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Some Classes of Difference Paranormed Sequence Spaces Defined by Orlicz Functions

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Abstract : In this article we introduce the difference paranormed sequence spaces $c_0\{M,\Delta,p,q\}$, $c\{M,\Delta,p,q\}$ and $\ell_\infty\{M,\Delta,p,q\}$ defined by Orlicz functions. We study its different properties like solidness, symmetricity, completeness etc. and prove some inclusion results.

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1 Introduction

Throughout the article w, c, c_0 and ℓ_∞ denote the spaces of all convergent, null and bounded sequences, respectively. The zero sequence (0,0,0,...) is denoted by θ and $p = (p_k)$ is a sequence of strictly positive real numbers. Further the sequence (p_k^{-1}) will be represented by (t_k). The notion of paranormed sequences was introduced by Nakano [6] and Simons [8]. It was further investigated by Maddox [5], Lascarides [4] and many others.

The notion of difference sequence space was introduced by Kizmaz [1] as follows:

$$Z(\Delta) = \Big\{x = (x_k) : (\Delta x_k) \in Z\Big\},$$

for $Z = c,\ c_0$ and ℓ_∞, where $(\Delta x_k) = (x_k - x_{k+1})$.

An *Orlicz function* is a function $M : [0,\infty) \to [0,\infty)$, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$, as $x \to \infty$. If the convexity of the Orlicz function M is replaced by

$$M(x+y) \le M(x) + M(y),$$

then this function is called modulus function, introduced by Nakano [6] and studied by Ruckle [7] and further investigated by many others.

Lindenstrauss and Tzafriri [3] used the idea of Orlicz function to construct the sequence space

$$\ell_M = \Big\{x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \ \text{ for some } \ \rho > 0\Big\}.$$

The space ℓ_M with the norm

$$||x|| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\Big\}.$$

becomes a Banach space, called as an Orlicz sequence space. The space ℓ_M is closely related to the space ℓ_p which is an Orlicz sequence space with $M(x) = |x|^p$ for $1 \leq p < \infty$.

An Orlicz function M is said to satisfy Δ_2-condition for all values of u, if there exists a constant $K > 0$, such that

$$M(2u) \leq KM(u), (u \geq 0).$$

The Δ_2-condition is equivalent to the inequality $M(Lu) \leq KLM(u)$ for all values of u and for $L > 1$ (see for instance Krasnosleskii and Rutitsky [2]).

2 Definitions and Preliminaries

A sequence space E is said to be *solid (or normal)* if $(\alpha_k x_k) \in E$, whenever $(x_k) \in E$, for all sequence (α_k) of scalars with $|\alpha_k| \leq 1$ for all $k \in N$.

A sequence space E is said to be *symmetric* if $(x_{\pi(n)}) \in E$, whenever $(x_n) \in E$, where π is a permutation on N.

Let $H = \sup p_k$ and $D = \max(1, 2^{H-1})$, then it is well known that

$$|a_k + b_k|^{p_k} \leq D\Big\{|a_k|^{p_k} + |b_k|^{p_k}\Big\}.$$

Remark 1 Let M be an Orlicz function and $0 < \lambda < 1$, then for all $x > 0$, $M(\lambda x) \leq \lambda M(x)$.

Let M be an Orlicz function, then we have the following known Orlicz sequence spaces:

$$c_0(M, \Delta, p) = \Big\{(x_k) \in w : \left[M\left(\frac{|\Delta x_k|}{\rho}\right)\right]^{p_k} \to 0, \text{ as } k \to \infty, \text{ for some } \rho > 0\Big\}.$$

$$c(M, \Delta, p) = \Big\{(x_k) \in w : \left[M\left(\frac{|\Delta x_k - L|}{\rho}\right)\right]^{p_k} \to 0, \text{ as } k \to \infty, \text{ for some } L \in C \text{ and for some } \rho > 0\Big\}.$$

$$\ell_\infty(M,\Delta,p)=\Big\{(x_k)\in w:\sup_k\Big[M\Big(\frac{|\Delta x_k|}{\rho}\Big)\Big]^{p_k}<\infty,\quad\text{for some}\quad\rho>0\Big\}.$$

