



Hyperidentities in $(x(yz))z$ with Loop Graph Varieties of Type $(2,0)$ ¹

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Abstract : Graph algebras establish a connection between directed graphs without multiple edges and special universal algebras of type $(2,0)$. We say that a graph G satisfies a term equation $s \approx t$ if the corresponding graph algebra $\underline{A}(G)$ satisfies $s \approx t$. A class of graph algebras \mathcal{V} is called a graph variety if $\mathcal{V} = \overline{\text{Mod}_g \Sigma}$ where Σ is a subset of $W_{(2)}(X) \times W_{(2)}(X)$. A graph variety $\mathcal{V}' = \overline{\text{Mod}_g \Sigma'}$ is called $(x(yz))z$ with loop graph variety if Σ' is a set of $(x(yz))z$ with loop term equations. A term equation $s \approx t$ is called an identity in a graph variety \mathcal{V} if $\underline{A}(G)$ satisfies $s \approx t$ for all $G \in \mathcal{V}$. An identity $s \approx t$ of a variety \mathcal{V} is called a hyperidentity of a graph algebra $\underline{A}(G)$, $G \in \mathcal{V}$ whenever the operation symbols occurring in s and t are replaced by any term operations of $\underline{A}(G)$ of the appropriate arity, the resulting identities hold in $\underline{A}(G)$. An identity $s \approx t$ of a variety \mathcal{V} is called a hyperidentity of \mathcal{V} if it is a hyperidentity of $\underline{A}(G)$ for all $G \in \mathcal{V}$.

In this paper we characterize all hyperidentities of each $(x(yz))z$ with loop graph variety. For identities, varieties and other basic concepts of universal algebra see e.g. [1].

Keywords : Varieties; $(x(yz))z$ with loop graph varieties; Term; Identities; Hyperidentities; Binary algebra; Graph algebra.

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1 Introduction

An identity $s \approx t$ of terms s, t of any type τ is called a *hyperidentity* of an algebra \underline{A} if whenever the operation symbols occurring in s and t are replaced by any term operations of \underline{A} of the appropriate arity, the resulting identity holds in \underline{A} . Hyperidentities can be defined more precisely by using the concept of a hypersubstitution, which was introduced by Denecke, Lau, Pöschel and Schweigert in [2].

We fix a type $\tau = (n_i)_{i \in I}$, $n_i > 0$ for all $i \in I$, and operation symbols $(f_i)_{i \in I}$, where f_i is n_i -ary. Let $W_\tau(X)$ be the set of all terms of type τ over some fixed alphabet X , and let $Alg(\tau)$ be the class of all algebras of type τ . Then, a mapping

$$\sigma : \{f_i | i \in I\} \longrightarrow W_\tau(X)$$

which assigns to every n_i -ary operation symbol f_i an n_i -ary term will be called a *hypersubstitution* of type τ (for short, a hypersubstitution). We denote the extension of the hypersubstitution σ by a mapping

$$\hat{\sigma} : W_\tau(X) \longrightarrow W_\tau(X).$$

The term $\hat{\sigma}[t]$ is defined inductively by

- (i) $\hat{\sigma}[x] = x$ for any variable x in the alphabet X , and
- (ii) $\hat{\sigma}[f_i(t_1, \dots, t_{n_i})] = \sigma(f_i)^{W_\tau(X)}(\hat{\sigma}[t_1], \dots, \hat{\sigma}[t_{n_i}]).$

Here $\sigma(f_i)^{W_\tau(X)}$ on the right hand side of (ii) is the operation induced by $\sigma(f_i)$ on the term algebra with the universe $W_\tau(X)$.

Graph algebras were invented by Shallon in [3], to obtain examples of infinitely based on finite algebras. To recall this concept, let $G = (V, E)$ be a (directed) graph with the vertex set V and the edge set $E \subseteq V \times V$. Define the *graph algebra* $A(G)$ corresponding to G with the underlying set $V \cup \{\infty\}$, where ∞ is a symbol outside V , and with two fundamental operations, namely a nullary operation pointing to ∞ and a binary one denoted by juxtaposition, i.e. for $u, v \in V \cup \{\infty\}$

$$uv = \begin{cases} u, & \text{if } (u, v) \in E, \\ \infty, & \text{otherwise.} \end{cases}$$

Pöschel and Wessel investigated graph varieties for finite undirected graphs in order to get graph theoretic results (structure theorems) from universal algebra via graph algebras [4]. These investigations were extended to arbitrary (finite) directed graphs where the authors ask for a graph theoretic characterization of graph varieties, i.e., of classes of graphs which can be defined by identities for their corresponding graph algebras [5]. The answer is a theorem of **Birkhoff-type**, which uses graph theoretic closure operations. *A class of finite directed graphs is equational (i.e., a graph variety) if and only if it is closed with respect to finite restricted pointed subproducts and isomorphic copies.*

Poomsa-ard et al. studied hyperidentities in the class of graph algebras which satisfy various term equations [6, 7, 8, 9]. For $(x(yz))z$ with loop term equations,

Anantpinitwatna and Poomsa-ard studied the properties of graph algebras in each $(x(yz))z$ with loop graph variety [10]. Further, they characterized identities in each $(x(yz))z$ with loop graph variety [11].

In this paper we characterized all hyperidentities in each $(x(yz))z$ with loop graph variety.

2 Terms, identities and graph varieties

Dealing with terms for graph algebras, the underlying formal language has to contain a binary operation symbol (juxtaposition) and a symbol for the constant ∞ (denoted by ∞ too).

