbb{R}$ and an integer k, $k \geq 2$ with $|2^k| < \alpha$ and $|2| \neq 0$ such that

$$\Psi(2^{-k}x, 2^{-k}y, t) \geq_L \Psi(x, y, \alpha t), \ \ \forall x \in X, \ t > 0, \tag{2.10}$$

$$\lim_{n\to\infty} T_{j=n}^{\infty} N\left(x, \frac{\alpha^j t}{|2|^{kj}}\right) = 1_\ell, \ \ \forall x \in X, \ t > 0,$$

then there exists a unique quartic mapping $Q : X \to Y$ such that

$$P(f(x) - Q(x), t) \geq T_{i=0}^{\infty} N\left(x, \frac{\alpha^{i+1} t}{|2|^{ki}}\right), \ \ \forall x \in X, \ t > 0, \tag{2.11}$$

where

$$N(x, t) := T(\Psi(0, x, t), \Psi(0, 2x, t), ..., \Psi(0, 2^{k-1}x, t))$$

for all $x \in X$, $t > 0$.

Proof. First, we show, by induction on j, that, for all $x \in X$, $t > 0$ and $j \geq 1$,

$$P(f(2^j x) - 16^j f(x), t) \geq_L N_j(x, t) := T(\Psi(0, x, t), ..., \Psi(0, 2^{j-1}x, t)). \tag{2.12}$$

Putting $x = 0$ in (2.9), we obtain

$$P(f(2y) - 16f(y), t) \geq_L \Psi(0, y, t), \tag{2.13}$$

for all $y \in X$ and $t > 0$. If we replace y in (2.13) by x, we get

$$P(f(2x) - 16f(x), t) \geq_L \Psi(0, x, t),$$

for all $x \in X$ and $t > 0$. This proves (2.12) for $j = 1$. Let (2.12) holds for some $j > 1$. Putting $x = 0$ and $y = 2^j x$ in (2.9), we get

$$P(f(2^{j+1}x) - 16f(2^j x), t) \geq_L \Psi(0, 2^j x, t),$$

for all $x \in X$ and $t > 0$. Since $|2| < 1$, it follows that

$$\begin{aligned}
&P(f(2^{j+1}x) - 16^{j+1} f(x), t) \\
&\geq_L T(P(f(2^{j+1}x) - 16f(2^j x), t), P(16f(2^j x) - 16^{j+1} f(x), t)) \\
&= T\left(P(f(2^{j+1}x) - 16f(2^j x), t), P\left(f(2^j x) - 16^j f(x), \frac{t}{|16|}\right)\right) \\
&\geq_L T(P(f(2^{j+1}x) - 16f(2^j x), t), P(f(2^j x) - 16^j f(x), t)) \\
&= T(\Psi(0, 2^j x, t), N_j(x, t)) \\
&= N_{j+1}(x, t),
\end{aligned}$$

for all $x \in X$ and $t > 0$. Thus (2.12) holds for all $j \geq 1$. In particular, we have

$$P(f(2^k x) - 16^k f(x), t) \geq_L N(x, t), \tag{2.14}$$

for all $x \in X$ and $t > 0$. Replacing x by $2^{-(kn+k)}x$ in (2.14) and using the inequality (2.10), we obtain

$$P\left(f\left(\frac{x}{2^{kn}}\right) - 16^k f\left(\frac{x}{2^{kn+k}}\right), t\right) \geq_L N\left(\frac{x}{2^{kn+k}}, t\right) \geq_L N\left(x, \alpha^{n+1} t\right)$$

for all $x \in X$, $t > 0$ and $n \geq 0$. Thus we have

$$\begin{aligned} P\left((2^{4k})^n f\left(\frac{x}{(2^k)^n}\right) - (2^{4k})^{n+1} f\left(\frac{x}{(2^k)^{n+1}}\right), t\right) &\geq_L N\left(x, \frac{\alpha^{n+1}}{|(2^{4k})^n|} t\right) \\ &\geq_L N\left(x, \frac{\alpha^{n+1}}{|(2^k)^n|} t\right) \end{aligned}$$

for all $x \in X$, $t > 0$ and $n \geq 0$. Hence it follows that

$$\begin{aligned} &P\left((2^{4k})^n f\left(\frac{x}{(2^k)^n}\right) - (2^{4k})^{n+p} f\left(\frac{x}{(2^k)^{n+p}}\right), t\right) \\ &\geq_L T_{j=n}^{n+p-1} P\left((2^{4k})^j f\left(\frac{x}{(2^k)^j}\right) - (2^{4k})^{j+1} f\left(\frac{x}{(2^k)^{j+1}}\right), t\right) \\ &\geq_L T_{j=n}^{n+p-1} N\left(x, \frac{\alpha^{j+1}}{|(2^k)^j|} t\right) \end{aligned}$$