Let (X,q) be a seminormed space seminormed by q. Now we introduce the following sequence spaces.

$$c_0\{M,\Delta,p,q\}=\Big\{(x_k)\in w:\Big[M\Big(\frac{q(\Delta x_k)}{\rho}\Big)\Big]^{p_k}t_k\to 0,\text{ as }k\to\infty,\text{ for some }\rho>0\Big\}.$$

$$c\{M,\Delta,p,q\}=\Big\{(x_k)\in w:\Big[M\Big(\frac{q(\Delta x_k-L)}{\rho}\Big)\Big]^{p_k}t_k\to 0,\text{ as }k\to\infty,\text{ for some } L\in X\text{ and for some }\rho>0\Big\}.$$

$$\ell_\infty\{M,\Delta,p,q\}=\Big\{(x_k)\in w:\sup_k\Big\{\Big[M\Big(\frac{q(\Delta x_k)}{\rho}\Big)\Big]^{p_k}t_k\Big\}<\infty,\text{ for some }\rho>0\Big\}.$$

Lascarides [4] has shown that the sequence spaces $c_0\{p\}$ and $\ell_\infty\{p\}$ are linear spaces for any positive sequence $p=(p_k)$.

Two sets of sequences E and F are said to be equivalent if there exists a sequence $u=(u_k)$ of strictly positive numbers such that the mapping $u:E\to F$ defined by $(u_kx_k)\in F$, whenever $(x_k)\in E$ is a one to one correspondence. We write $E\cong F(u)$. Clearly $E\cong F(u)$ implies $F\cong E(u^{-1})$, where $u^{-1}=(u_k^{-1})$.

The following results will be used for establishing some results of this article. It were proved in Lascarides [4] :

Lemma 2.1 *Let $h=\inf p_k$ and $H=\sup p_k$, then the following are equivalent :*

(i) $H<\infty$ *and* $h>0$.

(ii) $c_0(p)=c_0$ *or* $\ell_\infty(p)=\ell_\infty$.

(iii) $\ell_\infty\{p\}=\ell_\infty(p)$.

(iv) $c_0\{p\}=c_0(p)$.

(v) $\ell\{p\}=\ell(p)$.

Lemma 2.2 *Let p,q be two sequences of strictly positive numbers. Then $c_0\{p\}\cong c_0\{q\}$ if and only if there exists a sequence $u=(u_k)$ of strictly positive numbers such that*

$$\lim_N\limsup_k\frac{(u_kp_k^{p_k^{-1}}N^{-(1+\frac{1}{p_k})})^{q_k}}{q_k}=0 \tag{1}$$

and

$$\lim_N\limsup_k\frac{(u_kq_k^{q_k^{-1}}N^{-(1+\frac{1}{q_k})})^{p_k}}{p_k}=0. \tag{2}$$

Lemma 2.3 *Let the sequence* $a = (a_k) = (q_k^{q_k^{-1}} p_k^{-p_k^{-1}})$. *Then* $c_0\{p\} \cong c_0\{q\}$ *if and only if the following conditions hold*

$$\lim_N \limsup_k \ N^{q_k(1+p_k^{-1})} = 0 \tag{3}$$

and

$$\lim_N \limsup_k \ N^{-p_k(1+q_k^{-1})} = 0. \tag{4}$$

Lemma 2.4 *Let the sequence* $a = (a_k) = (q_k^{q_k^{-1}} p_k^{-p_k^{-1}})$. *Then*

$$\lim_{k\to\infty} \left(\frac{1}{p_k} - \frac{1}{q_k}\right) = 0 \quad \textit{implies} \ \ c_0\{p\} \cong c_0\{q\}.$$

Lemma 2.5 *Let* $f_k = \frac{p_k}{q_k}$ *for every* $k \in N$. *Let* (f_k) *and* (f_k^{-1}) *be both in* ℓ_∞. *Then* $\ell_\infty\{p\} \cong \ell_\infty\{q\}(f)$.