Definition 2.1. *The set $W_{(2)}(X)$ of all terms over the alphabet*

$$X = \{x_1, x_2, x_3, \dots\}$$

is defined inductively as follows:

- (i) *every variable $x_i, i = 1, 2, 3, \dots$, and ∞ are terms;*
- (ii) *if t_1 and t_2 are terms, then t_1t_2 is a term;*
- (iii) *$W_{(2)}(X)$ is the set of all terms which can be obtained from (i) and (ii) in finitely many steps.*

Terms built up from the two-element set $X_2 = \{x_1, x_2\}$ of variables are thus binary terms. We denote the set of all binary terms by $W_{(2)}(X_2)$. The leftmost variable of a term t is denoted by $L(t)$, the rightmost variable of a term t is denoted by $R(t)$. A term in which the symbol ∞ occurs is called a trivial term.

Definition 2.2. *For each non-trivial term t of type $\tau = (2, 0)$, one can define a directed graph $G(t) = (V(t), E(t))$, where the vertex set $V(t)$ is the set of all variables occurring in t and the edge set $E(t)$ is defined inductively by*

$$E(t) = \phi \text{ if } t \text{ is a variable and } E(t_1t_2) = E(t_1) \cup E(t_2) \cup \{(L(t_1), L(t_2))\}$$

where $t = t_1t_2$ is a compound term.

$L(t)$ is called the root of the graph $G(t)$, and the pair $(G(t), L(t))$ is the rooted graph corresponding to t . Formally, we assign the empty graph ϕ to every trivial term t .

Definition 2.3. *A non-trivial term t of type $\tau = (2, 0)$ is called $(x(yz))z$ with loop term if and only if $G(t)$ is a graph with $V(t) = \{x, y, z\}$ and $E(t) = E \cup E'$, where $E = \{(x, y), (x, z), (y, z)\}$, $E' \subseteq \{(x, x), (y, y), (z, z)\}$, $E' \neq \phi$. A term equation $s \approx t$ is called $(x(yz))z$ with loop term equation if s and t are $(x(yz))z$ with loop terms.*

Definition 2.4. We say that a graph $G = (V, E)$ satisfies a term equation $s \approx t$ if the corresponding graph algebra $A(G)$ satisfies $s \approx t$ (i.e., we have $s = t$ for every assignment $V(s) \cup V(t) \rightarrow V \cup \{\overline{\infty}\}$), and in this case, we write $G \models s \approx t$. Given a class \mathcal{G} of graphs and a set Σ of term equations (i.e., $\Sigma \subseteq W_{(2)}(X) \times W_{(2)}(X)$) we introduce the following notation:

$$\begin{aligned} G \models \Sigma &\text{ if } G \models s \approx t \text{ for all } s \approx t \in \Sigma, \\ \mathcal{G} \models s \approx t &\text{ if } G \models s \approx t \text{ for all } G \in \mathcal{G}, \\ \mathcal{G} \models \Sigma &\text{ if } G \models \Sigma \text{ for all } G \in \mathcal{G}, \\ Id\mathcal{G} &= \{s \approx t \mid s, t \in W_{(2)}(X), \mathcal{G} \models s \approx t\}, \\ Mod_g\Sigma &= \{G \mid G \text{ is a graph and } G \models \Sigma\}, \\ \mathcal{V}_g(\mathcal{G}) &= Mod_g Id\mathcal{G}. \end{aligned}$$

$\mathcal{V}_g(\mathcal{G})$ is called the graph variety generated by \mathcal{G} and \mathcal{G} is called graph variety if $\mathcal{V}_g(\mathcal{G}) = \mathcal{G}$. \mathcal{G} is called equational if there exists a set Σ' of term equations such that $\mathcal{G} = Mod_g\Sigma'$. Obviously $\mathcal{V}_g(\mathcal{G}) = \mathcal{G}$ if and only if \mathcal{G} is an equational class.

3 $(x(yz))z$ with loop graph varieties and identities

All $(x(yz))z$ with loop graph varieties were characterized in [10] which there are only ten graph varieties. Let $\mathcal{A} = \{\mathcal{K}_0, \mathcal{K}_1, \mathcal{K}_2, \dots, \mathcal{K}_9\}$ is the set of all $(x(yz))z$ with loop graph varieties.

In [11], Anantpinitwatna and Poomsa-ard characterized all identities in each $(x(yz))z$ with loop graph variety. The common properties of all identities $s \approx t$ in each $(x(yz))z$ with loop graph variety are (i) $L(s) = L(t)$, (ii) $V(s) = V(t)$ and (iii) for any $x, y \in V(s)$, $x \neq y$, $(x, y) \in E(s)$ if and only if $(x, y) \in E(t)$. Further, we see that if $s' \approx t'$ is a trivial term equation (s', t' are both trivial or $G(s') = G(t')$, and $L(s') = L(t')$), then $s' \approx t' \in Id\mathcal{K}$ for every $(x(yz))z$ with loop graph variety \mathcal{K} . Hence, we consider the case that $s \approx t$ is a non-trivial equation with $G(s) \neq G(t)$, $V(s) = V(t)$ and $L(s) = L(t)$. For the others properties, we will quote only which we need to be referred. At first, we give some notations, for any non-trivial term t and for any $x \in V(t)$, let