for all $x \in X$, $t > 0$ and $n \geq 0$. Since $\lim_{n\to\infty} T_{j=n}^{\infty} N(x, \frac{\alpha^{j+1}}{|(2^k)^j|} t) = 1_\ell$ for all $x \in X$ and $t > 0$, $\{(2^{4k})^n f(\frac{x}{(2^k)^n})\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the non-Archimedean ℓ-fuzzy Banach space (Y, P, T). Hence we can define a mapping $Q : X \to Y$ such that

$$\lim_{n\to\infty} P\left((2^{4k})^n f\left(\frac{x}{(2^k)^n}\right) - Q(x), t\right) = 1_\ell \tag{2.15}$$

for all $x \in X$ and $t > 0$. Next, for all $n \geq 1$, $x \in X$ and $t > 0$, we have

$$\begin{aligned} &P\left(f(x) - (2^{4k})^n f\left(\frac{x}{(2^k)^n}\right), t\right) \\ &= P\left(\sum_{i=0}^{n-1} \left[(2^{4k})^i f\left(\frac{x}{(2^k)^i}\right) - (2^{4k})^{i+1} f\left(\frac{x}{(2^k)^{i+1}}\right)\right], t\right) \\ &\geq_L T_{i=0}^{n-1} \left(P\left((2^{4k})^i f\left(\frac{x}{(2^k)^i}\right) - (2^{4k})^{i+1} f\left(\frac{x}{(2^k)^{i+1}}\right), t\right)\right) \\ &\geq_L T_{i=0}^{n-1} N\left(x, \frac{\alpha^{i+1}}{|2^k|^i} t\right) \end{aligned}$$

and so

$$
\begin{aligned}
&P(f(x)-Q(x),t)\\
&\geq_L T\left(P\left(f(x)-(2^{4k})^n f\left(\frac{x}{(2^k)^n}\right),t\right),P\left((2^{4k})^n f\left(\frac{x}{(2^k)^n}\right)-Q(x),t\right)\right)\\
&\geq_L T\left(T_{i=0}^{n-1}N\left(x,\frac{\alpha^{i+1}}{|2^k|^i}t\right),P\left((2^{4k})^n f\left(\frac{x}{(2^k)^n}\right)-Q(x),t\right)\right). \qquad (2.16)
\end{aligned}
$$

Taking the limit as $n\to\infty$ in (2.16), we obtain

$$
P(f(x)-Q(x),t)\geq_L T_{i=0}^{\infty}N\left(x,\frac{\alpha^{i+1}t}{|2^k|^i}\right),
$$

which proves (2.11). As T is continuous, from a well known result in ℓ-fuzzy (probabilistic) normed space, replacing x,y by $2^{-kn}x, 2^{-kn}y$ in (2.9) and (2.10), we get

$$
\begin{aligned}
P\Bigg((16^k)^n f\left(\frac{x+2y}{2^{kn}}\right)&+(16^k)^n f\left(\frac{x-2y}{2^{kn}}\right)-4(16^k)^n f\left(\frac{x+y}{2^{kn}}\right)-4(16^k)^n f\left(\frac{x-y}{2^{kn}}\right)\\
&+24(16^k)^n f\left(\frac{y}{2^{kn}}\right)+6(16^k)^n f\left(\frac{x}{2^{kn}}\right)-3(16^k)^n f\left(\frac{2y}{2^{kn}}\right),t\Bigg)\\
&\geq_L \Psi\left(2^{-kn}x,2^{-kn}y,\frac{t}{|2^{4k}|^n}\right)\\
&\geq_L \Psi\left(x,y,\frac{\alpha^n t}{|2^k|^n}\right)
\end{aligned}
$$

for all $x,y\in X$ and $t>0$. Since $\lim_{n\to\infty}\Psi(x,y,\frac{\alpha^n t}{|2^k|^n})=1_\ell$, we infer that Q is a quartic mapping. For the uniqueness of Q, let $Q^{'}:X\to Y$ be another quartic mapping such that

$$
P(Q^{'}(x)-f(x),t)\geq_L T_{i=0}^{\infty}N\left(x,\frac{\alpha^{i+1}t}{|2^k|^i}\right)
$$

for all $x\in X$ and $t>0$. Then we have, for all $x,y\in X$ and $t>0$,

$$
\begin{aligned}
&P(Q(x)-Q^{'}(x),t)\\
&\geq_L T\left(P\left(Q(x)-(2^{4k})^n f\left(\frac{x}{(2^k)^n}\right),t\right),P\left((2^{4k})^n f\left(\frac{x}{(2^k)^n}\right)-Q^{'}(x),t\right)\right).
\end{aligned}
$$