Lemma 2.6 *Let* $q \in \ell_\infty$. *Then* $\ell_\infty\{p\} \subseteq \ell_\infty\{q\}$ *if and only if*

$$\liminf_k \ q_k(Np_k)^{-q_k p_k^{-1}} > 0, \tag{5}$$

for every integer $N > 1$.

Lemma 2.7 *Let* $q \in \ell_\infty$ *and* $c_0\{p\} \cong c_0\{q\}$, *then* $c_0(p) \cong c_0(q)$.

3 Main Results

In this section we prove the results of this article.

Theorem 3.1 *The classes* $c_0\{M, \Delta, p, q\}$, $c\{M, \Delta, p, q\}$ *and* $\ell_\infty\{M, \Delta, p, q\}$ *are linear spaces, for any sequence* $p = (p_k)$ *of strictly positive numbers.*

Proof. We establish it for the case $c_0\{M, \Delta, p, q\}$ and rest of the cases will follow similarly. Let $(x_k),\ (y_k) \in c_0\{M, \Delta, p, q\}$ and $\alpha,\ \beta \in C$. Then there exists $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$\left[M\left(\frac{q(\Delta x_k)}{\rho_1}\right)\right]^{p_k} t_k \to 0, \text{ as } k \to \infty \tag{6}$$

and

$$\left[M\left(\frac{q(\Delta y_k)}{\rho_2}\right)\right]^{p_k} t_k \to 0, \text{ as } k \to \infty. \tag{7}$$

Let $\rho = \max\{2|\alpha|\rho_1,\ 2|\beta|\rho_2\}$. By (6) and (7), we then have

$$\left[M\left(\frac{q(\alpha\Delta x_k + \beta\Delta y_k)}{\rho}\right)\right]^{p_k} t_k \le D\left[M\left(\frac{q(\Delta x_k)}{\rho_1}\right)\right]^{p_k} t_k + D\left[M\left(\frac{q(\Delta y_k)}{\rho_2}\right)\right]^{p_k} t_k$$
$$\to 0, \text{ as } k \to \infty.$$

Hence $(\alpha x_k + \beta y_k) \in c_0\{M, \Delta, p, q\}$. Therefore $c_0\{M, \Delta, p, q\}$ is a linear space. □

Theorem 3.2 *The space* $\ell_\infty\{M,\Delta,p,q\}$ *is paranormed by*

$$g(x) = q(x_1) + \inf\left\{\rho^{\frac{p_k}{J}} : \sup_{k\geq 1}\left\{M\left(\frac{q(\Delta x_k)}{\rho}\right)t_k^{\frac{1}{p_k}}\right\} \leq 1, \quad \rho > 0\right\}$$

where $J = \max(1, H)$.

Proof. Clearly $g(\theta) = 0, g(-x) = g(x)$. Next let $x = (x_k), y = (y_k) \in \ell_\infty\{M,\Delta,p,q\}$. Then there exists some $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$M(\frac{q(\Delta x_k)}{\rho_1})t_k^{\frac{1}{p_k}} \leq 1 \text{ and } M(\frac{q(\Delta y_k)}{\rho_2})t_k^{\frac{1}{p_k}} \leq 1.$$

Let $\rho = \rho_1 + \rho_2$. Then we have

$$\begin{aligned}
&\sup_{k\geq 1}\left\{M\left(\frac{q(\Delta x_k + \Delta y_k)}{\rho}\right)t_k^{\frac{1}{p_k}}\right\} \\
&\leq \frac{\rho_1}{\rho_1+\rho_2}\sup_{k\geq 1}\left\{M\left(\frac{q(\Delta x_k)}{\rho}\right)t_k^{\frac{1}{p_k}}\right\} + \frac{\rho_2}{\rho_1+\rho_2}\sup_{k\geq 1}\left\{M\left(\frac{q(\Delta y_k)}{\rho}\right)t_k^{\frac{1}{p_k}}\right\} \\
&\leq 1.
\end{aligned}$$