$$\begin{aligned} N_i^t(x) &= \{x' \in V(t) \mid x' \text{ is an in-neighbor of } x \text{ in } G(t)\}, \\ N_o^t(x) &= \{x' \in V(t) \mid x' \text{ is an out-neighbor of } x \text{ in } G(t)\}, \\ A_x^0(t) &= \{x\}, A_x^1(t) = \{x' \in V(t) \mid x' \text{ is an out-neighbor of } x \text{ or } x' \text{ is an in-neighbor of } x \text{ which there exists } z' \text{ such that } (x, z'), (x', z') \in E(t)\}, \\ A_x^2(t) &= \bigcup_{y \in A_x^1(t)} A_y^1(t), \dots, A_x^n(t) = \bigcup_{y \in A_x^{n-1}(t)} A_y^1(t), A_x^*(t) = \bigcup_{i=0}^{\infty} A_x^i(t), \\ C_x^0(t) &= \{x\}, C_x^1(t) = \{x' \in V(t) \mid x' \text{ is both out and in-neighbor of } x \text{ or } x' \text{ is an out-neighbor of } x \text{ which there exists } z \text{ such that } (z, x), (z, x') \in E(t)\}, \\ C_x^2(t) &= \bigcup_{y \in C_x^1(t)} C_y^1(t), \dots, C_x^n(t) = \bigcup_{y \in C_x^{n-1}(t)} C_y^1(t), C_x^*(t) = \bigcup_{i=0}^{\infty} C_x^i(t). \end{aligned}$$

Then, all identities in some $(x(yz))z$ with loop graph varieties are characterized by the following table:

Table 1. The properties of identities in some $(x(yz))z$ with loop graph varieties.

Variety	Properties of s and t .
\mathcal{K}_1	There exists $y \in V(s)$ such that $(y, y) \in E(s)$ iff there exists $z \in V(t)$ such that $(z, z) \in E(t)$.
\mathcal{K}_2	For any $x \in V(s)$, there exists, $y \in A_x^*(s)$ such that $(y, y) \in E(s)$ iff there exists $z \in A_x^*(t)$ such that $(z, z) \in E(t)$.
\mathcal{K}_6	For any $x \in V(s)$, there exists, $y \in C_x^*(s)$ such that $(y, y) \in E(s)$ iff there exists $z \in C_x^*(t)$ such that $(z, z) \in E(t)$.

4 Hyperidentities in $(x(yz))z$ with loop graph varieties

Let \mathcal{K} be a graph variety. Now, we want to formulate precise the concept of a graph hypersubstitution for graph algebras.

Definition 4.1. *A mapping $\sigma : \{f, \infty\} \rightarrow W_{(2)}(X_2)$, where $X_2 = \{x_1, x_2\}$ and f is the operation symbol corresponding to the binary operation of a graph algebra is called a graph hypersubstitution if $\sigma(\infty) = \infty$ and $\sigma(f) = s \in W_{(2)}(X_2)$. The graph hypersubstitution with $\sigma(f) = s$ is denoted by σ_s .*

Definition 4.2. *An identity $s \approx t$ is a \mathcal{K} graph hyperidentity iff for all graph hypersubstitutions σ , the equations $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ are identities in \mathcal{K} .*

If we want to check that an identity $s \approx t$ is a hyperidentity in \mathcal{K} we can restrict our consideration to a (small) subset of $Hyp\mathcal{G}$ - the set of all graph hypersubstitutions. In [12], the following relation between hypersubstitutions was defined:

Definition 4.3. *Two graph hypersubstitutions σ_1, σ_2 are called \mathcal{K} -equivalent iff $\sigma_1(f) \approx \sigma_2(f)$ is an identity in \mathcal{K} . In this case we write $\sigma_1 \sim_{\mathcal{K}} \sigma_2$.*

The following lemma was proven in [13].

Lemma 4.4. *If $\hat{\sigma}_1[s] \approx \hat{\sigma}_1[t] \in Id\mathcal{K}$ and $\sigma_1 \sim_{\mathcal{K}} \sigma_2$, then $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in Id\mathcal{K}$.*

Therefore, it is enough to consider the quotient set $Hyp\mathcal{G}/\sim_{\mathcal{K}}$.

In [14], it was shown that any non-trivial term t over the class of graph algebras has a uniquely determined normal form term $NF(t)$ and there is an algorithm to construct the normal form term to a given term t . Without difficulties one shows $G(NF(t)) = G(t), L(NF(t)) = L(t)$.

The following definition was given in [15].

Definition 4.5. *The graph hypersubstitution $\sigma_{NF(t)}$, is called normal form graph hypersubstitution. Here $NF(t)$ is the normal form of the binary term t .*

Since for any binary term t the rooted graphs of t and $NF(t)$ are the same, we have $t \approx NF(t) \in Id\mathcal{K}$. Then for any graph hypersubstitution σ_t with $\sigma_t(f) = t \in W_{(2)}(X_2)$, one obtains $\sigma_t \sim_{\mathcal{K}} \sigma_{NF(t)}$.

In [15], all rooted graphs with at most two vertices were considered. Then, we formed the corresponding binary terms and used the algorithm to construct normal form terms. The result is given as the following table:

Table 2. Normal form terms of binary terms.

normal form term	graph hypers	normal form term	graph hypers
x_1x_2	σ_0	x_1	σ_1
x_2	σ_2	x_1x_1	σ_3
x_2x_2	σ_4	x_2x_1	σ_5
$(x_1x_1)x_2$	σ_6	$(x_2x_1)x_2$	σ_7
$x_1(x_2x_2)$	σ_8	$x_2(x_1x_1)$	σ_9
$(x_1x_1)(x_2x_2)$	σ_{10}	$(x_2(x_1x_1))x_2$	σ_{11}
$x_1(x_2x_1)$	σ_{12}	$x_2(x_1x_2)$	σ_{13}
$(x_1x_1)(x_2x_1)$	σ_{14}	$(x_2(x_1x_2))x_2$	σ_{15}
$x_1((x_2x_1)x_2)$	σ_{16}	$x_2((x_1x_1)x_2)$	σ_{17}
$(x_1x_1)((x_2x_1)x_2)$	σ_{18}	$(x_2((x_1x_1)x_2))x_2$	σ_{19}

Let $M_{\mathcal{G}}$ be the set of all normal form graph hypersubstitutions. Then we get,

$$M_{\mathcal{G}} = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8, \sigma_9, \sigma_{10}, \sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{15}, \sigma_{16}, \sigma_{17}, \sigma_{18}, \sigma_{19}\}.$$

We defined the product of two normal form graph hypersubstitutions in $M_{\mathcal{G}}$ as follows.