Therefore, from (2.15), we conclude that $Q=Q^{'}$. This completes the proof. □

Theorem 2.3. *Let K be a non-Archimedean field, X a vector space over K and (Y,P,T) a non-Archimedean ℓ-fuzzy Banach space over K. Suppose that $f:X\to Y$ is a mapping satisfying*

$$
P(f(x+2y)+f(x-2y)-4f(x+y)-4f(x-y)+24f(y)+6f(x)-3f(2y),t)\geq_L \Psi(x,y,t) \qquad (2.17)
$$

for all $x, y \in X$ and $t > 0$. If there exist an $\alpha \in \mathbb{R}$ and an integer k, $k \geq 2$ with $|2^k| < \alpha$ and $|2| \neq 0$ such that

$$\Psi(2^{-k}x, 2^{-k}y, t) \geq_L \Psi(x, y, \alpha t), \;\; \forall x \in X, \;\; t > 0, \tag{2.18}$$

$$\lim_{n\to\infty} T_{j=n}^{\infty}\left(T\left(M\left(x, \frac{2\alpha^j t}{|2|^{kj}}\right), M\left(-x, \frac{2\alpha^j t}{|2|^{kj}}\right)\right)\right) = 1_\ell,$$

and

$$\lim_{n\to\infty} T_{j=n}^{\infty}\left(T\left(N\left(x, \frac{2\alpha^j t}{|2|^{kj}}\right), N\left(-x, \frac{2\alpha^j t}{|2|^{kj}}\right)\right)\right) = 1_\ell,$$

then there exist a cubic mapping $C : X \to Y$ and a quartic mapping $Q : X \to Y$ such that

$$\begin{aligned} P(f(x) - C(x) - Q(x), t) \geq_L T\Bigg(&T_{i=0}^{\infty}\left(T\left(M\left(x, \frac{2\alpha^{i+1}t}{|2^k|^i}\right), M\left(-x, \frac{2\alpha^{i+1}t}{|2^k|^i}\right)\right)\right), \\ &T_{i=0}^{\infty}\left(T\left(N\left(x, \frac{2\alpha^{i+1}t}{|2^k|^i}\right), N\left(-x, \frac{2\alpha^{i+1}t}{|2^k|^i}\right)\right)\right)\Bigg), \end{aligned} \tag{2.19}$$

where

$$M(x, t) := T(\Psi(0, x, 3t), \Psi(0, 2x, 3t), ..., \Psi(0, 2^{k-1}x, 3t)),$$
$$N(x, t) := T(\Psi(0, x, t), \Psi(0, 2x, t), ..., \Psi(0, 2^{k-1}x, t)),$$

for all $x \in X$, $t > 0$.

Proof. Let $f_o(x) = \frac{1}{2}[f(x) - f(-x)]$ for all $x \in X$. Then $f_o(0) = 0$, $f_o(-x) = -f_o(x)$, and

$$\begin{aligned} &P(f_o(x+2y) + f_o(x-2y) - 4f_o(x+y) - 4f_o(x-y) + 24f_o(y) + 6f_o(x) - 3f_o(2y), t) \\ &\geq_L T\,(P\,((1/2)[f(x+2y) + f(x-2y) - 4f(x+y) - 4f(x-y) + 24f(y) + 6f(x) \\ &\quad -3f(2y)], t)\,, P\,((-1/2)[f(-x-2y) + f(-x+2y) - 4f(-x-y) - 4f(-x+y) \\ &\quad +24f(-y) + 6f(-x) - 3f(-2y)], t)) \\ &\geq_L T(\Psi(x, y, 2t), \Psi(-x, -y, 2t)) \end{aligned}$$

for all $x, y \in X$ and $t > 0$. By Theorem 2.1, it follows that there exists a unique cubic function $C : X \to Y$ satisfying

$$P(f_o(x) - C(x), t) \geq_L T_{i=0}^{\infty}\left(T\left(M\left(x, \frac{2\alpha^{i+1}t}{|2^k|^i}\right), M\left(-x, \frac{2\alpha^{i+1}t}{|2^k|^i}\right)\right)\right) \tag{2.20}$$

for all $x, y \in X$ and $t > 0$. Let $f_e(x) = \frac{1}{2}[f(x) + f(-x)]$ for all $x \in X$. Then $f_e(0) = 0$, $f_e(-x) = f_e(x)$, and

$$\begin{aligned} &P(f_e(x+2y) + f_e(x-2y) - 4f_e(x+y) - 4f_e(x-y) + 24f_e(y) + 6f_e(x) - 3f_e(2y), t) \\ &\geq_L T(P((1/2)[f(x+2y) + f(x-2y) - 4f(x+y) - 4f(x-y) + 24f(y) + 6f(x) \\ &\quad - 3f(2y)], t), P((1/2)[f(-x-2y) + f(-x+2y) - 4f(-x-y) - 4f(-x+y) \\ &\quad + 24f(-y) + 6f(-x) - 3f(-2y)], t)) \\ &\geq_L T(\Psi(x, y, 2t), \Psi(-x, -y, 2t)) \end{aligned}$$

for all $x, y \in X$ and $t > 0$. By Theorem 2.2, it follows that there exists a unique quartic function $Q : X \to Y$ satisfying

$$P(f_e(x) - Q(x), t) \geq_L T_{i=0}^{\infty} \left(T \left(N \left(x, \frac{2\alpha^{i+1}t}{|2^k|^i} \right), N \left(-x, \frac{2\alpha^{i+1}t}{|2^k|^i} \right) \right) \right) \quad (2.21)$$

for all $x, y \in X$ and $t > 0$. Hence (2.19) follows from (2.20) and (2.21). □

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