Now we have

$$\begin{aligned}
g(x+y) &= q(x_1+y_1) + \inf\left\{(\rho_1+\rho_2)^{\frac{p_k}{J}} : \sup_{k\geq 1}\left\{M\left(\frac{q(\Delta x_k + \Delta y_k)}{\rho}\right)\right\}t_k^{\frac{1}{p_k}} \leq 1\right\} \\
&\leq q(x_1) + \inf\left\{(\rho_1)^{\frac{p_k}{J}} : \sup_{k\geq 1}\left\{M\left(\frac{q(\Delta x_k)}{\rho_1}\right)\right\}t_k^{\frac{1}{p_k}} \leq 1\right\} \\
&\quad + q(y_1) + \inf\left\{(\rho_2)^{\frac{p_k}{J}} : \sup_{k\geq 1}\left\{M\left(\frac{q(\Delta y_k)}{\rho_2}\right)\right\}t_k^{\frac{1}{p_k}} \leq 1\right\} \\
&\leq g(x) + g(y).
\end{aligned}$$

Let $\eta \in C$, then the continuity of the product follows from the following equality.

$$\begin{aligned}
g(\eta x) &= q(\eta x_1) + \inf\left\{\rho^{\frac{p_k}{J}} : \sup_{k\geq 1}\left\{M\left(\frac{q(\eta\Delta x_k)}{\rho}\right)\right\}t_k^{\frac{1}{p_k}} \leq 1, \quad \rho > 0\right\} \\
&= |\eta|q(x_1) + \inf\left\{(|\eta|r)^{\frac{p_k}{J}} : \sup_{k\geq 1}\left\{M\left(\frac{q(\Delta x_k)}{r}\right)\right\}t_k^{\frac{1}{p_k}} \leq 1, \quad r > 0\right\},
\end{aligned}$$

where $\frac{1}{r} = \frac{|\eta|}{\rho}$ □

Theorem 3.3 *Let* $p \in \ell_\infty$, *then the spaces* $c_0\{M,\Delta,p,q\}$, $c\{M,\Delta,p,q\}$ *and* $\ell_\infty\{M,\Delta,p,q\}$ *are complete paranormed spaces, paranormed by* g.

Proof. We prove it for the case $\ell_\infty\{M,\Delta,p,q\}$ and the other cases can be established similarly. Let (x^n) be a Cauchy sequence in $\ell_\infty\{M,\Delta,p,q\}$, where $x^n = (x_k^n)_{k=1}^\infty$ for all $n \in \mathbb{N}$. Then $g(x^i - x^j) \to 0$, as $i,\ j \to \infty$.

For a given $\varepsilon > 0$, let r and x_0 be such that $\frac{\varepsilon}{rx_0} > 0$ and $M(\frac{rx_0}{2}) \geq \sup_{k\geq 1}(p_k)^{t_k}$. Now $g(x^i - x^j) \to 0$, as $i,\ j \to \infty$ implies that there exists $m_0 \in \mathbb{N}$ such that

$$g(x^i - x^j) < \frac{\varepsilon}{rx_0},\ \text{for all } i,\ j \geq m_0.$$

Then we obtain $q(x_1^i - x_1^j) < \frac{\varepsilon}{rx_0}$ and

$$\inf\left\{\rho^{\frac{p_k}{J}} : \sup_{k\geq 1}\left\{M\left(\frac{q(\Delta x_k^i - \Delta x_k^j)}{\rho}\right) t_k^{\frac{1}{p_k}}\right\} \leq 1,\quad \rho > 0\right\} < \frac{\varepsilon}{rx_0}. \qquad (8)$$

This shows that (x_1^i) is a Cauchy sequence in X. Since X is complete then (x_1^i) is convergent in X.