Definition 4.6. *The product $\sigma_{1N} \circ_N \sigma_{2N}$ of two normal form graph hypersubstitutions is defined by $(\sigma_{1N} \circ_N \sigma_{2N})(f) = NF(\hat{\sigma}_{1N}[\sigma_{2N}(f)])$.*

The concept of a proper hypersubstitution of a class of algebras was introduced in [13].

Definition 4.7. *A hypersubstitution σ is called proper with respect to a class \mathcal{K} of algebras if $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}$ for all $s \approx t \in Id\mathcal{K}$.*

A graph hypersubstitution with the property that $\sigma(f)$ contains both variables x_1 and x_2 is called *regular*. It is easy to check that the set of all regular graph hypersubstitutions M_{reg} forms a groupoid.

The following lemma was proved in [15].

Lemma 4.8. *For each non-trivial term $s, (s \neq x \in X)$ and for all $u, v \in X$, we have*

$$E(\hat{\sigma}_6[s]) = E(s) \cup \{(u, u) | (u, v) \in E(s)\},$$

$$E(\hat{\sigma}_8[s]) = E(s) \cup \{(v, v) | (u, v) \in E(s)\},$$

and

$$E(\hat{\sigma}_{12}[s]) = E(s) \cup \{(v, u) | (u, v) \in E(s)\}.$$

In the similar way we prove that,

$$E(\hat{\sigma}_{10}[s]) = E(s) \cup \{(u, u), (v, v) | (u, v) \in E(s)\}.$$

Let $PM_{\mathcal{K}}$ is the set of all proper graph hypersubstitutions with respect to the class \mathcal{K} . In [2], it proved for the class of all graph algebras \mathcal{G} that

$$PM_{\mathcal{G}} = \{\sigma_0, \sigma_6, \sigma_8, \sigma_{10}, \sigma_{12}, \sigma_{14}, \sigma_{16}, \sigma_{18}\}.$$

We want to find all proper graph hypersubstitutions with respect to each $(x(yz))z$ with loop graph variety. Before to do this we have some remark.

Remark 4.9.

- (i) σ_0 is a proper graph hypersubstitution with respect to every $(x(yz))z$ with loop graph variety.
- (ii) For any $(x(yz))z$ with loop graph variety \mathcal{K} , suppose $s \approx t \in Id\mathcal{K}$ and is a trivial equation, then for any $\sigma \in \{\sigma_6, \sigma_8, \sigma_{10}, \sigma_{12}\}$, we have $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}$. Further, if s and t are non-trivial terms, $L(s) = L(t)$, $V(s) = V(t)$, then we have $L(\hat{\sigma}[s]) = L(s) = L(t) = L(\hat{\sigma}[t])$. Since σ is a regular, we get $V(\hat{\sigma}[s]) = V(s) = V(t) = V(\hat{\sigma}[t])$.
- (iii) If s and t are trivial terms with different leftmost and different rightmost, then $\hat{\sigma}_1[s] \approx \hat{\sigma}_1[t] \notin Id\mathcal{K}$, $\hat{\sigma}_3[s] \approx \hat{\sigma}_3[t] \notin Id\mathcal{K}$, $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \notin Id\mathcal{K}$ and $\hat{\sigma}_4[s] \approx \hat{\sigma}_4[t] \notin Id\mathcal{K}$. If $s = x_1(x_2x_1)$, $t = x_1(x_2(x_1x_2))$, then $G(s) = G(t)$ and $L(s) = L(t)$. Hence $s \approx t \in Id\mathcal{K}$. For any σ which $L(\sigma(f)) = x_2$, we see that $L(\hat{\sigma}[s]) = x_1$ and $L(\hat{\sigma}[t]) = x_2$. Thus, $\hat{\sigma}[s] \approx \hat{\sigma}[t] \notin Id\mathcal{K}$. Therefore, $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ and σ which $L(\sigma(f)) = x_2$ are not proper graph hypersubstitutions with respect to \mathcal{K} .

Now, we use Table 1, to consider the relation between graph hypersubstitutions for each \mathcal{K}_i , $i = 1, 2, \dots, 9$ and find $M_{\mathcal{K}_i}$, $i = 1, 2, \dots, 9$. Then, use Lemma 4.2 and Remark 4.1 to find $PM_{\mathcal{K}_i}$, $i = 1, 2, \dots, 9$.

For \mathcal{K}_1 , we have the following relations:

- (i) $\sigma_6 \sim_{\mathcal{K}_1} \sigma_8 \sim_{\mathcal{K}_1} \sigma_{10}$,
- (ii) $\sigma_7 \sim_{\mathcal{K}_1} \sigma_9 \sim_{\mathcal{K}_1} \sigma_{11}$,
- (iii) $\sigma_{14} \sim_{\mathcal{K}_1} \sigma_{16} \sim_{\mathcal{K}_1} \sigma_{18}$,
- (iv) $\sigma_{15} \sim_{\mathcal{K}_1} \sigma_{17} \sim_{\mathcal{K}_1} \sigma_{19}$.