Let $\lim_{i\to\infty} x_1^i = x_1$, thus we have $\lim_{j\to\infty} q(x_1^i - x_1^j) < \dfrac{\varepsilon}{rx_0}$, which imply that

$$q(x_1^i - x_1) < \frac{\varepsilon}{rx_0}.$$

Again from (8), we have

$$M\left(\frac{q(\Delta x_k^i - \Delta x_k^j)}{g(x^i - x^j)}\right) t_k^{\frac{1}{p_k}} \leq 1.$$

These implies that

$$M\left(\frac{q(\Delta x_k^i - \Delta x_k^j)}{g(x^i - x^j)}\right) \leq (p_k)^{t_k} \leq M\left(\frac{rx_0}{2}\right).$$

Thus we obtain

$$q(\Delta x_k^i - \Delta x_k^j) < \frac{rx_0}{2}.\frac{\varepsilon}{rx_0} < \frac{\varepsilon}{2}.$$

Therefore, (Δx_k^i) is a Cauchy sequence in X for all $k \in \mathbb{N}$.

Hence (Δx_k^i) converges in X. Let $\lim_{i\to\infty} \Delta x_k^i = y_k$ for all $k \in \mathbb{N}$. Thus we have $\lim_{i\to\infty} \Delta x_2^i = y_1 - x_1$. Proceeding in this way, $\lim_{i\to\infty} \Delta x_{k+1}^i = y_k - x_k$ for all $k \in \mathbb{N}$.

Next we have by continuity of M,

$$\lim_{j\to\infty} \sup_{k\geq 1} M\left(\frac{q(\Delta x_k^i - \Delta x_k^j)}{\rho}\right) t_k^{\frac{1}{p_k}} \leq 1,$$

which implies that

$$\sup_{k\geq 1} M\left(\frac{q(\Delta x_k^i - \Delta x_k)}{\rho}\right) t_k^{\frac{1}{p_k}} \leq 1.$$

Let $i \geq m_0$, then taking infimum of such ρ's we have $g(x^i - x) < \varepsilon$.

Thus $(x^i - x) \in \ell_\infty\{M, \Delta, p, q\}$. Hence $x = x^i - (x^i - x) \in \ell_\infty\{M, \Delta, p, q\}$, since $\ell_\infty\{M, \Delta, p, q\}$ is a linear space. Therefore $\ell_\infty\{M, \Delta, p, q\}$ is complete. $\square$

Using the technique applied in establishing the above result, one can prove the following result.

Proposition 3.4 *The spaces* $c_0\{M, \Delta, p, q\}$*,* $c\{M, \Delta, p, q\}$ *and* $\ell_\infty\{M, \Delta, p, q\}$ *are K-spaces.*

Since the inclusions $c_0\{M, \Delta, p, q\} \subset \ell_\infty\{M, \Delta, p, q\}$ and $c\{M, \Delta, p, q\} \subset \ell_\infty\{M, \Delta, p, q\}$ are proper, in view of Theorem 3.3 we have the following result.

Proposition 3.5 *The spaces* $c_0\{M, \Delta, p, q\}$ *and* $c\{M, \Delta, p, q\}$ *are nowhere dense subsets of* $\ell_\infty\{M, \Delta, p, q\}$.

Theorem 3.6 *The spaces* $c_0\{M, \Delta, p, q\}$*,* $c\{M, \Delta, p, q\}$ *and* $\ell_\infty\{M, \Delta, p, q\}$ *are not solid in general.*

The spaces $c_0\{M, \Delta, p, q\}$, $c\{M, \Delta, p, q\}$ and $\ell_\infty\{M, \Delta, p, q\}$ are not solid follow from the following examples.

Example 3.1 Let $X = c, M(x) = x$ and $p_k = 1$ for all $k \in \mathbb{N}$. Let the sequence (x_k) be defined by $x_k = (x_k^i)$ where $x_k^i = (1, 1, 1, \ldots)$ for all $k \in \mathbb{N}$ which is in $c_0\{M, \Delta, p, q\}$. Now consider the sequence (α_k) defined by $\alpha_k = (-1)^k$ for all $k \in \mathbb{N}$. Then $(\alpha_k x_k)$ does not belong to $c_0\{M, \Delta, p, q\}$. Hence the space $c_0\{M, \Delta, p, q\}$ is not solid in general.