Then, we get,

$$M_{\mathcal{K}_1} = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{15}\}.$$

Theorem 4.10. $PM_{\mathcal{K}_1} = \{\sigma_0, \sigma_6, \sigma_{12}, \sigma_{14}\}$.

Proof. Assume that $s \approx t$ is a non-trivial equation and $s \approx t \in Id\mathcal{K}_1$. For σ_6 : Since $s \approx t$ is a non-trivial equation and $s \approx t \in Id\mathcal{K}_1$, by Lemma 4.2, we have $(L(s), L(s)) \in E(\hat{\sigma}_6[s])$, $(L(t), L(t)) \in E(\hat{\sigma}_6[t])$ and for any $x, y \in V(s)$, $x \neq y$, $(x, y) \in E(\hat{\sigma}_6[s])$ if and only if $(x, y) \in E(\hat{\sigma}_6[t])$. Then by Table 1 and Remark 4.1, we get $\hat{\sigma}_6[s] \approx \hat{\sigma}_6[t] \in Id\mathcal{K}_1$.

For σ_{12} : Suppose that there exists $y \in V(s)$ such that $(y, y) \in E(\hat{\sigma}_{12}[s])$. By Lemma 4.2, we have $(y, y) \in E(s)$. Since $s \approx t \in Id\mathcal{K}_1$, we get there exists $z \in V(s)$ such that $(z, z) \in E(t)$. Hence, $(z, z) \in E(\hat{\sigma}_{12}[t])$. Since $s \approx t \in Id\mathcal{K}_1$, we have for any $x, y \in V(s)$, $x \neq y$, $(x, y) \in E(s)$ if and only if $(x, y) \in E(t)$. By Lemma 4.2, we get for any $x, y \in V(s)$, $x \neq y$, $(x, y) \in E(\hat{\sigma}_{12}[s])$ if and only if $(x, y) \in E(\hat{\sigma}_{12}[t])$. Then by Table 1 and Remark 4.1, we get $\hat{\sigma}_{12}[s] \approx \hat{\sigma}_{12}[t] \in Id\mathcal{K}_1$.

Because of $\sigma_{12} \circ_N \sigma_6 = \sigma_{14}$. We have $\hat{\sigma}_{14}[s] \approx \hat{\sigma}_{14}[t] \in Id\mathcal{K}_1$. \square

For \mathcal{K}_2 , we have the following relations:

$$(i) \sigma_8 \sim_{\mathcal{K}_2} \sigma_{10}, \quad (ii) \sigma_9 \sim_{\mathcal{K}_2} \sigma_{11}, \quad (iii) \sigma_{14} \sim_{\mathcal{K}_2} \sigma_{16} \sim_{\mathcal{K}_2} \sigma_{18}, \quad (iv) \sigma_{15} \sim_{\mathcal{K}_2} \sigma_{17} \sim_{\mathcal{K}_2} \sigma_{19}.$$

Then we get,

$$M_{\mathcal{K}_2} = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8, \sigma_9, \sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{15}\}.$$

For \mathcal{K}_3 , we have the following relations:

$$(i) \sigma_8 \sim_{\mathcal{K}_3} \sigma_{10}, \quad (ii) \sigma_9 \sim_{\mathcal{K}_3} \sigma_{11}, \quad (iii) \sigma_{14} \sim_{\mathcal{K}_3} \sigma_{16} \sim_{\mathcal{K}_3} \sigma_{18}, \quad (iv) \sigma_{15} \sim_{\mathcal{K}_3} \sigma_{17} \sim_{\mathcal{K}_3} \sigma_{19}.$$

Then we get,

$$M_{\mathcal{K}_3} = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8, \sigma_9, \sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{15}\}.$$

For \mathcal{K}_4 , we have the following relations:

$$(i) \sigma_6 \sim_{\mathcal{K}_4} \sigma_{10}, \quad (ii) \sigma_7 \sim_{\mathcal{K}_4} \sigma_{11}, \quad (iii) \sigma_{14} \sim_{\mathcal{K}_4} \sigma_{16} \sim_{\mathcal{K}_4} \sigma_{18}, \quad (iv) \sigma_{15} \sim_{\mathcal{K}_4} \sigma_{17} \sim_{\mathcal{K}_4} \sigma_{19}.$$

Then we get,

$$M_{\mathcal{K}_4} = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8, \sigma_9, \sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{15}\}.$$

For \mathcal{K}_5 , we have the following relations:

$$(i) \sigma_6 \sim_{\mathcal{K}_5} \sigma_{10}, \quad (ii) \sigma_7 \sim_{\mathcal{K}_5} \sigma_{11}, \quad (iii) \sigma_{14} \sim_{\mathcal{K}_5} \sigma_{16} \sim_{\mathcal{K}_5} \sigma_{18}, \quad (iv) \sigma_{15} \sim_{\mathcal{K}_5} \sigma_{17} \sim_{\mathcal{K}_5} \sigma_{19}.$$

Then we get,

$$M_{\mathcal{K}_5} = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8, \sigma_9, \sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{15}\}.$$

Theorem 4.11.

$$(i) PM_{\mathcal{K}_2} = \{\sigma_0, \sigma_6, \sigma_8, \sigma_{12}, \sigma_{14}\}. \quad (ii) PM_{\mathcal{K}_3} = \{\sigma_0, \sigma_6, \sigma_8, \sigma_{12}, \sigma_{14}\}.$$

$$(iii) \ PM_{\mathcal{K}_4} = \{\sigma_0, \sigma_6, \sigma_8, \sigma_{12}, \sigma_{14}\}. \quad (iv) \ PM_{\mathcal{K}_5} = \{\sigma_0, \sigma_6, \sigma_8, \sigma_{12}, \sigma_{14}\}.$$

Proof. Assume that $s \approx t$ is a non-trivial equation and $s \approx t \in Id\mathcal{K}_2$. Since $s \approx t \in Id\mathcal{K}_2$, by Lemma 4.2, we have for any $x \in V(s)$,

$$A_x^*(\hat{\sigma}_8[s]) = A_x^*(\hat{\sigma}_6[s]) = A_x^*(s) = A_x^*(t) = A_x^*(\hat{\sigma}_6[t]) = A_x^*(\hat{\sigma}_8[t]), \ A_x^*(\hat{\sigma}_{12}[s]) = V(s) = V(t) = A_x^*(\hat{\sigma}_{12}[t]).$$

For σ_6 : For any $x \in V(s)$, suppose that there exists $y \in A_x^*(\hat{\sigma}_6[s])$ such that $(y, y) \in E(\hat{\sigma}_6[s])$. If $(y, y) \in E(s)$, then $y \in A_x^*(s)$ such that $(y, y) \in E(s)$. We get there exists $z \in A_x^*(t)$ such that $(z, z) \in E(t)$. Hence, $z \in A_x^*(\hat{\sigma}_6[t])$ such that $(z, z) \in E(\hat{\sigma}_6[t])$. If $(y, y) \notin E(s)$, then there exists $y' \in V(s)$ such that $(y, y') \in E(s)$. Hence, $(y, y'), (y, y) \in E(\hat{\sigma}_6[t])$. Further, we have, for any $x, y \in V(s)$, $x \neq y$, $(x, y) \in E(\hat{\sigma}_6[s])$ if and only if $(x, y) \in E(\hat{\sigma}_6[t])$. Therefore, $\hat{\sigma}_6[s] \approx \hat{\sigma}_6[t] \in Id\mathcal{K}_2$.

For σ_8 : By Lemma 4.2, we have $(x, x) \in E(\hat{\sigma}_8[s])$ and $(x, x) \in E(\hat{\sigma}_8[t])$ for all $x \in V(s)$, $x \neq L(s)$. For $L(s)$, we get $y \in V(s)$, $y \neq L(s)$, $y \in A_{L(s)}^*(\hat{\sigma}_8[s])$ and $y \in A_{L(t)}^*(\hat{\sigma}_8[t])$ such that $(y, y) \in E(\hat{\sigma}_8[s])$ and $(y, y) \in E(\hat{\sigma}_8[t])$. Further, we have for any $x, y \in V(s)$, $x \neq y$, $(x, y) \in E(\hat{\sigma}_8[s])$ if and only if $(x, y) \in E(\hat{\sigma}_8[t])$. Therefore, $\hat{\sigma}_8[s] \approx \hat{\sigma}_8[t] \in Id\mathcal{K}_2$.

For σ_{12} : By Lemma 4.2 and $s \approx t \in Id\mathcal{K}_2$, we have for any $x, y \in V(s)$, $x \neq y$, $(x, y) \in E(\hat{\sigma}_{12}[s])$ if and only if $(x, y) \in E(\hat{\sigma}_{12}[t])$. For any $x \in V(s)$ suppose that there exists $z \in A_x^*(\hat{\sigma}_{12}[s])$ such that $(z, z) \in E(\hat{\sigma}_{12}[s])$. If $(z, z) \in E(\hat{\sigma}_{12}[t])$, then $z \in A_x^*(\hat{\sigma}_{12}[t])$ such that $(z, z) \in E(\hat{\sigma}_{12}[t])$. Suppose that $(z, z) \notin E(\hat{\sigma}_{12}[t])$. Since $(z, z) \in E(s)$ and $s \approx t \in Id\mathcal{K}_2$. We have there exists $z' \in A_z^*(t)$ such that $(z', z') \in E(t)$. So $z' \in A_x^*(\hat{\sigma}_{12}[t])$ such that $(z', z') \in E(\hat{\sigma}_{12}[t])$. Therefore, $\hat{\sigma}_{12}[s] \approx \hat{\sigma}_{12}[t] \in Id\mathcal{K}_2$.

Because of $\sigma_{12} \circ_N \sigma_6 = \sigma_{14}$. We have $\hat{\sigma}_{14}[s] \approx \hat{\sigma}_{14}[t] \in Id\mathcal{K}_2$. Then by Remark 4.1, we get $PM_{\mathcal{K}_2} = \{\sigma_0, \sigma_6, \sigma_8, \sigma_{12}, \sigma_{14}\}$.

The proof of (ii) – (iv) are similar to the proof of (i). □

For \mathcal{K}_6 , we have the following relations:

$$(i) \ \sigma_{14} \sim_{\mathcal{K}_6} \sigma_{16} \sim_{\mathcal{K}_6} \sigma_{18}, \quad (ii) \ \sigma_{15} \sim_{\mathcal{K}_6} \sigma_{17} \sim_{\mathcal{K}_6} \sigma_{19}.$$

Then, we get

$$M_{\mathcal{K}_6} = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8, \sigma_9, \sigma_{10}, \sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{15}\}.$$

For \mathcal{K}_7 , we have the following relations:

$$(i) \ \sigma_{14} \sim_{\mathcal{K}_7} \sigma_{16} \sim_{\mathcal{K}_7} \sigma_{18}, \quad (ii) \ \sigma_{15} \sim_{\mathcal{K}_7} \sigma_{17} \sim_{\mathcal{K}_7} \sigma_{19}.$$