Example 3.2 Let $X = c,\ M(x) = x$ and $p_k = 1$ for all $k \in \mathbb{N}$. Let the sequence (x_k) be defined by $x_k = (x_k^i)$ where $x_k^i = (k, k+1, k+2, \ldots)$ for all $k \in \mathbb{N}$. Then the sequence (x_k) is in $c\{M, \Delta, p, q\}$ as well as in $\ell_\infty\{M, \Delta, p, q\}$. Now consider the sequence (α_k) defined by $\alpha_k = (-1)^k$ for all $k \in \mathbb{N}$. Then $(\alpha_k x_k)$ belong to neither $\ell_\infty\{M, \Delta, p, q\}$ nor $c\{M, \Delta, p, q\}$. Hence the spaces $c\{M, \Delta, p, q\}$ and $\ell_\infty\{M, \Delta, p, q\}$ are not solid in general.

Theorem 3.7 *The spaces* $c_0\{M, \Delta, p, q\}$*,* $c\{M, \Delta, p, q\}$ *and* $\ell_\infty\{M, \Delta, p, q\}$ *are not symmetric in general.*

To show that the spaces are not symmetric in general, consider the following example.

Example 3.3 Let $X = c,\ M(x) = x$ and $p_k = 2$ for all $k \in \mathbb{N}$. Let the sequence (x_k) be defined by $x_k = (x_k^i)$ where $x_k^i = (k, k+1, k+2, \ldots)$ for all $k \in \mathbb{N}$. Then the sequence (x_k) is in $c\{M, \Delta, p, q\}$ as well as in $\ell_\infty\{M, \Delta, p, q\}$. Now consider the rearrangement (y_k) of (x_k) defined as

$$(y_k) = (x_1^i, x_4^i, x_2^i, x_9^i, x_3^i, x_{16}^i, x_5^i, \ldots)$$

Then (y_k) neither belongs to $c\{M, \Delta, p, q\}$ nor to $\ell_\infty\{M, \Delta, p, q\}$.

Theorem 3.8 *Let M_1 and M_2 be two Orlicz functions satisfying the Δ_2-condition then*

(i) if $(p_k) \in \ell_\infty$ then $Z\{M_1, \Delta, p, q\} \subseteq Z\{M_2 \circ M_1, \Delta, p, q\}$ for $Z = c,\ c_0$ and ℓ_∞.

(ii) $Z\{M_1, \Delta, p, q\} \cap Z\{M_2, \Delta, p, q\} \subseteq Z\{M_1 + M_2, \Delta, p, q\}$ for $Z = c,\ c_0$ and ℓ_∞.

Proof. (i) Let $(x_k) \in c_0\{M, \Delta, p, q\}$. Then from the definition we have, there exists $\rho > 0$ such that

$$\left\{\left[M_1\left(\frac{q(\Delta x_k)}{\rho}\right)\right]^{p_k} t_k\right\} \to 0, \text{ as } k \to \infty.$$

Let $y_k = M_1\left(\frac{q(\Delta x_k)}{\rho}\right)$ for all $k \in \mathbb{N}$. Let $0 < \delta < 1$ be chosen. For $y_k \geq \delta$ we have

$$y_k < \frac{y_k}{\delta} < 1 + \frac{y_k}{\delta}.$$

Since M_2 satisfies Δ_2-condition, therefore there exists a $K \geq 1$ such that

$$M_2(y_k) < \frac{K y_k}{2\delta} M_2(2) + \frac{K y_k}{2\delta} M_2(2) = K M_2(2) \frac{y_k}{\delta}.$$

Then we have

$$\begin{aligned}\left[(M_2 \circ M_1)\left(\frac{q(\Delta x_k)}{\rho}\right)\right]^{p_k} t_k &= \left[M_2\Big\{M_1(\frac{q(\Delta x_k)}{\rho})\Big\}\right]^{p_k} t_k \\ &= [M_2(y_k)]^{p_k}\, t_k \\ &\leq \max\left\{\sup_k([M_2(1)]^{p_k}),\ \sup_k([K M_2(2)\delta^{-1}]^{p_k})\right\}[y_k]^{p_k} t_k \\ &\to 0, \quad \text{as } k \to \infty.\end{aligned}$$

The other cases can be proved following the above technique.