Then, we get

$$M_{\mathcal{K}_7} = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8, \sigma_9, \sigma_{10}, \sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{15}\}.$$

For \mathcal{K}_8 , we have the following relations:

$$(i) \ \sigma_{14} \sim_{\mathcal{K}_8} \sigma_{16} \sim_{\mathcal{K}_8} \sigma_{18}, \quad (ii) \ \sigma_{15} \sim_{\mathcal{K}_8} \sigma_{17} \sim_{\mathcal{K}_8} \sigma_{19}.$$

Then, we get

$$M_{\mathcal{K}_8} = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8, \sigma_9, \sigma_{10}, \sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{15}\}.$$

For \mathcal{K}_9 , we have the following relations:

$$(i) \sigma_{14} \sim_{\mathcal{K}_9} \sigma_{16} \sim_{\mathcal{K}_9} \sigma_{18}, \quad (ii) \sigma_{15} \sim_{\mathcal{K}_9} \sigma_{17} \sim_{\mathcal{K}_9} \sigma_{19}.$$

Then, we get

$$M_{\mathcal{K}_9} = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8, \sigma_9, \sigma_{10}, \sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{15}\}.$$

Theorem 4.12.

- (i) $PM_{\mathcal{K}_6} = \{\sigma_0, \sigma_6, \sigma_8, \sigma_{10}, \sigma_{12}, \sigma_{14}\}$.
- (ii) $PM_{\mathcal{K}_7} = \{\sigma_0, \sigma_6, \sigma_8, \sigma_{10}, \sigma_{12}, \sigma_{14}\}$.
- (iii) $PM_{\mathcal{K}_8} = \{\sigma_0, \sigma_6, \sigma_8, \sigma_{10}, \sigma_{12}, \sigma_{14}\}$.
- (iv) $PM_{\mathcal{K}_9} = \{\sigma_0, \sigma_6, \sigma_8, \sigma_{10}, \sigma_{12}, \sigma_{14}\}$.

Proof. Assume that $s \approx t$ is a non-trivial equation and $s \approx t \in Id\mathcal{K}_6$. Since $s \approx t \in Id\mathcal{K}_6$, we have for any $x \in V(s)$,

$$C_x^*(\hat{\sigma}_{10}[s]) = C_x^*(\hat{\sigma}_8[s]) = C_x^*(\hat{\sigma}_6[s]) = C_x^*(s) = C_x^*(t) = C_x^*(\hat{\sigma}_6[t]) = C_x^*(\hat{\sigma}_8[t]) = C_x^*(\hat{\sigma}_{10}[t]), \quad C_x^*(\hat{\sigma}_{12}[s]) = V(s) = V(t) = C_x^*(\hat{\sigma}_{12}[t]).$$

For σ_6 : Suppose that there exists $y \in C_x^*(\hat{\sigma}_6[s])$ such that $(y, y) \in E(\hat{\sigma}_6[s])$. We see that $(x, x) \in E(\hat{\sigma}_6[s])$. Suppose that $(x, x) \in E(s)$. Since $s \approx t \in Id\mathcal{K}_6$, we have there exists $z \in C_x^*(t)$ such that $(z, z) \in E(t)$. Hence, $z \in C_x^*(\hat{\sigma}_6[t])$ such that $(z, z) \in E(\hat{\sigma}_6[t])$. If $(x, x) \notin E(s)$, then there exists $z' \neq x$ such that $(x, z') \in E(s)$. We have $(x, z') \in E(t)$, too. Hence, $(x, x) \in E(\hat{\sigma}_6[t])$. Further, we have, for any $x, y \in V(s)$, $x \neq y$, $(x, y) \in E(\hat{\sigma}_6[s])$ if and only if $(x, y) \in E(\hat{\sigma}_6[t])$. Therefore, $\hat{\sigma}_6[s] \approx \hat{\sigma}_6[t] \in Id\mathcal{K}_6$.

For σ_8 : Suppose that there exists $y \in C_x^*(\hat{\sigma}_8[s])$ such that $(y, y) \in E(\hat{\sigma}_8[s])$. Suppose that $(y, y) \in E(s)$. Since $s \approx t \in Id\mathcal{K}_6$, we have there exists $z \in C_x^*(t)$ such that $(z, z) \in E(t)$. Hence, $z \in C_x^*(\hat{\sigma}_8[t])$ such that $(z, z) \in E(\hat{\sigma}_8[t])$. If $(y, y) \notin E(s)$, then there exists $z' \neq x$ such that $(z', y) \in E(s)$. We have $(z', y) \in E(t)$, too. Hence, $(y, y) \in E(\hat{\sigma}_8[t])$. Further, we have, for any $x, y \in V(s)$, $x \neq y$, $(x, y) \in E(\hat{\sigma}_8[s])$ if and only if $(x, y) \in E(\hat{\sigma}_8[t])$. Therefore, $\hat{\sigma}_8[s] \approx \hat{\sigma}_8[t] \in Id\mathcal{K}_6$.

For σ_{10} : By Lemma 4.2, we have $(x, x) \in E(\hat{\sigma}_{10}[s])$ and $(x, x) \in E(\hat{\sigma}_{10}[t])$ for all $x \in V(s)$. Hence, for any $x \in V(s)$, we get $x \in C_x^*(\hat{\sigma}_{10}[s])$ such that $(x, x) \in E(\hat{\sigma}_{10}[s])$ and $x \in C_x^*(\hat{\sigma}_{10}[t])$ such that $(x, x) \in E(\hat{\sigma}_{10}[t])$. Further, we have, for any $x, y \in V(s)$, $x \neq y$, $(x, y) \in E(\hat{\sigma}_{10}[s])$ if and only if $(x, y) \in E(\hat{\sigma}_{10}[t])$. Therefore, $\hat{\sigma}_{10}[s] \approx \hat{\sigma}_{10}[t] \in Id\mathcal{K}_6$.