(ii) Let $(x_k) \in c_0\{M_1, \Delta, p, q\} \cap c_0\{M_2, \Delta, p, q\}$, then there exists $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$\left\{\left[M_1\left(\frac{q(\Delta x_k)}{\rho_1}\right)\right]^{p_k} t_k\right\} \to 0, \quad \text{as } k \to \infty$$

and

$$\left\{\left[M_2\left(\frac{q(\Delta x_k)}{\rho_2}\right)\right]^{p_k} t_k\right\} \to 0, \quad \text{as } k \to \infty.$$

Let $\rho = \max\{\rho_1, \rho_2\}$. The rest follows from the following inequality.

$$\left\{\left[(M_1 + M_2)\left(\frac{q(\Delta x_k)}{\rho}\right)\right]^{p_k} t_k\right\} \leq D\left\{\left[M_1\left(\frac{q(\Delta x_k)}{\rho_1}\right)\right]^{p_k} t_k + \left[M_2\left(\frac{q(\Delta x_k)}{\rho_2}\right)\right]^{p_k} t_k\right\}.$$

Thus $c_0\{M_1, \Delta, p, q\} \cap c_0\{M_2, \Delta, p, q\} \subset c_0\{M_1 + M_2, \Delta, p, q\}$. The cases $c\{M, \Delta, p, q\}$ and $\ell_\infty\{M, \Delta, p, q\}$ can be proved in a similar way. □

It can be easily shown that:

Proposition 3.9 *$Z\{M,p,q\} \subseteq Z\{M,\Delta,p,q\}$ for $Z = c,\ c_0$ and ℓ_∞.*

The following results can be obtained from the lemmas listed in section 2.

Proposition 3.10 *Let $h = \inf p_k$ and $H = \sup p_k$ then the following are equivalent*

(i) $H < \infty$ *and* $h > 0$,

(ii) $c_0\{M,\Delta,p,q\} = c_0(M,\Delta,p,q)$

(iii) $\ell_\infty\{M,\Delta,p,q\} = \ell_\infty(M,\Delta,p,q)$.

Proposition 3.11 *Let $p,\ s$ be two sequences of strictly positive numbers. Then $c_0\{M,\Delta,p,q\} \cong c_0\{M,\Delta,s,q\}$ if and only if there exists a sequence $u = (u_k)$ of strictly positive numbers such that eq.(1) and eq.(2) hold.*

Proposition 3.12 *Let the sequence $a = (a_k) = (s_k^{s_k^{-1}} p_k^{-p_k^{-1}})$. Then $c_0\{M,\Delta,p,q\} \cong c_0\{M,\Delta,s,q\}$ if and only if eq.(3) and eq.(4) hold.*

Proposition 3.13 *Let the sequence $a = (a_k) = (s_k^{s_k^{-1}} p_k^{-p_k^{-1}})$. Then*

$$\lim_{k\to\infty}\left(\frac{1}{p_k} - \frac{1}{s_k}\right) = 0 \ \text{ implies } \ c_0\{M,\Delta,p,q\} \cong c_0\{M,\Delta,s,q\}.$$

Proposition 3.14 *Let $f_k = \frac{p_k}{s_k}$ for every $k \in \mathbb{N}$. Let (f_k) and (f_k^{-1}) both be in ℓ_∞. Then $\ell_\infty\{M,\Delta,p,q\} \cong \ell_\infty\{M,\Delta,s,q\}(f)$.*

Proposition 3.15 *Let $s = (s_k) \in \ell_\infty$. Then $\ell_\infty\{M,\Delta,p,q\} \subseteq \ell_\infty\{M,\Delta,s,q\}$ if and only if eq.(5) holds.*

Proposition 3.16 *Let $s = (s_k) \in \ell_\infty$ and $c_0\{M,\Delta,p,q\} \cong c_0\{M,\Delta,s,q\}$ then $c_0(M,\Delta,p) \cong c_0(M,\Delta,q)$.*

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