For σ_{12} : By Lemma 4.2 and $s \approx t \in Id\mathcal{K}_6$, we have for any $x, y \in V(s)$, $x \neq y$, $(x, y) \in E(\hat{\sigma}_{12}[s])$ if and only if $(x, y) \in E(\hat{\sigma}_{12}[t])$. For any $x \in V(s)$, suppose that there exists $z \in C_x^*(\hat{\sigma}_{12}[s])$ such that $(z, z) \in E(\hat{\sigma}_{12}[s])$. If $(z, z) \in E(\hat{\sigma}_{12}[t])$, then $z \in C_x^*(\hat{\sigma}_{12}[t])$ such that $(z, z) \in E(\hat{\sigma}_{12}[t])$. Suppose that $(z, z) \notin E(\hat{\sigma}_{12}[t])$. Since $(z, z) \in E(s)$ and $s \approx t \in Id\mathcal{K}_6$. We have there exists $z' \in C_z^*(t)$ such that $(z', z') \in E(t)$. So $z' \in C_x^*(\hat{\sigma}_{12}[t])$ such that $(z', z') \in E(\hat{\sigma}_{12}[t])$. Therefore, $\hat{\sigma}_{12}[s] \approx \hat{\sigma}_{12}[t] \in Id\mathcal{K}_6$.

Because of $\sigma_{12} \circ_N \sigma_6 = \sigma_{14}$. We have $\hat{\sigma}_{14}[s] \approx \hat{\sigma}_{14}[t] \in Id\mathcal{K}_6$. Then by Remark 4.1, we get $PM_{\mathcal{K}_6} = \{\sigma_0, \sigma_6, \sigma_8, \sigma_{10}, \sigma_{12}, \sigma_{14}\}$.

The proof of (ii) – (iv) are similar to the proof of (i). \square

Now, we apply our results to characterize all hyperidentities in each $(x(yz))z$ with loop graph variety. Clearly, if s and t are trivial terms, then $s \approx t$ is a hyperidentity in each $(x(yz))z$ with loop graph variety if and only if $L(s) = L(t)$, $R(s) = R(t)$ and $s \approx t$ which $G(s) = G(t)$, $L(s) = L(t)$ is a hyperidentity in each $(x(yz))z$ with loop graph variety, too. So, we consider the case that $s \approx t$ is a non-trivial equation.

Theorem 4.13. *An identity $s \approx t$ in $\mathcal{K} \in \{\mathcal{K}_0, \mathcal{K}_1, \mathcal{K}_2, \dots, \mathcal{K}_9\}$, where $s \approx t$ is a non-trivial equation, is a hyperidentity in \mathcal{K} if and only if $\hat{\sigma}_5[s] \approx \hat{\sigma}_5[t]$ is also an identity in \mathcal{K} .*

Proof. For \mathcal{K}_0 : It was proven in [15]. Consider for \mathcal{K}_1 . If $s \approx t$ is a hyperidentity in \mathcal{K}_1 , then $\hat{\sigma}_5[s] \approx \hat{\sigma}_5[t]$ is an identity in \mathcal{K}_1 . Conversely, assume that $s \approx t$ is an identity in \mathcal{K}_1 and that $\hat{\sigma}_5[s] \approx \hat{\sigma}_5[t]$ is an identity in \mathcal{K}_1 , too. We have to prove that $s \approx t$ is closed under all graph hypersubstitutions from $M_{\mathcal{K}_1}$.

If σ is a proper, then we get $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}_1$. By assumption, $\hat{\sigma}_5[s] \approx \hat{\sigma}_5[t]$ is an identity in \mathcal{K}_1 .

For $\sigma_1, \sigma_2, \sigma_3$ and σ_4 , we have $\hat{\sigma}_1[s] = L(s) = L(t) = \hat{\sigma}_1[t]$, $\hat{\sigma}_2[s] = L(\hat{\sigma}_5[s]) = L(\hat{\sigma}_5[t]) = \hat{\sigma}_2[t]$, $\hat{\sigma}_3[s] = L(s)L(s) = L(t)L(t) = \hat{\sigma}_3[t]$ and $\hat{\sigma}_4[s] = L(\hat{\sigma}_5[s])L(\hat{\sigma}_5[s]) = L(\hat{\sigma}_5[t])L(\hat{\sigma}_5[t]) = \hat{\sigma}_4[t]$.

Since $\sigma_6 \circ_N \sigma_5 = \sigma_7$, $\sigma_{12} \circ_N \sigma_5 = \sigma_{13}$, $\sigma_{14} \circ_N \sigma_5 = \sigma_{15}$ and $\hat{\sigma}[\hat{\sigma}_5[t']] = \hat{\sigma}'[t'^d]$ for all $\sigma \in M_{\mathcal{K}_1}$, $t' \in T(X)$. We have that $\hat{\sigma}_7[s] \approx \hat{\sigma}_7[t]$, $\hat{\sigma}_{13}[s] \approx \hat{\sigma}_{13}[t]$, $\hat{\sigma}_{15}[s] \approx \hat{\sigma}_{15}[t]$ are identities in \mathcal{K}_1 .

The proof of $\mathcal{K} \in \{\mathcal{K}_2, \mathcal{K}_3, \dots, \mathcal{K}_9\}$ are similar to the proof of \mathcal{K}_1 . \square

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